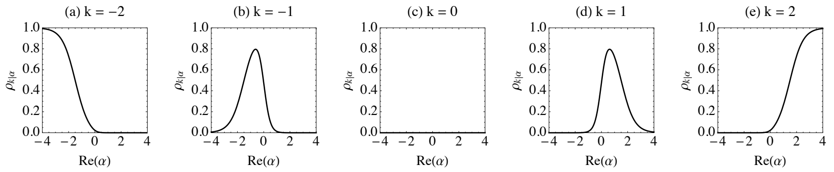

Figure 3.8: The conditional populations in Eq. (3.134) for the example state (3.126) and $gt = 1$.

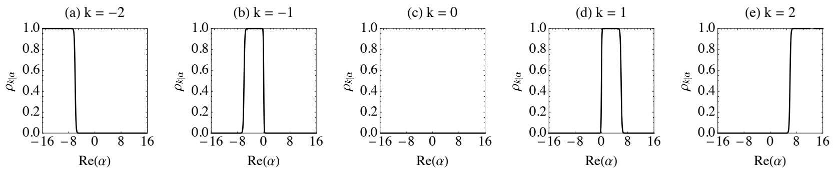

Figure 3.9: Same as Fig. 3.8 but for $gt = 4$.

3.7 Lindblad dynamics for the quantum harmonic oscillator

We already briefly touched upon the idea of a Lindblad master equation in Sec. 2.2, particularly in Eq. (2.22). The Lindblad master equation is a modification of von Neumann's equation to model *open quantum systems*. That is, the contact of the system with an external bath. Next chapter will be dedicated solely to open quantum systems. But here, I want to take another quick look at this problem, focusing on continuous variables. What I propose is to just show you what is the most widely used Lindblad equation in this case. Then we can just play with it a bit and get a feeling of what it means. The *derivation* of this master equation, together with a deeper discussion of what it means, will be done in the next chapter.

We return to the pumped cavity model described in Fig. 3.2. We assume the optical cavity contains only a single mode of radiation a, of frequency ω_c, which is pumped externally by a laser at a frequency ω_p. The Hamiltonian describing this system is given by Eq. (3.22):

$$H = \omega_c a^\dagger a + \epsilon a^\dagger e^{-i\omega_p t} + \epsilon^* a e^{i\omega_p t}. \tag{3.135}$$

In addition to this, we now include also the loss of photons through the semi-transparent mirror. This is modeled by the following master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H,\rho] + D(\rho), \tag{3.136}$$

where $D(\rho)$ is called the **Lindblad dissipator** and is given by

$$D(\rho) = \gamma\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right]. \tag{3.137}$$

Here $\gamma > 0$ is a constant which quantifies the loss rate of the cavity. Recall that the pump term ϵ in Eq. (3.135) was related to the laser power P by $|\epsilon|^2 = \gamma P/\hbar\omega_p$, which therefore depends on γ. This is related to the fact that the mechanism allowing for the photons to get in is the same that allows them to get out, which is the semi-transparent mirror. I should also mention that sometimes Eq. (3.137) is written instead with another constant, $\gamma = 2\kappa$. There is a sort of unspoken rule that if Eq. (3.137) has a 2 in front, the constant should be named κ. If there is no factor of 2, it should be named γ. If you ever want to be mean to a referee, try changing that order.

For qubits the dimension of the Hilbert space is finite so we can describe the master equation by simply solving for the density matrix. Here things are not so easy. Finding a general solution for any density matrix is a more difficult task. Instead, we need to learn alternative ways of dealing with (and understanding) this type of equation.

Before we do anything else, it is important to understand the meaning of the structure of the dissipator, in particular the meaning of a term such as $a\rho a^\dagger$. Suppose at $t = 0$ we prepare the system with certainty in a number state so $\rho(0) = |n\rangle\langle n|$. Then

$$D(|n\rangle\langle n|) = \gamma n\bigg\{|n-1\rangle\langle n-1| - |n\rangle\langle n|\bigg\}.$$

The first term, which comes from $a\rho a^\dagger$, represents a state with one photon less. This is precisely the idea of a *loss process.* But this process must also preserve probability, which is why we also have another term to compensate. The structure of the dissipator (3.137) represents a very finely tuned equation, where the system looses photons, but does so in such a way that the density matrix remains positive and normalized at all times. We also see from this result that

$$D(|0\rangle\langle 0|) = 0. \tag{3.138}$$

Thus, if you start with zero photons, nothing happens with the dissipator term . We say that the the vacuum is a **fixed point** of the dissipator (it is not necessarily a fixed point of the unitary evolution).

