Chapter 3

Continuous variables

Continuous variables is a fancy name we give to harmonic oscillators. So far we have talked about systems with a finite number of states, so that everything is discrete. Now we will talk about harmonic oscillators, which have an infinite number of levels. Of course, these levels are also discrete. However, it turns out that many things can be described in terms of continuous variables, such as coherent states and the quantum phase space representation.

Continuous variables systems occur naturally in many platforms. The most important example is quantum optics, where, it turns out, the quantum properties of the electromagnetic field can be represented in terms of harmonic oscillators. Other continuous variable platforms include trapped ions, nano- or micro-mechanical oscillators and Bose-Einstein condensates.

This chapter provides a first look into continuous variables. After this, we will start to work with both discrete and continuous variable systems, side by side. More advanced properties will be discussed later, or can be found in the excellent book by Alessio Serafini entitled *"Quantum Continuous Variables"*.

3.1 Creation and annihilation operators

The starting point of continuous variable systems is an operator a called the *annihilation operator* and its Hermitian conjugate $a^\dagger$, called the *creation operator*. They are defined so as to satisfy the following algebra:

$$\boxed{[a, a^\dagger] = 1.} \tag{3.1}$$

All properties of these operators and the Hilbert space they represent follow from this simple commutation relation, as we will see below. Another set of operators which can be used as the starting point of the discussion are the

position and momentum operators q and p. They satisfy

$$\boxed{[q,p]=i.} \tag{3.2}$$

In quantum optics, they no longer represent position and momentum, but are related to the electric and magnetic fields. In this case they are usually called *quadrature operators*. We define q and p to be dimensionless. Then they are related to the creation and annihilation operators according to

$$\begin{aligned} q &= \frac{1}{\sqrt{2}}(a^\dagger + a) & \qquad a &= \frac{1}{\sqrt{2}}(q+ip) \\ & \iff & & \\ p &= \frac{i}{\sqrt{2}}(a^\dagger - a) & \qquad a^\dagger &= \frac{1}{\sqrt{2}}(q-ip)\,. \end{aligned} \tag{3.3}$$

From this it can be clearly seen that q and p are Hermitian operators, even though a is not. Also, please take a second to verify that with this relation Eq. (3.1) indeed implies (3.2) and vice-versa.

Mechanical oscillators

The operators a, $a^\dagger$, q and p appear in two main contexts: mechanical oscillators and second quantization. The latter will be discussed below. A mechanical oscillator is specified by the Hamiltonian

$$H = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2, \tag{3.4}$$

where m is the mass and ω is the frequency. Moreover Q and P are the position and momentum operators satisfying

$$[Q,P] = i\hbar. \tag{3.5}$$

Now define the dimensionless operators

$$q = \sqrt{\frac{m\omega}{\hbar}}Q, \qquad p = \frac{P}{\sqrt{m\hbar\omega}}. \tag{3.6}$$

Then Eq. (3.5) implies that q and p will satisfy (3.2). In terms of q and p, the Hamiltonian (3.4) becomes

$$H = \frac{\hbar\omega}{2}(p^2+q^2), \tag{3.7}$$

which, you have to admit, is way more elegant than (3.4). Using now Eq. (3.3) we finally write the Hamiltonian as

$$H = \hbar\omega(a^\dagger a + 1/2). \tag{3.8}$$

Eqs. (3.7) and (3.8) show very well why $\hbar$ is not important: it simply redefines the energy scale. If we set $\hbar = 1$, as we shall henceforth do, we are simply measuring energy in units of frequency.

In the days of Schrödinger, harmonic oscillators were usually used either as toy models or as an effective description of some other phenomena such as, for instance, the vibration of molecules. In the last two decades this has change and we are now able to observe quantum effects in actual mechanical mesoscopic- (nano- or micro-) oscillators. This is usually done by engineering thin suspended membranes, which can then undergo mechanical vibrations. This field is usually known as **optomechanics** since most investigations involve the contact of the membranes with radiation. I find it absolutely fascinating that in our day and age we can observe quantum effects as awesome as entanglement and coherence in these mechanical objects. I love the century we live in!

An algebraic problem

In Eq. (3.8) we see the appearance of the Hermitian operator $a^\dagger a$, called the **number operator**. To find the eigenstuff of H we therefore only need to know the eigenstuff of $a^\dagger a$. We have therefore arrived at a very clean mathematical problem: *given a non-Hermitian operator a, satisfying $[a, a^\dagger] = 1$, find the eigenvalues and eigenvectors of $a^\dagger a$.* This is a really important problem that appears often in all areas of quantum physics: given an algebra, find the eigenstuff. Maybe you have seen this before, but I will nonetheless do it again, because I think this is one of those things that everyone should know.

Here we go. Since $a^\dagger a$ is Hermitian, its eigenvalues must be real and its eigenvectors can be chosen to form an orthonormal basis. Let us write them as

$$\boxed{a^\dagger a|n\rangle = n|n\rangle.} \tag{3.9}$$

Our goal is to find the allowed n and the corresponding $|n\rangle$. The first thing we notice is that $a^\dagger a$ must be positive semi-definite operator, so n cannot be negative:

$$n = \langle n|a^\dagger a|n\rangle \geq 0.$$

Next we use Eq. (3.1) to show that

$$\boxed{[a^\dagger a, a] = -a, \qquad [a^\dagger a, a^\dagger] = a^\dagger.} \tag{3.10}$$

This type of structure is a signature of a *ladder* like spectrum (that is, when the eigenvalues are equally spaced). To see that, we use these commutation relations to compute:

$$(a^\dagger a)a|n\rangle = [a(a^\dagger a) - a]|n\rangle = a(a^\dagger a - 1)|n\rangle = (n-1)a|n\rangle.$$

Hence, we conclude that if $|n\rangle$ is an eigenvector with eigenvalue n, then $a|n\rangle$ is also an eigenvector, but with eigenvalue $(n-1)$ [This is the key argument. Make sure you understand what this sentence means.]. However, I wouldn't call this $|n-1\rangle$ just yet because $a|n\rangle$ is not normalized. Thus we need to write

$$|n-1\rangle = \gamma a|n\rangle,$$

where γ is a normalization constant. To find it we simply write

$$1 = \langle n-1|n-1\rangle = |\gamma|^2\langle n|a^\dagger a|n\rangle = |\gamma|^2 n.$$

