the non-JC terms will contain $\lambda/(\Omega-\omega_c)$. If we are close to resonance ($\Omega \sim \omega_c$) and if λ is small the first term will be very large and the second very small. Consequently, the second term may be neglected.

3.4 Coherent states

Coherent states are a very special set of states which form the basis of continuous variables quantum information. In this section we will discuss some of its basic properties. If you ever need more advanced material, I recommend the paper by K. Cahill and R. Glauber in *Phys. Rev.* **177**, 1857-1881 (1969).

We begin by defining the **displacement operator**

$$\boxed{D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}.} \tag{3.74}$$

where α is an arbitrary complex number and α^* is its complex conjugate. The reason why it is called a "displacement" operator will become clear soon. A coherent state is defined as the action of $D(\alpha)$ into the vacuum state:

$$\boxed{|\alpha\rangle = D(\alpha)|0\rangle.} \tag{3.75}$$

We sometimes say that **"a coherent state is a displaced vacuum"**. This sounds like a typical Star Trek sentence: "Oh no! He displaced the vacuum. Now the entire planet will be annihilated!"

$D(\alpha)$ displaces a and $a^\dagger$

Let us first try to understand why $D(\alpha)$ is called a displacement operator. First, one may verify directly from Eq. (3.74) that

$$D^\dagger(\alpha)D(\alpha) = D(\alpha)D^\dagger(\alpha) = 1 \qquad \text{(it is unitary)}, \tag{3.76}$$

$$D^\dagger(\alpha) = D(-\alpha). \tag{3.77}$$

This means that if you displace by a given α and then displace back by $-\alpha$, you return to where you started. Next I want to compute $D^\dagger(\alpha)aD(\alpha)$. To do that we use the BCH formula (1.70):

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots. \tag{3.78}$$

with $B = a$ and $A = \alpha^* a - \alpha a^\dagger$. Using the commutation relations $[a, a^\dagger] = 1$ we get

$$[\alpha^* a - \alpha a^\dagger, a] = \alpha.$$

But this is a c-number, so that all higher order commutators in the BCH expansion will be zero. We therefore conclude that

$$\boxed{D^\dagger(\alpha) a D(\alpha) = a + \alpha.} \tag{3.79}$$

This is why we call D the displacement operator: it displacements the operator by an amount α. Since $D^\dagger(\alpha) = D(-\alpha)$ it follows that

$$D(\alpha) a D^\dagger(\alpha) = a - \alpha. \tag{3.80}$$

The action on $a^\dagger$ is similar: you just need to take the adjoint: For instance

$$D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^*. \tag{3.81}$$

The coherent state is an eigenstate of a

What I want to do now is apply a to the coherent state $|\alpha\rangle$ in Eq. (3.75). Start with Eq. (3.79) and multiply by $D(\alpha)$ on the left. Since D is unitary we get $aD(\alpha) = D(\alpha)(a+\alpha)$. Thus

$$a|\alpha\rangle = aD(\alpha)|0\rangle = D(\alpha)(a+\alpha)|0\rangle = D(\alpha)(\alpha)|0\rangle = \alpha|\alpha\rangle,$$

where I used the fact that $a|0\rangle = 0$. Hence we conclude that **the coherent state is the eigenvector of the annihilation operator**:

$$\boxed{a|\alpha\rangle = \alpha|\alpha\rangle.} \tag{3.82}$$

The annihilation operator is not Hermitian so its eigenvalues do not have to be real. In fact, this equation shows that the eigenvalues of a are all complex numbers.

Alternative way of writing D

It is possible to express D in a different way, which may be more convenient for some computations. Using the Zassenhaus formula (3.60) we see that, if it happens that $[A, B]$ commute with both A and B, then

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{3.83}$$

Since $[a, a^\dagger] = 1$, we may write

$$\boxed{D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{|\alpha|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger}.} \tag{3.84}$$

This result is useful because now the exponentials of a and $a^\dagger$ are completely separated.

From this result it follows that

$$\boxed{D(\alpha)D(\beta) = e^{(\beta^*\alpha - \alpha^*\beta)/2} D(\alpha+\beta).} \tag{3.85}$$

This means that if you do two displacements in a sequence, it is almost the same as doing just a single displacement; the only thing you get is a phase factor (the quantity in the exponential is purely imaginary).