The case of zero pump, $\epsilon = 0$

Let us consider the case $\epsilon = 0$, so that the Hamiltonian (3.135) becomes simply $H = \omega_c a^\dagger a$. This means the photons can never be injected, but only lost. As a consequence, if our intuition is correct, the system should eventually relax to the vacuum. That is, we should expect that

$$\lim_{t\to\infty} \rho(t) = |0\rangle\langle 0|. \tag{3.139}$$

We are going to try to verify this in several ways. The easiest way is to simply verify that if $\rho^* = |0\rangle\langle 0|$ then

$$-i\omega_c[a^\dagger a, \rho^*] + D(\rho^*) = 0,$$

so the vacuum is indeed a steady-state of the equation. If it is unique (it is) and if the system will always converge to it (it will), that is another question.

Next let us look at the populations in the Fock basis

$$p_n = \langle n|\rho|n\rangle. \tag{3.140}$$

They represent the probability of finding the system in the Fock state $|n\rangle$. We can find an equation for $p_n(t)$ by sandwiching Eq. (3.136) in $\langle n|\ldots|n\rangle$. The unitary part turns out to give zero since $|n\rangle$ is an eigenstate of $H = \omega_c a^\dagger a$. As for $\langle n|D(\rho)|n\rangle$, I will leave for you to check that we get

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = \gamma\Big[(n+1)p_{n+1} - np_n\Big]. \tag{3.141}$$

This is called a **Pauli master equation** and is nothing but a **rate equation**, specifying how the population $p_n(t)$ changes with time. Positive terms increase p_n and negative terms decrease it. So the first term in Eq. (3.141) describes the increase in p_n due to populations coming from p_{n+1}. This represents the decays from higher levels. Similarly, the second term in Eq. (3.141) is negative and so describes how p_n decreases due to populations at p_n that are falling down to p_{n-1}.

The steady-state of Eq. (3.141) is obtained by setting $\mathrm{d}p_n/\mathrm{d}t = 0$, which gives

$$p_{n+1} = \frac{n}{n+1}p_n, \tag{3.142}$$

In particular, if $n=0$ we get $p_1 = 0$. Then plugging this in $n=1$ gives $p_2 = 0$ and so on. Thus, the steady-state correspond to all $p_n = 0$. The only exception is p_0 which, by normalization, must then be $p_0 = 1$.

Evolution of observables

Another useful thing to study is the evolution of observables, such as $\langle a\rangle$, $\langle a^\dagger a\rangle$, etc. Starting from the master equation (3.136), the expectation value of any observables is

$$\frac{\mathrm{d}\langle\mathcal{O}\rangle}{\mathrm{d}t} = \mathrm{tr}\left\{\mathcal{O}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right\} = -i\,\mathrm{tr}\left\{\mathcal{O}[H,\rho]\right\} + \mathrm{tr}\left\{\mathcal{O}D(\rho)\right\}.$$

Rearranging the first term we may write this as

$$\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}t} = i\langle[H,\mathcal{O}]\rangle + \mathrm{tr}\left\{\mathcal{O}D(\rho)\right\}. \tag{3.143}$$

The first term is simply Heisenberg's equation (3.48) for the unitary part. What is new is the second term. It is convenient to write this as the trace of ρ times "something", so that we can write this as an expectation value. We can do this using the cyclic property of the trace:

$$\mathrm{tr}\left\{\mathcal{O}\left[a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right]\right\} = \langle a^\dagger\mathcal{O}a - \frac{1}{2}a^\dagger a\mathcal{O} - \frac{1}{2}\mathcal{O}a^\dagger a\rangle. \tag{3.144}$$