Thus $|\gamma|^2| = 1/n$. The actual sign of γ is arbitrary so we choose it for simplicity as being real and positive. We then get

$$|n-1\rangle = \frac{a}{\sqrt{n}}|n\rangle.$$

From this analysis we conclude that a reduces the eigenvalues by unity:

$$a|n\rangle = \sqrt{n}|n-1\rangle.$$

We can do a similar analysis with $a^\dagger$. We again use Eq. (3.10) to compute

$$(a^\dagger a)a^\dagger|n\rangle = (n+1)a^\dagger|n\rangle.$$

Thus $a^\dagger$ raises the eigenvalue by unity. The normalization factor is found by a similar procedure: we write $|n+1\rangle = \beta a^\dagger|n\rangle$, for some constant β, and then compute

$$1 = \langle n+1|n+1\rangle = |\beta|^2\langle n|aa^\dagger|n\rangle = |\beta|^2\langle n|(1+a^\dagger a)|n\rangle = |\beta|^2(n+1).$$

Thus

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.$$

These results are important, so let me summarize them in a boxed equation:

$$\boxed{a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle}. \tag{3.11}$$

From this formula we can see why the operators a and $a^\dagger$ also receive the name *lowering and raising operators*.

Now comes the trickiest (and most beautiful) argument. We have seen that if n is an eigenvalue, then $n \pm 1$, $n \pm 2$, etc., will all be eigenvalues. But this doesn't mean that n itself should be an integer. Maybe we find one eigenvalue which is 42.42 so that the eigenvalues are 41.42, 43.42 and so on. Of course, you know that is not true and n must be integer. To show that, we proceed as follows. Suppose we start with some eigenstate $|n\rangle$ and keep on applying a a bunch of times. At each application we will lower the eigenvalue by one tick:

$$a^\ell|n\rangle = \sqrt{n(n-1)\dots(n-\ell+1)}|n-\ell\rangle.$$

But this crazy party cannot continue forever because, as we have just discussed, the eigenvalues of $a^\dagger a$ cannot be negative. They can, at most, be zero. The only way for this to happen is if there exists a certain integer ℓ for which $a^\ell|n\rangle \neq 0$ but $a^{\ell+1}|n\rangle = 0$. And this can only happen if $\ell = n$ because, then

$$a^{\ell+1}|n\rangle = \sqrt{n(n-1)\dots(n-\ell+1)(n-\ell)}|n-\ell-1\rangle = 0,$$

and the term $n-\ell$ will vanish. Since ℓ is an integer, we therefore conclude that n must also be an integer. Thus, we finally conclude that

$$\text{eigs}(a^\dagger a) = n \in \{0, 1, 2, 3, \dots\}. \tag{3.12}$$

It is for this reason that $a^\dagger a$ is called the number operator: we usually say $a^\dagger a$ counts the number of quanta in a given state: given a state $|n\rangle$, you first apply a to annihilate one quant and then $a^\dagger$ to create it back again. The proportionality factor is the eigenvalue n. Curiously, this analysis seem to imply that if you want to count how many people there are in a room, you first need to annihilate one person and then create a fresh new human. Quantum mechanics is indeed strange.

This analysis also serves to define the state with $n = 0$, which we call the **vacuum**, $|0\rangle$. It is defined by

$$\boxed{a|0\rangle = 0.} \tag{3.13}$$

We can build all states starting from the vacuum and applying $a^\dagger$ successively:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{3.14}$$

Using this and the algebra of a and $a^\dagger$ it then follows that the states $|n\rangle$ form an orthonormal basis, as expected:

$$\langle n|m\rangle = \delta_{n,m}.$$

The states $|n\rangle$ are called **Fock states**, although this nomenclature is more correctly employed in the case of multiple modes, as we will now discuss.

Multiple modes and second quantization

It is straightforward to generalize the idea of creation and annihilation operators to composite systems. We simply define a set of annihilation operators a_i, where $i = 1, 2, \dots, N$. It is customary to use the word **mode** to label each i. Thus we say things like "mode a_2". These operators are defined to satisfy

$$[a_i, a_j^\dagger] = \delta_{i,j}, \qquad [a_i, a_j] = 0. \tag{3.15}$$

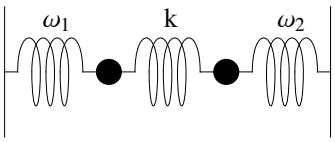

Figure 3.1: Two harmonic oscillators coupled by a harmonic spring.

That is, a_i with $a_i^\dagger$ behaves just like before, whereas a_i with $a_j^\dagger$ commute if $j \neq i$. Moreover annihilation operators always commute among themselves. Taking the adjoint of $[a_i, a_j] = 0$ we see that the same will be true for the creation operators $[a_i^\dagger, a_j^\dagger] = 0$. Using the same transformation as in Eq. (3.3), but with indices everywhere, we can also define quadrature operators q_i and p_i, which will then satisfy

$$[q_i, p_j] = i\delta_{i,j}, \qquad [q_i, q_j] = [p_i, p_j] = 0. \tag{3.16}$$

Multi-mode systems can appear in mechanical contexts. For instance, consider two mechanical oscillators coupled by springs, as in Fig. 3.1. Each oscillator has a natural frequency ω_1 and ω_2 and they are coupled by a spring constant k. Assuming unit mass, the Hamiltonian will then be

$$H = \left(p_1^2 + \frac{\omega_1^2}{2} q_1^2\right) + \left(p_2^2 + \frac{\omega_2^2}{2} q_2^2\right) + \frac{k}{2}(q_1 - q_2)^2. \tag{3.17}$$

If we want we can also transform this into a_i and $a_i^\dagger$, or we can extend it to multiple oscillators forming a chain. In fact, these "harmonic chains" are a widely studied topic in the literature because they can always be solved analytically and they are the starting point for a series of interesting quantum effects. We will have the opportunity to practice with some of these solutions later on.