Poisson statistics

Let us use Eq. (3.84) to write the coherent state a little differently. Since $a|0\rangle = 0$ it follows that $e^{-\alpha a}|0\rangle = |0\rangle$. Hence we may also write Eq. (3.75) as

$$\boxed{|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle.} \tag{3.86}$$

Now we may expand the exponential and use Eq. (3.14) to write $(a^\dagger)^n|0\rangle$ in terms of the number states. We get

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.87}$$

Thus we find that

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}. \tag{3.88}$$

The probability of finding it in a given state $|n\rangle$, given that it is in a coherent state, is therefore

$$|\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{(|\alpha|^2)^n}{n!}. \tag{3.89}$$

This is a Poisson distribution with parameter $\lambda = |\alpha|^2$. The photons in a laser are usually in a coherent state and the Poisson statistics of photon counts can be measured experimentally. If you measure this statistics for thermal light you will find that it is not Poisson (usually it follows a geometric distribution). Hence, Poisson statistics is a signature of coherent states.

Orthogonality

Coherent states are not orthogonal. To figure out the overlap between two coherent states $|\alpha\rangle$ and $|\beta\rangle$ we use Eq. (3.86):

$$\langle\beta|\alpha\rangle = e^{-|\beta|^2/2} e^{-|\alpha|^2/2} \langle 0|e^{\beta^* a} e^{\alpha a^\dagger}|0\rangle.$$

We need to exchange the two operators because we know how a acts on $|0\rangle$ and how $a^\dagger$ acts on $\langle 0|$. To do that we use Eq. (3.83):

$$e^{\beta^* a} e^{\alpha a^\dagger} = e^{\alpha a^\dagger} e^{\beta^* a} e^{\beta^* \alpha}. \tag{3.90}$$

We therefore conclude that

$$\boxed{\langle \beta|\alpha\rangle = \exp\bigg\{\beta^*\alpha - \frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2}\bigg\}.} \tag{3.91}$$

The overlap of the two states, squared, can be simplified to read:

$$\boxed{|\langle \beta|\alpha\rangle|^2 = \exp\bigg\{ - |\alpha - \beta|^2\bigg\}.} \tag{3.92}$$

Hence, the overlap between two coherent states decays *exponentially* with their distance. For large α and β they therefore become approximately orthogonal. Also, as a sanity check, if $\beta = \alpha$ then

$$\langle \alpha|\alpha\rangle = 1, \tag{3.93}$$

which we already knew from Eq. (3.75) and the fact that D is unitary. Coherent states are therefore normalized, but they do *not* form an orthonormal basis. In fact, they form an overcomplete basis in the sense that there are more states than actually needed.

Completeness

Even though the coherent states do not form an orthonormal basis, we can still write down a completeness relation for them. However, it looks a little different:

$$\boxed{\int \frac{\mathrm{d}^2\alpha}{\pi}\, |\alpha\rangle\langle\alpha| = 1.} \tag{3.94}$$

This integral is over the entire complex plane. That is, if $\alpha = x + iy$ then $\mathrm{d}^2\alpha = \mathrm{d}x\,\mathrm{d}y$. This is, therefore, just your old-fashioned integral over two variables. The proof of Eq. (3.94) is a little bit cumbersome. You can find it in Gardiner and Zoller.

Trace of a displacement operator

Due to the orthogonality (3.94), you can also use the coherent state basis to compute traces:

$$\mathrm{tr}(\mathcal{O}) = \int \frac{\mathrm{d}^2\alpha}{\pi}\, \langle\alpha|\mathcal{O}|\alpha\rangle. \tag{3.95}$$

As an example, let us compute the trace of the displacement operator:

$$\text{tr}\, D(\lambda) = \int \frac{\mathrm{d}^2\alpha}{\pi} \langle\alpha|D(\lambda)|\alpha\rangle = \int \frac{\mathrm{d}^2\alpha}{\pi} \langle 0|D^\dagger(\alpha)D(\lambda)D(\alpha)|0\rangle.$$

But since $D(\alpha)$ is unitary, it infiltrates everywhere:

$$D^\dagger(\alpha)D(\lambda)D(\alpha) = \exp\left\{D^\dagger(\alpha)(\lambda a^\dagger - \lambda^* a)D(\alpha)\right\} = e^{\lambda\alpha^* - \lambda^*\alpha} D(\lambda).$$