Using this result for $\mathcal{O} = a$ and $\mathcal{O} = a^\dagger a$ gives, playing with the algebra a bit,

$$\text{tr}\Big\{a\mathcal{D}(\rho)\Big\} = -\frac{\gamma}{2}\langle a\rangle, \qquad \text{tr}\Big\{a^\dagger a\mathcal{D}(\rho)\Big\} = -\gamma\langle a^\dagger a\rangle. \tag{3.145}$$

Using these results in Eq. (3.143) then gives

$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = -(i\omega + \gamma/2)\langle a\rangle, \tag{3.146}$$

$$\frac{\mathrm{d}\langle a^\dagger a\rangle}{\mathrm{d}t} = -\gamma\langle a^\dagger a\rangle. \tag{3.147}$$

Thus, both the first and the second moments will relax exponentially with a rate γ, except that $\langle a\rangle$ will also oscillate:

$$\langle a\rangle_t = e^{-(i\omega+\gamma/2)t}\langle a\rangle_0, \tag{3.148}$$

$$\langle a^\dagger a\rangle_t = e^{-\gamma t}\langle a^\dagger a\rangle_0 \tag{3.149}$$

As $t \to \infty$ the average number of photons $\langle a^\dagger a\rangle$ tends to zero, no matter which state you begin at. Looking at a handful of observables is a powerful way to have an idea about what the density matrix is doing.

Evolution in the presence of a pump

Let us now go back to the full master Eq. (3.136). We can move to the interaction picture exactly as was done in Eq. (3.31), defining

$$\tilde{\rho}_t = S(t)\rho S^\dagger(t), \qquad S(t) = e^{i\omega_p t a^\dagger a}.$$

This transforms the Hamiltonian (3.135) into the detuned time-independent Hamiltonian (3.33):

$$\boxed{\tilde{H} = \Delta a^\dagger a + \epsilon a^\dagger + \epsilon^* a,} \tag{3.150}$$

where $\Delta = \omega_c - \omega_p$ is the detuning. Moreover, I will leave for you as an exercise to check that this does not change in any way the dissipative term. Thus, $\tilde{\rho}$ will evolve according to

$$\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t} = -i[\tilde{H}, \tilde{\rho}] + D(\tilde{\rho}). \tag{3.151}$$

To get a feeling of what is going on, let us use Eq. (3.143) to compute the evolution of $\langle a\rangle$. Everything is identical, except for the new pump term that appears. As a result we get

$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = -(i\Delta + \gamma/2)\langle a\rangle - i\epsilon. \tag{3.152}$$

As before, $\langle a \rangle$ will evolve as a damped oscillation. However, now it will not tend to zero in the long-time limit, but instead will tend to

$$\langle a \rangle_{\rm ss} = -\frac{i\epsilon}{i\Delta + \gamma/2}. \tag{3.153}$$

I think this summarizes well the idea of a *pumped cavity*: the steady-state is a competition of how much we pump (unitary term) and how much we drain (the dissipator). Interestingly, the detuning Δ also affects this competition, so for a given ϵ and γ, we get more photons in the cavity if we are at resonance, $\Delta = 0$.

We can also try to ask the more difficult question of what is the density matrix ρ^* in the steady-state. It turns out it is a coherent state set exactly at the value of $\langle a \rangle$:

$$\tilde{\rho}^* = |\alpha\rangle\langle\alpha|, \qquad \alpha = -\frac{i\epsilon}{i\Delta + \gamma/2}. \tag{3.154}$$

One way to check this is to take the coherent state as an ansatz and then try to find what is the value of α which solves Eq. (3.151). The average number of photons will then be

$$\langle a^\dagger a \rangle = |\alpha|^2 = \frac{\epsilon^2}{\Delta^2 + \gamma^2/4}. \tag{3.155}$$

The purpose of this section was to show you a practical use of master equations and open quantum systems. This "cavity loss" dissipator is present in literally every quantum optics setup which involves a cavity. In fact, I know of several papers which sometimes even forget to tell that this dissipator is there, but it always is. We will now turn to a more detailed study of open quantum systems.