But by far the most important use of multi-mode systems is in **second quantization**. Since operators pertaining to different modes commute, the Hilbert space of a multi-mode system will be described by a basis

$$|\boldsymbol{n}\rangle = |n_1, n_2, \ldots, n_N\rangle, \qquad n_i = 0, 1, 2, \ldots. \tag{3.18}$$

These are called **Fock states** and are the eigenstates of the number operators $a_i^\dagger a_i$:

$$a_i^\dagger a_i |\boldsymbol{n}\rangle = n_i |\boldsymbol{n}\rangle. \tag{3.19}$$

Thus, $a_i^\dagger a_i$ counts the number of quanta in mode i.

Second quantization is essentially a change of perspective from "quanta" to "particles". After all, what the hell is a quanta anyway? In second quantization we say $a_i^\dagger a_i$ is the operator counting the number of *particles* in mode i. Then $a_i^\dagger$ is the operator which creates a particle in mode i, whereas $a_i^\dagger$ annihilates. You may also be wondering what is a "mode" in this case. Well, there is actually an infinite number of choices. We could take for instance $i = x$, the position

in space. Then $a_x^\dagger$ is the operator which creates a particle at position x. In quantum field theory we call it $\psi^\dagger(x)$ instead. But it's the same thing.

According to Eq. (3.19) each mode can have an arbitrary number n of particles. We then call a_i a **bosonic mode**. So whenever someone says "consider a set of bosonic modes" they mean a set of operators a_i satisfying (3.15). This is to be contrasted with Fermionic systems, for which the only allowed Fock states are $n = 0$ and $n = 1$ (due to the Pauli exclusion principle). We will not discuss much of fermionic systems in this course, but the idea is somewhat similar. We also define creation and annihilation operators, except that now they satisfy a different algebra:

$$\{c_i, c_j^\dagger\} = \delta_{i,j}, \qquad \{c_i, c_j\} = 0, \tag{3.20}$$

where $\{A, B\} = AB + BA$ is the anti-commutator. If we repeat the diagonalization procedure of the last section for this kind of algebra we will find a similar "Fock structure" but with the only allowed eigenvalues being $n_i = 0$ and $n_i = 1$.

The most important bosonic system is the electromagnetic field. The excitations are then the photons and the modes are usually chosen to be the momentum and polarization. Hence, we usually write an annihilation operator as $a_{\boldsymbol{k},\lambda}$ where $\boldsymbol{k} = (k_x, k_y, k_z)$ is the momentum and $\lambda = \pm 1$ is the polarization. Moreover, the Hamiltonian of the electromagnetic field is written as

$$H = \sum_{\boldsymbol{k},\lambda} \omega_{\boldsymbol{k}} a_{\boldsymbol{k},\lambda}^\dagger a_{\boldsymbol{k},\lambda}, \tag{3.21}$$

where $\omega_{\boldsymbol{k}}$ is the frequency of each mode and is given by[1] $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ where c is the speed of light.

You have noticed that my discussion of second quantization was rather shallow. I apologize for that. But I have to do it like this, otherwise we would stray too far. Second quantization is covered in many books on condensed matter, quantum many-body and quantum field theory. A book which I really like is Feynman's *Statistical mechanics: a set of lectures*".

3.2 Some important Hamiltonians

In this section we briefly discuss some important Hamiltonians that appear often in controlled quantum experiments.

Optical cavities

Many controlled experiments take place inside optical cavities, like the one represented in my amazing drawing in Fig. 3.2 (it took me 30 minutes to draw it!). The cavity is made up of highly reflective mirrors allowing the photons to survive for some time, forming standing wave patterns. Unlike in free space, where all radiation modes can exist equally, the confinement inside the cavity

[1] If we define $\omega = 2\pi\nu$ and $|\boldsymbol{k}| = 2\pi/\lambda$ we see that this is nothing but the relation $\nu = \lambda c$ that you learned in high school.

Figure 3.2: An optical cavity of frequency ω_c, pumped from the outside by a laser of frequency ω_p.

favors those radiation modes whose frequencies are close to the *cavity frequency* ω_c, which is related to the geometry of the cavity. It is therefore common to consider only one radiation mode, with operator a and frequency ω_c.

The photons always have a finite lifetime so more photons need to be injected all the time. This is usually done by making one of the mirrors semi-transparent and pumping it with a laser from the outside, with frequency ω_p. Of course, since photons can come in, they can also leak out. This leakage is an intrinsically irreversible process and can only be described using the theory of open quantum systems, which we will get to in the next chapter. Hence, we will omit the process of photon losses for now. The Hamiltonian describing a single mode pumped externally by a laser then has the form

$$H = \omega_c a^\dagger a + \epsilon a^\dagger e^{-i\omega_p t} + \epsilon^* a e^{i\omega_p t}, \tag{3.22}$$

where ϵ is the pump amplitude and is related to the laser power P according to $|\epsilon|^2 = 2\kappa P/\hbar\omega_p$ where κ is the cavity loss rate (the rate at which photons can go through the semi-transparent mirror). This Hamiltonian is very simple, but is time-dependent. Lucky for us, however, this time dependence can be eliminated using the concept of a rotating frame, as will be discussed below.

Jaynes-Cummings and Rabi models

Quantum information has always been intimately related with quantum optics and atomic physics, so *light-matter interaction* is an essential topic in the field. The two most important models in this sense are the Jaynes-Cummings and Rabi models, both of which describe the interaction of a single radiation mode with a single atom, approximated as a two-level system. The basic idea of both models is the exchange of quanta between the two systems; that is, sometimes the atom absorbs a photon and jumps to an excited state and sometimes it emits a photon and drops down to the ground-state. These effects of course take place on free space, but we are usually interested in controlled experiments performed inside optical cavities. The situation is then like that of Fig. 3.3.