Thus we get

$$\text{tr}\, D(\lambda) = \int \frac{\mathrm{d}^2\alpha}{\pi} = e^{\lambda\alpha^* - \lambda^*\alpha}\langle 0|D(\lambda)|0\rangle = e^{-|\lambda|^2/2} \int \frac{\mathrm{d}^2\alpha}{\pi} e^{\lambda\alpha^* - \lambda^*\alpha} \tag{3.96}$$

where I used the fact that $\langle 0|D(\lambda)|0\rangle = \langle 0|\lambda\rangle = e^{-|\lambda|^2/2}$ [Eq. (3.88)].

The remaining integral is actually an important one. Let us write $\alpha = x + iy$ and $\lambda = u + iv$. Then

$$\lambda\alpha^* - \lambda^*\alpha = 2ixv - 2iuy.$$

Thus

$$\int \frac{\mathrm{d}^2\alpha}{\pi} e^{\lambda\alpha^* - \lambda^*\alpha} = \int \mathrm{d}x e^{2ixv} \int \mathrm{d}y e^{-2iuy}$$

But each one is now a Dirac delta

$$\int\limits_{-\infty}^{\infty} \mathrm{d}x e^{ixk} = 2\pi\delta(k).$$

Whence

$$\boxed{\int \frac{\mathrm{d}^2\alpha}{\pi} e^{\lambda\alpha^* - \lambda^*\alpha} = \pi\delta(\lambda).} \tag{3.97}$$

where $\delta(\lambda) = \delta(\text{Re}(\lambda))\delta(\text{Im}(\lambda))$. This integral is therefore nothing but the two-dimensional Fourier transform in terms of the complex variable α.

Substituting this in Eq. (3.96) we finally conclude that

$$\boxed{\text{tr}\, D(\lambda) = \pi\ \delta(\lambda),} \tag{3.98}$$

where I omitted the factor of $e^{-|\lambda|^2/2}$ since the Dirac delta make it irrelevant. Using this and Eq. (3.85) also allows us to write the neat formula

$$\text{tr}\left[D(\alpha)D^\dagger(\beta)\right] = \pi\delta(\alpha - \beta). \tag{3.99}$$

This is a sort of orthogonality relation, but between operators.

$D(\alpha)$ as a basis for operators

Due to Eqs. (3.98) and (3.99), it turns out that the displacement operators form a basis for the Hilbert space, in the sense that any operator F may be decomposed as

$$F = \int \frac{\mathrm{d}^2\alpha}{\pi} \, f(\alpha) D^\dagger(\alpha) \tag{3.100}$$

where

$$f(\alpha) := \mathrm{tr}\Big[F D(\alpha)\Big]. \tag{3.101}$$

This is just like decomposing a state in a basis, but we are actually decomposing an operator.

3.5 The Husimi-Q function

A big part of dealing with continuous variables systems is the idea of *quantum phase space*, similarly to the classical coordinate-momentum phase space in classical mechanics. There are many ways to represent continuous variables in phase space. The three most important are the Husimi-Q function, the Wigner function and the Glauber-Sudarshan P function. Each has its own advantages and disadvantages. Since this chapter is meant to be a first look into this topic, we will focus here on the simplest one of them, the Q function.

The Husimi-Q function is defined as the expectation value of the density matrix in a coherent state

$$Q(\alpha^*, \alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle. \tag{3.102}$$

Here α and α^* are to be interpreted as independent variables. If that confuses you, define $\alpha = x + iy$ and interpret Q as a function of x and y. In fact, following the transformation between $a, a^\dagger$ and the quadrature operators q, p in Eq. (3.3), $x/\sqrt{2}$ represents the position in phase space, whereas $y/\sqrt{2}$ represents the momentum.

Using Eq. (3.95) for the trace in the coherent state basis, we get

$$1 = \mathrm{tr}\,\rho = \int \frac{\mathrm{d}^2\alpha}{\pi} \langle \alpha | \rho | \alpha \rangle.$$

Thus, we conclude that the Husimi Q function is normalized as

$$\int \mathrm{d}^2\alpha \, Q(\alpha^*, \alpha) = 1 \tag{3.103}$$

which resembles the normalization of a probability distribution.