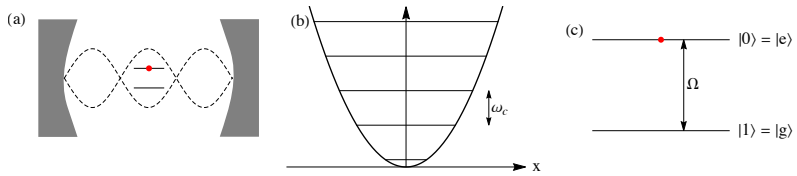

Figure 3.3: (a) Typical scenario for light-matter interaction: an atom, modeled as a two-level system, is placed inside a cavity in which there is only one cavity mode. The atom then absorbs and emits photons jumping up and down from the ground-state to the excited state. (b) The cavity field is represented by a harmonic oscillator of frequency ω_c. (c) The atom is represented as a two-level system (qubit) with energy gap Ω. When the atom Hamiltonian is $+\sigma_z$ then the ground-state will be $|1\rangle$ and the excited state will be $|0\rangle$.

The Jaynes-Cummings model reads

$$H = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z + \lambda(a\sigma_+ + a^\dagger\sigma_-). \tag{3.23}$$

The first two terms are the free Hamiltonians of the cavity field, with frequency ω_c, and the atom, with energy gap Ω. Whenever the atom Hamiltonian is written as $+\sigma_z$, the ground-state will be $|g\rangle = |1\rangle$ and the excited state will be $|e\rangle = |0\rangle$ [see Fig. 3.3(c)]. Finally, the last term in (3.23) is the light-atom coupling. The term $a\sigma_+$ describes the process where a photon is annihilated and the atom jumps to the excited state. Similarly, $a^\dagger\sigma_-$ describes the opposite process. The Hamiltonian must always be Hermitian so every time we include a certain type of process, we must also include its reverse.

The type of interaction in Eq. (3.23) introduces a special *symmetry* to the problem. Namely, it conserves the number of quanta in the system:

$$[H, a^\dagger a + \sigma_z] = 0. \tag{3.24}$$

This means that if you start the evolution with 7 photons and the atom in the ground-state, then at all times you will either have 7 photons + ground-state or 6 photons and the atom in the excited state. This is a very special symmetry and is the reason why the Jaynes-Cummings model turns out to be easy to deal with.

However, if we start with a physical derivation of the light-atom interaction, we will see that it is not exactly like the Jaynes-Cummings Hamiltonian (3.23). Instead, it looks more like the Rabi model

$$H = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z + \lambda(a + a^\dagger)\sigma_x. \tag{3.25}$$

The difference is only in the last term. In fact, if we recall that $\sigma_x = \sigma_+ + \sigma_-$, we get

$$(a + a^\dagger)\sigma_x = (a\sigma_+ + a^\dagger\sigma_-) + (a^\dagger\sigma_+ + a\sigma_-).$$

The first term in parenthesis is exactly the Jaynes-Cummings interaction, so the new thing here is the term $(a^\dagger\sigma_+ + a\sigma_-)$. It describes a process where the atom jumps to the excited state *and* emits a photon, something which seems rather strange at first. Moreover, this new term destroys the pretty symmetry (3.24), making the Rabi model much more complicated to deal with, but also much richer from a physical point of view. Notwithstanding, as we will see below, if λ is small compared to ω_c, Ω this new term becomes negligible and the Rabi model approximately tends to the JC Hamiltonian.

3.3 Rotating frames and interaction picture

In this section I want to introduce the concept of *rotating frames*, which is a small generalization of the interaction and Heisenberg pictures that you may have learned in quantum mechanics. Consider a system with density matrix ρ evolving according to von Neumann's equation (we could do the same with Schrödinger's equation)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H(t), \rho], \tag{3.26}$$

where $H(t)$ is a possibly time-dependent Hamiltonian. We can always move to a *rotating frame* by defning a new density matrix

$$\tilde{\rho}_t = S(t)\rho S^\dagger(t), \tag{3.27}$$

where $S(t)$ is an arbitrary unitary. I will leave to you as an exercise to show that $\tilde{\rho}$ will also obey a von Neumann equation

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -i[\tilde{H}(t), \tilde{\rho}], \tag{3.28}$$

but with an effective Hamiltonian[2]

$$\boxed{\tilde{H}(t) = i\frac{\mathrm{d}S}{\mathrm{d}t}S^\dagger + SHS^\dagger.} \tag{3.29}$$

Thus, we see that in any rotating frame the system always obeys von Neumann's (or Schrödinger's) equation, but the Hamiltonian changes from $H(t)$ to $\tilde{H}(t)$. Note that this result is absolutely general and holds for any unitary $S(t)$. Of course, whether it is useful or not will depend on your smart choice for $S(t)$.

Before we move to applications, I need to mention that computing the first term in Eq (3.29) can be tricky. Usually we write unitaries as $S(t) = e^{iK(t)}$ where

[2] To derive this equation it is necessary to use the following trick: since $SS^\dagger = 1$ then

$$0 = \frac{\mathrm{d}SS^\dagger}{\mathrm{d}t} = S\frac{\mathrm{d}S^\dagger}{\mathrm{d}t} + \frac{\mathrm{d}S}{\mathrm{d}t}S^\dagger \qquad \longrightarrow \qquad S\frac{\mathrm{d}S^\dagger}{\mathrm{d}t} = -\frac{\mathrm{d}S}{\mathrm{d}t}S^\dagger.$$

K is Hermitian. Then, one may easily verify the following BCH expansion

$$\boxed{\frac{\mathrm{d}e^{iK}}{\mathrm{d}t}e^{-iK} = i\frac{\mathrm{d}K}{\mathrm{d}t} + \frac{i^2}{2}[K, \frac{\mathrm{d}K}{\mathrm{d}t}] + \frac{i^3}{3!}[K, [K, \frac{\mathrm{d}K}{\mathrm{d}t}]] + \dots.} \tag{3.30}$$

The important point here is whether or not K commutes with $\mathrm{d}K/\,\mathrm{d}t$. If that is the case then only the first term survives and things are easy and pretty. Otherwise, you may get an infinite series. I strongly recommend you always use this formula, because then you are always sure you will not get into trouble.

Eliminating time-dependences

A simple yet useful application of rotating frames is to eliminate the time-dependence of certain simple Hamiltonians, such as the pumped cavity (3.22). In this case the unitary that does the job is

$$S(t) = e^{i\omega_p t a^\dagger a}. \tag{3.31}$$

That is, we move to a frame that is rotating at the same frequency as the pump laser ω_p. Using the BCH expansion (1.70) one may show that

$$\boxed{e^{i\alpha a^\dagger a} a e^{-i\alpha a^\dagger a} = e^{-i\alpha} a, \qquad e^{i\alpha a^\dagger a} a^\dagger e^{-i\alpha a^\dagger a} = e^{i\alpha} a^\dagger,} \tag{3.32}$$

which are easy to remember: a goes with negative α and $a^\dagger$ with positive α. It then follows that

$$S(t)\left[\epsilon a^\dagger e^{-i\omega_p t} + \epsilon a e^{i\omega_p t}\right] S^\dagger(t) = \epsilon a^\dagger + \epsilon a,$$

while $S(t)$ has no effect on $a^\dagger a$. Moreover, this is one of those cases where only the first term in (3.30) contributes:

$$\frac{\mathrm{d}S}{\mathrm{d}t} S^\dagger = i\omega_p a^\dagger a.$$

Thus Eq. (3.29) becomes

$$\boxed{\tilde{H} = (\omega_c - \omega_p) a^\dagger a + \epsilon a^\dagger + \epsilon^* a.} \tag{3.33}$$

We therefore conclude that in this rotating frame the Hamiltonian is time-independent, but evolves according to the **detuned frequency** $\Delta = \omega_c - \omega_p$. This idea of *detuning* a frequency is extremely important in quantum optics applications since it is an easy way to change the parameters in the problem.

For more general bosonic Hamiltonians containing a pump term, the time-dependence can be eliminated by the same transformation, provided the remainder of the Hamiltonian conserves the number of quanta (i.e., when all operators have an equal number of as and $a^\dagger$s). This is due to the simple rule imposed by (3.32), which says that every a gets a term $e^{-i\omega_p t}$ while every $a^\dagger$ gets a $e^{i\omega_p t}$. Thus, a Hamiltonian such as

$$H = \omega_a^\dagger a + \frac{U}{2} a^\dagger a^\dagger a a + \epsilon a^\dagger e^{-i\omega_p t} + \epsilon^* a e^{i\omega_p t},$$

will lead to a rotating frame Hamiltonian

$$\tilde{H} = (\omega - \omega_p) a^\dagger a + \frac{U}{2} a^\dagger a^\dagger a a + \epsilon a^\dagger + \epsilon^* a.$$

Once you get the hang of it, it is quite easy: detune the frequency and get rid of the exponential. But be careful. This can only be done if the number of as and $a^\dagger$s is the same. For instance,

$$H = \omega_a^\dagger a + \chi(a + a^\dagger)^4 + \epsilon a^\dagger e^{-i\omega_p t} + \epsilon^* a e^{i\omega_p t},$$

would not have a time-independent rotating frame under the transformation (3.31) because if you expand $(a + a^\dagger)^4$ there will be terms with a unbalanced number of as and $a^\dagger$s.

A similar rotating frame transformation also works for qubit systems of the form

$$H = \frac{\Omega}{2}\sigma_z + \frac{\lambda}{2}(\sigma_+ e^{-i\omega_p t} + \sigma_- e^{i\omega_p t}) \tag{3.34}$$

$$= \frac{\Omega}{2}\sigma_z + \frac{\lambda}{2}[\sigma_x \cos(\omega_p t) + \sigma_y \sin(\omega_p t)]. \tag{3.35}$$

This Hamiltonian appears often in magnetic resonance because it represents a spin 1/2 particle subject to a constant field Ω in the z direction and a rotating field λ in the xy plane. Remarkably, the transformation here is almost exactly as in the bosonic case:

$$S(t) = e^{i\omega_p t \sigma_z/2}. \tag{3.36}$$

In this case the idea of a rotating frame becomes a bit more intuitive: the Hamiltonian is time-dependent because there is a field rotating in the xy plane. So to get rid of it, we go to a frame that is rotating around the z axis by an angle $\omega_p t$. I will leave for you to check that this $S(t)$ indeed does the job. One thing that is useful to know is that Eq. (3.32) is translated almost literally to the spin case:

$$e^{i\alpha\sigma_z/2}\sigma_- e^{-i\alpha\sigma_z/2} = e^{-i\alpha}\sigma_-, \qquad e^{i\alpha\sigma_z/2}\sigma_+ e^{-i\alpha\sigma_z/2} = e^{i\alpha}\sigma_+, \tag{3.37}$$

Interaction picture

Now let us consider another scenario. Suppose the Hamiltonian is time-independent but can be written in the standard perturbation-theory-style

$$H = H_0 + V, \tag{3.38}$$

where H_0 is easy to handle but V is not. Then choose

$$S(t) = e^{iH_0 t}. \tag{3.39}$$

Eq. (3.29) then becomes

$$\boxed{\tilde{H}(t) = S(t) V S^\dagger(t).} \tag{3.40}$$

This is the interaction picture: we eliminate the dependence on H_0 at the cost of transforming a time-independent Hamiltonian $H_0 + V$ into a time-dependent Hamiltonian $SVS^\dagger$.

The interaction picture is usually employed as the starting point of time-dependent perturbation theory. We will learn a bit more about this below. But to get a first glimpse, consider the Rabi Hamiltonian (3.25) and let us move to the interaction picture with respect to $H_0 = \omega a^\dagger a + \frac{\Omega}{2}\sigma_z$. Using Eqs. (3.32) and (3.37) we then find

$$\tilde{H}(t) = \lambda\left[a\sigma_+ e^{i(\Omega-\omega_c)t} + a^\dagger \sigma_- e^{-i(\Omega-\omega_c)t}\right] + \lambda\left[a^\dagger\sigma_+ e^{i(\Omega+\omega_c)t} + a\sigma_- e^{-i(\Omega+\omega_c)t}\right]. \tag{3.41}$$

In the interaction picture we see more clearly the difference between the two types of couplings. The first term, which is the Jaynes-Cummings coupling, oscillates in time with a frequency $\Omega - \omega_c$, which will be very small when Ω is close to ω_c. The second term, on the other hand, oscillates quickly with frequency $\omega_c + \Omega$, which is in general a much faster frequency that $\omega_c - \Omega$. We therefore see the appearance of two time scales, the JC term, which is slow, and the Rabi dude which give rise to fast oscillations.