If we know Q we can also use it to compute the expectation value of operators. For instance, since $a|\alpha\rangle = \alpha|\alpha\rangle$ it follows that

$$\langle a \rangle = \text{tr}(\rho a) = \int \frac{\mathrm{d}^2\alpha}{\pi} \langle \alpha|\rho a|\alpha\rangle = \int \mathrm{d}^2\alpha\, Q(\alpha, \alpha^*)\alpha,$$

which is intuitive. As another example, recalling that $\langle \alpha|a^\dagger = \langle \alpha|\alpha^*$, we get

$$\langle aa^\dagger \rangle = \text{tr}(a^\dagger \rho a) = \int \frac{\mathrm{d}^2\alpha}{\pi} \langle \alpha|a^\dagger \rho a|\alpha\rangle = \int \mathrm{d}^2\alpha\, Q(\alpha, \alpha^*)|\alpha|^2.$$

It is interesting to see here how the ordering of operators play a role. Suppose you want to compute $\langle a^\dagger a\rangle$. Then you should first reorder it as $\langle a^\dagger a\rangle = \langle aa^\dagger\rangle - 1$ and then use the above result for $\langle aa^\dagger\rangle$.

More generally, we may obtain a rule for computing the expectation values of *anti-normally ordered* operators. That is, operators which have all $a^\dagger$s to the right. If this is the case then we can easily write

$$\langle a^k (a^\dagger)^\ell \rangle = \int \mathrm{d}^2\alpha \;\; \alpha^k (\alpha^*)^\ell Q(\alpha^*, \alpha). \tag{3.104}$$

Thus, to compute the expectation value of an arbitrary operator, we should first use the commutation relations to put it in anti-normal order and then use this result.

The Q function is always non-negative. But not all Q functions correspond to valid states. For instance, $\delta(\alpha)$ is not a valid Husimi function since it would lead to

$$\langle aa^\dagger \rangle = \int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha|^2 \delta^2(\alpha) = 0, \tag{3.105}$$

which is impossible since $\langle aa^\dagger\rangle = \langle a^\dagger a\rangle + 1$ and $\langle a^\dagger a\rangle \geq 0$.

Let us now turn to some examples of Q functions.

Example: coherent state

If the state is a coherent state $|\mu\rangle$, then $\rho = |\mu\rangle\langle\mu|$ and we get from (3.92) and (3.102):

$$Q(\alpha^*, \alpha) = \frac{1}{\pi}\langle \alpha|\mu\rangle\langle\mu|\alpha\rangle = \frac{1}{\pi}\exp\bigg\{ -|\alpha-\mu|^2 \bigg\} \tag{3.106}$$

This is a *Gaussian distribution in the complex plane*, centered around μ and with unit variance (see Fig. 3.5). The ground-state of the harmonic oscillator is also a coherent state, but with $\mu = 0$. It will therefore also be a unit-variance Gaussian, but centered at zero. This is why we say *the coherent state is a displaced vacuum*: it has the same distribution, but simply displaced in the complex plane by μ.

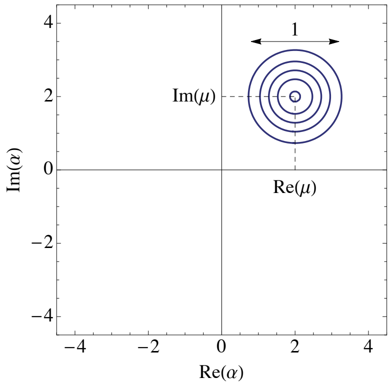

Figure 3.5: Example of the Husimi function (3.106) for $\mu = 2 + 2i$.

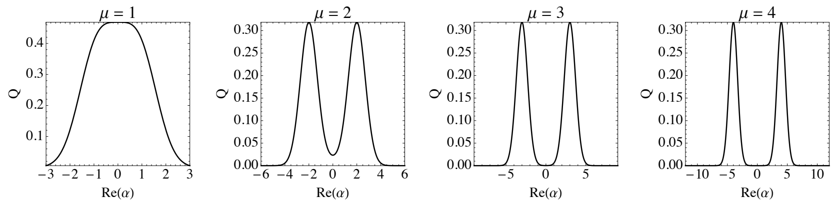

Figure 3.6: Example of the Husimi function (3.108) for a Schrödinger cat state (3.107), assuming μ real. The plots correspond to a cut at $\text{Im}(\alpha) = 0$.