Eq. (3.41) is frequently used as the starting point to justify why sometimes we can throw away the last term (and hence obtain the Jaynes-Cummings model (3.23) from the Rabi model). The idea is called the **rotating-wave approximation (RWA)** and is motivated by the fact that if $\Omega + \omega$ is very large, the last terms will oscillate rapidly around zero average and hence will have a small contribution to the dynamics. But this explanation is only partially convincing, so be careful. In the end of the day, the RWA is really an argument on time-dependent perturbation theory. Hence, it will only be good when λ is small compared to ω_c and Ω. Thus, the RWA is better stated as follows: *if $\lambda \ll \omega_c, \Omega$ and $\omega_c \sim \Omega$, it is reasonable to throw away the fast oscillating terms in the interaction picture.* For an interesting discussion connection with perturbation theory, see the Appendix in `arXiv 1601.07528`.

Heisenberg picture

In the interaction picture we started with a Hamiltonian $H = H_0 + V$ and went to a rotating frame with H_0. In the Heisenberg picture, we go all the way through. That is, we go to a rotating frame (3.29) with $S(t) = e^{iHt}$. For now I will assume H is time-independent, but the final result also holds in the time-dependent case. As a result we find

$$\tilde{H} = 0 \tag{3.42}$$

Consequently, the solution of the rotating frame Eq. (3.28) will be simply

$$\tilde{\rho}(t) = \tilde{\rho}(0) = \rho(0). \tag{3.43}$$

But by Eq. (3.27) we have $\tilde{\rho}(t) = S(t)\rho(t)S^\dagger(t)$ so we get

$$\rho(t) = S^\dagger(t)\rho(0)S(t) = e^{-iHt}\rho(0)e^{iHt}. \tag{3.44}$$

You may now be thinking "DUH! This is is just the solution of the of von Neumann's equation!". Yes, that's exactly the point. The solution of von Neumann's equation is exactly that special rotating frame where time stands still (like in the Rush song!).

In the Heisenberg picture we usually transfer the time-dependence to the operators, instead of the states. Recall that given an arbitrary operator A, its expectation value will be $\langle A\rangle = \mathrm{tr}(A\rho)$. Using Eq. (3.44) we then get

$$\langle A\rangle = \mathrm{tr}\left\{Ae^{-iHt}\rho(0)e^{iHt}\right\} = \mathrm{tr}\left\{e^{iHt}Ae^{iHt}\rho(0)\right\}. \tag{3.45}$$

This formula summarizes well the Schrödinger vs. Heisenberg ambiguity. It provides two equivalent ways to compute $\langle A\rangle$. In the first, which is the usual Schrödinger picture approach, the state $\rho(t)$ evolves in time and A is time-independent. In the second, the state ρ is fixed at $\rho(0)$ and we transfer the time evolution to the operator. It is customary to define the *Heisenberg operator*

$$A_H(t) = A(t) = e^{iHt}Ae^{-iHt}. \tag{3.46}$$

Some people write $A_H(t)$ to emphasize that this is different from A What I usually do is just be careful to always write the time argument in $A(t)$.

By direct differentiation one may verify that the operator $A(t)$ satisfies the **Heisenberg equation**

$$\boxed{\frac{\mathrm{d}A(t)}{\mathrm{d}t} = i[H, A(t)].} \tag{3.47}$$

This is to be interpreted as an equation for the evolution of the operator $A(t)$. If what you are interested is instead the evolution of the expectation value

$\langle A\rangle_t$, then it doesn't matter which picture you use. In the Heisenberg picture, Eq. (3.47) directly gives you

$$\boxed{\frac{\mathrm{d}\langle A\rangle}{\mathrm{d}t} = i\langle[H, A]\rangle.} \tag{3.48}$$

But you can also get the same equation in the Schrödinger picture using the von Neumann equation:

$$\frac{\mathrm{d}\langle A\rangle}{\mathrm{d}t} = \mathrm{tr}\left\{A\frac{\mathrm{d}\rho}{\mathrm{d}t}\right\} = -i\,\mathrm{tr}\left\{A[H,\rho]\right\} = i\,\mathrm{tr}\left\{[H,A]\rho\right\},$$

where, in the last line, all I did was rearrange the commutator using the cyclic property of the trace.

About time-dependent Hamiltonians

The solution of Schrödinger's or von Neumann's equation for time-independent Hamiltonians is very easy, being simply e^{-iHt}. However, when the Hamiltonian is time-dependent this solution no longer works. Let us then see how to write down the solution in this case. I will do so for the case of Schödinger's equation, simply because it looks a little bit cuter. It is straightforward to generalized to von Neumann's equation. Our starting point is thus the equation

$$\partial_t|\psi_t\rangle = -iH(t)|\psi_t\rangle. \tag{3.49}$$

In order to figure out what the solution will be in this case, we follow the maxim of Polish mathematician Marc Kac: **"be wise, discretize!"** That is, we assume that the Hamiltonian $H(t)$ is actually piecewise constant at intervals Δt, having the value $H(n\Delta t)$ during the interval between $n\Delta t$ and $(n+1)\Delta t$ (something like what is shown in Fig. 3.4, but for the operator $H(t)$). We can then solve Eq. (3.49) exactly for one interval:

$$|\psi((n+1)\Delta t)\rangle = e^{-i\Delta t H(n\Delta t)}|\psi(n\Delta t)\rangle. \tag{3.50}$$

From this we can proceed sequentially, using the solution for a given interval as the initial condition for the next. This allows us to glue together a solution between $t_0 = M\Delta t$ and $t = (N+1)\Delta t$ (with M, N integers and $N > M$):

$$|\psi_t\rangle = \left[e^{-i\Delta t H(N\Delta t)}e^{-i\Delta t H((N-1)\Delta t)}\ldots e^{-i\Delta t H(M\Delta t)}\right]|\psi_{t_0}\rangle. \tag{3.51}$$

Of course, this discretization is just a trick. We can now take $\Delta t \to 0$ and we will have solved for the most general time-dependent Hamiltonian.

If we define the time-evolution operator according to

$$|\psi_t\rangle = U(t, t_0)|\psi_{t_0}\rangle, \tag{3.52}$$

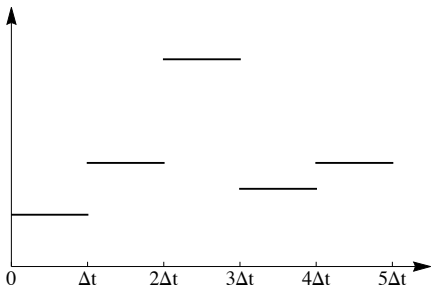

Figure 3.4: A silly example of a piecewise constant function.

then we see that

$$\boxed{U(t,t_0) = e^{-i\Delta t H(N\Delta t)} e^{-i\Delta t H((N-1)\Delta t)} \dots e^{-i\Delta t H(M\Delta t)}.} \tag{3.53}$$

Since this becomes exact when $\Delta t \to 0$, we conclude that this is *the* general solution of the time-dependent problem. Admittedly, this solution is still quite a mess and part of our effort below will be to clean it up a bit. But if you ever wonder "what is the solution with a time-dependent Hamiltonian?", I recommend you think about (3.53).

It is interesting to note that this operator $U(t,t_0)$ satisfies all properties of its time-independent cousin:

$$U(t_0,t_0) = 1, \tag{3.54}$$

$$U(t,t_1)U(t_1,t_0) = U(t,t_0), \qquad t_0 < t_1 < t, \tag{3.55}$$

$$U(t,t_0)U^\dagger(t,t_0) = 1, \tag{3.56}$$

$$U^\dagger(t,t_0) = U(t_0,t). \tag{3.57}$$

Eq. (3.55) is particularly important, because it shows that even in the time-dependent case the solution can still be broken down in pieces.

The important point that must be remembered concerning Eq. (3.53) is that in general you *cannot* recombine the exponentials since the Hamiltonian at different times may not commute:

$$\text{in general} \quad [H(t), H(t')] \neq 0. \tag{3.58}$$

If this happens to be the case, then the problem is very easy and Eq. (3.53)

becomes

$$U(t,t_0) = \exp\left\{ -i\Delta t \sum_{n=M}^{N} H(n\Delta t) \right\}$$
$$= \exp\left\{ -i \int_{t_0}^{t} H(t')\,\mathrm{d}t' \right\},$$

where, in the last line, I already took the limit $\Delta t \to 0$ and transformed the sum to an integral.

However, if $H(t)$ does not commute at different times, this solution is incorrect. Instead, we can use a trick to write down the solution in a way that looks formally similar. We define the *time-ordering* operator $\mathcal{T}$ such that, when acting on any set of time-dependent operators, it always puts later times to the left:

$$\mathcal{T}A(t_1)A(t_2) = \begin{cases} A(t_1)A(t_2) & \text{if } t_1 > t_2 \\ A(t_2)A(t_1) & \text{if } t_2 > t_1 \end{cases} \tag{3.59}$$

This time-ordering operator can now be used to combine exponentials. If we recall the Zassenhaus (BCH) formula (1.68):

$$e^{t(A+B)} = e^{tA}e^{tB}e^{-\frac{t^2}{2}[A,B]}e^{\frac{t^3}{3!}(2[B,[A,B]]+[A,[A,B]])}\ldots, \tag{3.60}$$

we see that the combination-recombination of exponentials involves only commutators. Now suppose $t_2 > t_1$. Then

$$\mathcal{T}[A(t_2),B(t_1)] = \mathcal{T}\Big(A(t_2)B(t_1) - B(t_1)A(t_2)\Big) = A(t_2)B(t_1) - A(t_2)B(t_1) = 0.$$

Consequently, if we expand $e^{A(t_2)+B(t_1)}$ and then apply $\mathcal{T}$, the only term that will survive will be $e^{A(t_2)}e^{B(t_1)}$. Hence,

$$e^{A(t_2)}e^{B(t_1)} = \mathcal{T}e^{A(t_2)+B(t_1)}. \tag{3.61}$$

Within the protection of the time-ordering operator, we can freely recombine exponentials.

Using this time-ordering trick we may now recombine all terms in the product (3.53), leading to

$$\boxed{U(t,t_0) = \mathcal{T}\exp\left\{ -i \int_{t_0}^{t} H(t')\,\mathrm{d}t' \right\},} \tag{3.62}$$

where I already transformed this into an integral. This is the way we usually write the formal solution of a time-dependent problem. The time-ordering operator $\mathcal{T}$ is just a compact way to write down the solution in Eq. (3.53). If you

are ever confused about how to operate with it, go back to Eq. (3.53). Finally, let me mention that Eq. (3.62) can also be viewed as the solution of the initial value problem

$$\frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t} = -iH(t)U(t,t_0), \qquad U(t_0,t_0) = 1. \tag{3.63}$$

This may not be so evident from Eq. (3.62), but it is if we substitute Eq. (3.52) into (3.49).

Magnus expansion

We are now in a good point to discuss time-dependent perturbation theory. The scenario is as follows. We start with $H_0 + V$ and move to the interaction picture where the rotating frame Hamiltonian becomes the time-independent operator (3.40). We then try to solve the von Neumann equation for this operator. Or, what is equivalent, we try to find the time-evolution operator $\tilde{U}(t,t_0)$ which, as in (3.63), will be the solution of

$$\frac{\mathrm{d}\tilde{U}(t,t_0)}{\mathrm{d}t} = -i\tilde{H}(t)\tilde{U}(t,t_0), \qquad \tilde{U}(t_0,t_0) = 1. \tag{3.64}$$

There are many ways to do this. Sometimes the perturbation theory is done in terms of states and sometimes it is done in terms of operators (in which case it is called a **Dyson series**).