Example: Schrödinger cat state

In the context of continuous variables, we sometimes call the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\mu\rangle + |-\mu\rangle\Big), \tag{3.107}$$

a Schrödinger cat state. Using Eq. (3.91) we then get

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} e^{-|\alpha-\mu|^2} \left\{ 1 + \frac{e^{-2\mu^*\alpha} + e^{-2\mu\alpha^*}}{2} \right\}. \tag{3.108}$$

An example of this function is shown in Fig. 3.6. It corresponds to roughly two Gaussians superposed. If μ is small then the two peaks merge into one, but as μ increases they become more distinguishable.

Example: thermal state

Next let us consider a thermal Gibbs state

$$\rho_{\text{th}} = \frac{e^{-\beta\omega a^\dagger a}}{Z}, \tag{3.109}$$

where

$$Z = \text{tr}(e^{-\beta\omega a^\dagger a}) = (1 - e^{-\beta\omega})^{-1}, \tag{3.110}$$

is the partition function. The Husimi function will be

$$Q(\alpha^*, \alpha) = \frac{(1 - e^{-\beta\omega})}{\pi} \sum_{n=0}^{\infty} e^{-\beta\omega n} \langle\alpha|n\rangle\langle n|\alpha\rangle.$$

This is a straightforward and fun calculation, which I will leave for you as an exercise. All you need is the overlap formula (3.88). The result is

$$Q(\alpha^*, \alpha) = \frac{1}{\pi(\bar{n}+1)} \exp\left\{ -\frac{|\alpha|^2}{\bar{n}+1} \right\}, \tag{3.111}$$

where

$$\bar{n} = \frac{1}{e^{\beta\omega} - 1}, \tag{3.112}$$

is the Bose-Einstein thermal occupation of the harmonic oscillator. Thus, we see that the thermal state is also a Gaussian distribution, centered at zero but with a variance proportional to $\bar{n}+1$. At zero temperature we get $\bar{n} = 0$ and we recover the Q function for the vacuum $\rho = |0\rangle\langle 0|$. The width of the Gaussian distribution can be taken as a measure of the fluctuations in the system. At high temperatures $\bar{n}$ becomes large and so does the fluctuations. Thus, in the classical limit we get a big fat Gaussian. But even at $T = 0$ there is still a finite width, which is a consequence of quantum fluctuations.

The two examples above motivate us to consider a **displaced thermal state**. It is defined in terms of the displacement operator (3.74) as

$$\rho = D(\mu) \frac{e^{-\beta H}}{Z} D^\dagger(\mu). \tag{3.113}$$

The corresponding Q function, as you can probably expect, is

$$Q(\alpha^*, \alpha) = \frac{1}{\pi(\bar{n}+1)} \exp\left\{ -\frac{|\alpha - \mu|^2}{\bar{n}+1} \right\}, \tag{3.114}$$

which is sort of a mixture of Eqs. (3.106) and (3.111): it represents a thermal Gaussian displaced in the complex plane by an amount μ.

Heterodyne measurements

The Husimi-Q function allows for an interesting interpretation in terms of measurements in the coherent state basis $|\alpha\rangle$, which is called heterodyne measurements. Recall that the basis $|\alpha\rangle$ is not orthonormal and therefore such a measurement is not a projective measurement. Instead, it is a generalized measurement in the same spirit of Sec. 2.10. In particular, please recall Eqs. (2.166)-(2.168). In our case, the set of measurement operators are

$$M_\alpha = \frac{1}{\sqrt{\pi}}|\alpha\rangle\langle\alpha|. \tag{3.115}$$

They are appropriately normalized as

$$\int \mathrm{d}^2\alpha\; M_\alpha^\dagger M_\alpha = \int \frac{\mathrm{d}^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = 1,$$

which is nothing but the completeness relation (3.94).

If outcome α is obtained, then the state after the measurement will collapse to $|\alpha\rangle\langle\alpha|$. And the probability of obtaining outcome α is, by Eq. (2.167),

$$\boxed{p_\alpha = \operatorname{tr} M_\alpha \rho M_\alpha^\dagger = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = Q(\alpha,\alpha^*).} \tag{3.116}$$

Thus, we see that the Husimi-Q function is nothing but the probability outcome if we were to perform a heterodyne measurement. This gives a nice interpretation to Q: whenever you see a plot of $Q(\alpha,\alpha^*)$ you can imagine "that is what I would get if I were to measure in the coherent state basis".

3.6 von Neumann's measurement model

In this section I want to use what we learned about continuous variables to discuss a more realistic measurement model. The calculations we are going to do here are a variation of an original proposal given by von Neumann. Suppose we have a system S that has been prepared in some state $|\psi\rangle$ and we wish to measure some observable K in this state. We write the eigenstuff of K as

$$K = \sum_k k|k\rangle\langle k|. \tag{3.117}$$

In order to measure this observable, what we are going to do is couple the system to an ancilla, consisting of a single continuous variable bosonic mode a, according to the interaction Hamiltonian

$$H = igK(a^\dagger - a). \tag{3.118}$$

This Hamiltonian represents a displacement of the bosonic mode which is proportional to the operator K. We could also do the same with $(a + a^\dagger)$ which looks more like a coordinate q. But doing it for $i(a^\dagger - a)$ turns out to be a bit simpler.

We assume the ancila starts in the vacuum so the initial state is

$$|\Phi(0)\rangle_{SA} = |\psi\rangle_S \otimes |0\rangle_A. \tag{3.119}$$

We then compute the time evolution of S+A under the interaction Hamiltonian (3.118). We will not worry here about the free part of the Hamiltonian. Including it would complicate the analysis, but will not lead to any new physics. Our goal then is to compute the state at time t

$$|\Phi(t)\rangle_{SA} = e^{-iHt}|\Phi(0)\rangle_{SA}. \tag{3.120}$$

To evaluate the matrix exponential we expand it in a Taylor series

$$e^{-iHt} = 1 - iHt + \frac{(-i)^2}{2}H^2t^2 + \ldots$$

We now note that, using the eigenstuff (3.117), we can write (being a bit sloppy with the $\otimes$):

$$\begin{aligned}
H &= \sum_k |k\rangle\langle k|(igk)(a + a^\dagger), \\
H^2 &= \sum_k |k\rangle\langle k|(igk)^2(a + a^\dagger)^2, \\
&\vdots \\
H^n &= \sum_k |k\rangle\langle k|(igk)^n(a + a^\dagger)^n.
\end{aligned}$$

Thus we may write

$$e^{-iHt} = \sum_k |k\rangle\langle k| e^{gtk(a+a^\dagger)} = \sum_k |k\rangle\langle k| \otimes D(gtk), \tag{3.121}$$

where I introduced here displacement operator $D(\alpha_k) = e^{\alpha_k a^\dagger - \alpha_k^* a}$ [Eq. (3.74)].

It is now easy to apply the evolution operator to the initial state, as in Eq. (3.120). We simply get

$$|\Phi(t)\rangle_{SA} = \sum_k \Big(|k\rangle\langle k| \otimes D(gtk)\Big)\Big(|\psi\rangle_S \otimes |0\rangle_A\Big),$$

or

$$\boxed{|\Phi(t)\rangle_{SA} = \sum_k \big[\langle k|\psi\rangle\big]\; |k\rangle_S \otimes |gtk\rangle_A,} \tag{3.122}$$

where $|gtk\rangle_A = D(gtk)|0\rangle_A$ is the coherent state at position $\alpha = gtk$. This result is quite important. It says that after a time t the combined S+A system will be in an entangled state, corresponding to a superposition of the system being in $|k\rangle$ and the ancilla being in $|gtk\rangle$.