Here I will try to do it in a slightly different way, using something called a Magnus expansion. Parametrize the time evolution operator as

$$\tilde{U}(t,t_0) = e^{-i\Omega(t,t_0)}, \qquad \Omega(t_0,t_0) = 0, \tag{3.65}$$

where $\Omega(t,t_0)$ is an operator to be determined. To find an equation for it, we first multiply Eq. (3.64) bu $U^\dagger$ on the left, leading to

$$\frac{\mathrm{d}e^{-i\Omega}}{\mathrm{d}t}e^{i\Omega} = -i\tilde{H}(t).$$

Then we use Eq. (3.30) to find

$$\dot{\Omega} - \frac{i}{2}[\Omega,\dot{\Omega}] - \frac{1}{3!}[\Omega,[\Omega,\dot{\Omega}]] + \ldots = \tilde{H}(t), \tag{3.66}$$

which is a really weird equation for $\Omega(t,t_0)$.

We now write this in perturbation-theory-style by assuming that $\tilde{H}(t) \to \epsilon\tilde{H}(t)$ where ϵ is a small parameter. Moreover, we expand Ω as

$$\Omega = \epsilon\Omega_1 + \epsilon^2\Omega_2 + \epsilon^3\Omega_3 + \ldots. \tag{3.67}$$

Substituting in Eq. (3.66) and collecting terms of the same order in ϵ we are then led to a system of equations

$$\dot{\Omega}_1 = \tilde{H}(t), \tag{3.68}$$

$$\dot{\Omega}_2 = \frac{i}{2}[\Omega_1, \dot{\Omega}_1], \tag{3.69}$$

$$\dot{\Omega}_3 = \frac{i}{2}[\Omega_1, \dot{\Omega}_2] + \frac{i}{2}[\Omega_2, \dot{\Omega}_] + \frac{1}{3!}[\Omega_1, [\Omega_1, \dot{\Omega}_1]]. \tag{3.70}$$

and so on. These can now be solved sequentially, leading to

$$\Omega_1(t) = \int_{t_0}^{t} \mathrm{d}t_1 \; \tilde{H}(t_1), \tag{3.71}$$

$$\Omega_2(t) = -\frac{i}{2}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \; [\tilde{H}(t_1), \tilde{H}(t_2)], \tag{3.72}$$

$$\Omega_3(t) = -\frac{1}{6}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \int_{t_0}^{t_2} \mathrm{d}t_3 \left\{ [\tilde{H}(t_1), [\tilde{H}(t_2), \tilde{H}(t_3)]] + [\tilde{H}(t_3), [\tilde{H}(t_2), \tilde{H}(t_1)]] \right\}. \tag{3.73}$$

This is the Magnus expansion. Higher order terms become more and more cumbersome. From this one may obtain the Dyson series expanding Eq. (3.65) in a Taylor series.

It is also important to note that if the Hamiltonian commutes at different times, then the series truncates at the first term. If this were always the case, there would be no need for perturbation theory at all. The need for time-dependent perturbation theory is really a consequence of the non-commutativity of $\tilde{H}$ at different times.

Rotating wave approximation

Consider once again the interaction picture Rabi Hamiltonian (3.41) and let us compute the first order term in the Magnus expansion, Eq. (3.71). We get, assuming $t_0 = 0$,

$$\int_0^t \mathrm{d}t_1 \; \tilde{H}(t_1) = \frac{\lambda}{i(\Omega - \omega_c)}\left[a\sigma_+(e^{i(\Omega-\omega_c)t} - 1) - a^\dagger\sigma_-(e^{-i(\Omega-\omega_c)t} - 1)\right]$$
$$+\frac{\lambda}{i(\Omega + \omega_c)}\left[a^\dagger\sigma_+(e^{i(\Omega+\omega_c)t} - 1) - a\sigma_-(e^{-i(\Omega+\omega_c)t} - 1)\right].$$

The Rotating-wave approximation scenario is now apparent: when we do perturbation theory, the Jaynes-Cummings terms will multiply $\lambda/(\Omega - \omega_c)$ whereas

the non-JC terms will contain $\lambda/(\Omega-\omega_c)$. If we are close to resonance ($\Omega \sim \omega_c$) and if λ is small the first term will be very large and the second very small. Consequently, the second term may be neglected.

3.4 Coherent states

Coherent states are a very special set of states which form the basis of continuous variables quantum information. In this section we will discuss some of its basic properties. If you ever need more advanced material, I recommend the paper by K. Cahill and R. Glauber in *Phys. Rev.* **177**, 1857-1881 (1969).

We begin by defining the **displacement operator**

$$\boxed{D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}.} \tag{3.74}$$

where α is an arbitrary complex number and α^* is its complex conjugate. The reason why it is called a "displacement" operator will become clear soon. A coherent state is defined as the action of $D(\alpha)$ into the vacuum state:

$$\boxed{|\alpha\rangle = D(\alpha)|0\rangle.} \tag{3.75}$$

We sometimes say that **"a coherent state is a displaced vacuum"**. This sounds like a typical Star Trek sentence: "Oh no! He displaced the vacuum. Now the entire planet will be annihilated!"

$D(\alpha)$ displaces a and $a^\dagger$

Let us first try to understand why $D(\alpha)$ is called a displacement operator. First, one may verify directly from Eq. (3.74) that

$$D^\dagger(\alpha)D(\alpha) = D(\alpha)D^\dagger(\alpha) = 1 \qquad \text{(it is unitary)}, \tag{3.76}$$

$$D^\dagger(\alpha) = D(-\alpha). \tag{3.77}$$

This means that if you displace by a given α and then displace back by $-\alpha$, you return to where you started. Next I want to compute $D^\dagger(\alpha)aD(\alpha)$. To do that we use the BCH formula (1.70):

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots. \tag{3.78}$$

with $B = a$ and $A = \alpha^* a - \alpha a^\dagger$. Using the commutation relations $[a, a^\dagger] = 1$ we get

$$[\alpha^* a - \alpha a^\dagger, a] = \alpha.$$