Reduced density matrix of the ancilla

Since the states $|k\rangle$ form an orthonormal basis, the reduced density matrix of the ancilla will be simply

$$\boxed{\rho_A(t) = \text{tr}_S\, |\Phi(t)\rangle\langle\Phi(t)| = \sum_k |\langle k|\psi\rangle|^2 |gtk\rangle\langle gtk|.} \tag{3.123}$$

This is just an incoherent combination of coherent states, with the coherent state $|gtk\rangle$ occurring with probability

$$p_k = |\langle k|\psi\rangle|^2. \tag{3.124}$$

The corresponding Q function will then be simply a sum of terms of the form (3.106):

$$\boxed{Q(\alpha, \alpha^*) = \frac{1}{\pi}\sum_k p_k\; e^{-|\alpha - gtk|^2}.} \tag{3.125}$$

To give an example, suppose our system is a spin 2 particle with dimension $d = 5$ and suppose that the eigenvalues k in Eq. (3.117) are some spin component which can take on the values $k = 2, 1, 0, -1, -2$ [there is nothing special about this example; I'm just trying to give an example that is not based on qubits!]. Suppose also that the state of the system was prepared in

$$|\psi\rangle = \frac{1}{2}\bigg\{|2\rangle - |1\rangle - |-1\rangle + |-2\rangle\bigg\}, \tag{3.126}$$

where the states here refer to the basis $|k\rangle$ in (3.117). Some examples of the Q function for this state and different values of gt are shown in Fig. 3.7. Remember that the Q function represents a heterodyne detection on the ancilla. These examples show that if gt is small then the different peaks become blurred so such a measurement would not be able to appropriately distinguish between the different peaks. Conversely, as gt gets larger (which means a longer interaction time or a stronger interaction) the peak separation becomes clearer. Thus, the more S and A interact (or, what is equivalent, the more entangled they are) the larger is the amount of information that you can learn about S by performing a heterodyne detection on A.

Reduced density matrix of the system

Next let us compute the reduced density matrix of the system, staring with the composite state (3.122). We get

$$\rho_S(t) = \text{tr}_A\, |\Phi(t)\rangle\langle\Phi(t)| = \sum_{k,k'} \bigg(\langle k|\psi\rangle\langle\psi|k'\rangle\langle gtk|gtk'\rangle\bigg)|k\rangle\langle k'|.$$

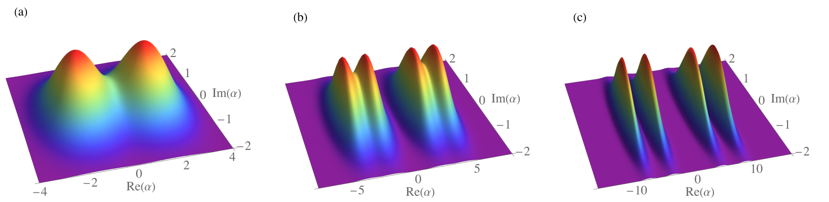

Figure 3.7: Example of the Q function (3.125) computed for the example state (3.126) for different values of gt. Namely (a) 1, (b) 2 and (c) 4.

We can simplify this using the orthogonality relation between coherent states, Eq. (3.91), which gives

$$\langle gtk|gtk'\rangle = \exp\left\{-\frac{(gt)^2}{2}(k-k')^2\right\}.$$

Thus, the reduced density matrix of S becomes

$$\rho_S(t) = \sum_{k,k'} \rho_{k,k'}(t)|k\rangle\langle k'|, \tag{3.127}$$

where

$$\boxed{\rho_{k,k'}(t) = \langle k|\psi\rangle\langle\psi|k'\rangle \exp\left\{-\frac{(gt)^2}{2}(k-k')^2\right\}.} \tag{3.128}$$

Just as a sanity check, at $t=0$ we recover the pure state $\rho_S(0) = |\psi\rangle\langle\psi|$.

What is really interesting about Eq. (3.128) is that the diagonal entries of ρ_S in the basis $|k\rangle$ are not effected:

$$\rho_{kk}(t) = \langle k|\psi\rangle\langle\psi|k\rangle = \rho_{k,k}(0). \tag{3.129}$$

Conversely, the off-diagonal coherences are exponentially damped and if we never turn off the S+A interaction we will eventually end up with

$$\lim_{t\to\infty} \rho_{k,k'}(t) = 0, \qquad k' \neq k. \tag{3.130}$$

Thus, the system initially started in a state $|\psi\rangle$ which was a superposition of the states $|k\rangle$. But, if we allow the system and ancilla to interact for a really long time, the system will end up in a incoherent mixture of states. It is also cool to note how the damping of the coherences is stronger for k and k' which are farther apart.

This analysis shows the emergence of a **preferred basis**. Before we turned on the S+A interaction, the system had no preferred basis. But once that interaction was turned on, the basis of the operator K, which is the operator we chose to couple to the ancila in Eq. (3.118), becomes a preferred basis, in the sense that populations and coherences behave differently in this basis.

Our model also allows us to interpolate between *weak measurements* and *strong measurements*. If gt is small then we perturb the system very little but we also don't learn a lot about it by measuring A. Conversely, if gt is large then we can learn a great deal more, but we also damage the system way more.

Conditional state given measurement outcome

Finally, let us analyze what happens if at time t we perform an actual heterodyne measurement with the operator set M_α in Eq. (3.115). Then if outcome α is obtained, the composite state of S+A will collapse so

$$|\Phi(t)\rangle\langle\Phi(t)| \to \frac{M_\alpha|\Phi(t)\rangle\langle\Phi(t)|M_\alpha^\dagger}{Q(\alpha,\alpha^*)}, \tag{3.131}$$

where I already used Eq. (3.116) to relate the outcome probability p_α with the Husimi function. After the measurement the ancilla will collapse to the coherent state $|\alpha\rangle\langle\alpha|$. Taking the partial trace of Eq. (3.131) over A we then get the reduced density matrix of S, given that the measurement outcome was α. I will leave the details of this calculation to you. The result is

$$\rho_{S|\alpha}(t) = \sum_{k,k'} \rho_{k,k'|\alpha}(t)|k\rangle\langle k'|, \tag{3.132}$$

where

$$\rho_{k,k'|\mu} = \frac{1}{\pi Q(\alpha,\alpha^*)}\langle k|\psi\rangle\langle\psi|k'\rangle\langle\alpha|gtk\rangle\langle gtk'|\alpha\rangle. \tag{3.133}$$

In particular, we can look at the diagonal elements $\rho_{k,k|\alpha}$

$$\rho_{k|\alpha}(t) = \frac{p_k e^{-|\alpha-gtk|^2}}{\sum\limits_{k'} p_{k'} e^{-|\alpha-gtk'|^2}}. \tag{3.134}$$

These quantities represent *the populations in the $|k\rangle$ basis, given that the measurement outcome was α.*

An example of these conditional populations is shown in Fig. 3.8, which represent $\rho_{k|\alpha}$ for different values of k as a function of $\text{Re}(\alpha)$ for the example state (3.126). We can read this as follows. Consider Fig. 3.8(a), which represents $\rho_{-2|\alpha}$. What we see is that if $\text{Re}(\alpha) \ll -2$ then it is very likely that the system is found in $k = -2$. Similarly, if $\text{Re}(\alpha)$ is around -2, as in Fig. 3.8(b), there is a large probability that the system is found in $k = -1$.

The results in Fig. 3.8 correspond to $gt = 1$ and therefore are not strong measurements. Conversely, in Fig. 3.9) we present the results for $gt = 4$. Now one can see a much shaper distinction of the probabilities. For instance, if $\text{Re}(\alpha) = 5$ then it is almost certain that the system is in $k = 1$, as in Fig. 3.9(d).

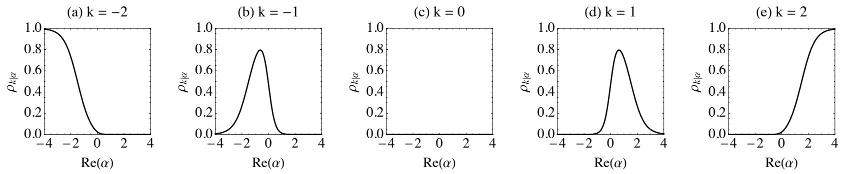

Figure 3.8: The conditional populations in Eq. (3.134) for the example state (3.126) and $gt = 1$.

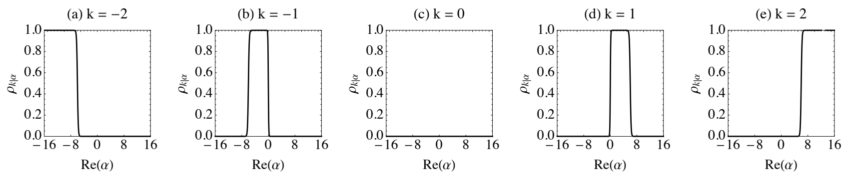

Figure 3.9: Same as Fig. 3.8 but for $gt = 4$.