Chapter 4

Open quantum systems

4.1 Quantum operations

Let's go back for a second to the basic postulates of quantum mechanics. Recall that when we first establish the theory, we begin by postulating that a system can be represented by an abstract state $|\psi\rangle$. Then we also postulate that the time evolution of $|\psi\rangle$ must be given by a **map** which is (i) linear and (ii) preserves probability, $\langle\psi_t|\psi_t\rangle =$ const. This is the entry point for the **unitaries**: any evolution in quantum mechanics can be represented by a unitary operator

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle. \tag{4.1}$$

However, after a while we realized that the state $|\psi\rangle$ is not the most general state of a system. Instead, the general state is the density matrix ρ.

We can then rethink the evolution postulate: *what is the most general evolution which is (i) linear and (ii) maps density matrices into density matrices?* We already saw that unitary evolutions are translated to density matrices as maps of the form

$$\rho \to \rho' = U\rho U^\dagger. \tag{4.2}$$

This is certainly a linear map and if ρ is a valid density matrix, then so will ρ'. But is it the most general kind of map satisfying these properties? The answer is no. The most general map is actually called a **quantum operation**, $\mathcal{E}(\rho)$, and has the form:

$$\boxed{\rho \to \rho' = \mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger, \qquad \text{with} \qquad \sum_k M_k^\dagger M_k = 1.} \tag{4.3}$$

This way of representing the map $\mathcal{E}(\rho)$ in terms of a set of operators M_k is called the **operator-sum representation**. If there is only one M_k then it must be unitary and we recover (4.2). A set of operators $\{M_k\}$ satisfying $\sum_k M_k^\dagger M_k = 1$ are called **Kraus operators**.

The take-home message I want you to keep is that quantum operations are *the* most general evolution map a density matrix can have. This chapter will be all about quantum operations and their ramifications, so we will have quite a lot to discuss about this. But for now let us start slow. In this section we will do two things: first I will show you that quantum operations are the natural language for describing *open quantum systems*. Any evolution of a system connected to an external environment can be written as a quantum operation. Second, we will prove the claim surrounding Eq. (4.3); that is, that any linear map which takes density matrices into density matrices can be written in the form (4.3).

Example: amplitude damping

Consider a qubit system and let

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & \sqrt{\lambda} \\ 0 & 0 \end{pmatrix}, \tag{4.4}$$

with $\lambda \in [0,1]$. This is a valid set of Kraus operators since $M_0^\dagger M_0 + M_1^\dagger M_1 = 1$. Its action on a general qubit density matrix reads:

$$\rho = \begin{pmatrix} p & q \\ q^* & 1-p \end{pmatrix} \to \rho' = \begin{pmatrix} \lambda + p(1-\lambda) & q\sqrt{1-\lambda} \\ q^*\sqrt{1-\lambda} & (1-\lambda)(1-p) \end{pmatrix}. \tag{4.5}$$

If $\lambda = 0$ nothing happens, $\rho' = \rho$. Conversely, if $\lambda = 1$ then

$$\rho \to \rho' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.6}$$

This is why this is called an **amplitude damping**: no matter where you start, the map tries to push the system towards $|0\rangle$. It does so by destroying coherences, $q \to q\sqrt{1-\lambda}$, and by affecting the populations, $p \to \lambda + p(1-\lambda)$. The larger the value of λ, the stronger is the effect.

Amplitude damping from a master equation

Consider a quantum master equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \gamma\left[\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right]. \tag{4.7}$$

We have briefly touched upon this type of equation in Secs. 2.2 and 3.7. And we will have a lot more to say about it below. Applying this equation to a general density matrix yields the pair of equations

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \gamma(1-p) \qquad \to \qquad p(t) = p_0 e^{-\gamma t} + (1 - e^{-\gamma t}),$$
$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\gamma q}{2} \qquad \to \qquad q(t) = q_0 e^{-\gamma t/2}.$$

Comparing this with Eq. (4.5) we see that the solution of the differential Eq. (4.7) can be viewed, at any given time t, as a map

$$\rho(t) = \sum_k M_k \rho(0) M_k^\dagger, \tag{4.8}$$

with the same Kraus operators (4.4) and

$$\lambda = 1 - e^{-\gamma t}. \tag{4.9}$$

If $t = 0$ then $\lambda = 0$ and nothing happens. If $t \to \infty$ then $\lambda \to 1$ and the system collapses completely towards $|0\rangle$, as in Eq. (4.6).

Amplitude damping from system-environment interactions

Let us now label our system S and suppose it interacts with an *environment ancilla* E by means of the Hamiltonian

$$H = g(\sigma_+^S \sigma_-^E + \sigma_-^S \sigma_+^E), \tag{4.10}$$

where g is some parameter. The corresponding unitary evolution matrix will be

$$U = e^{-iHt} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos gt & -i \sin gt & 0 \\ 0 & -i \sin gt & \cos gt & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

Suppose that the ancila starts in the state $|0\rangle_E$ whereas the system starts in an arbitrary state ρ_S. Then we compute

$$\rho'_{SE} = U\Big[\rho_S \otimes |0\rangle_E\langle 0|\Big]U^\dagger,$$

and finally take the partial trace over E to obtain $\rho'_S = \mathrm{tr}_E \rho'_{SE}$. I will leave this task for you as an exercise. The result is

$$\rho'_S = \begin{pmatrix} p + (1-p)\sin^2(gt) & q\cos(gt) \\ q^* \cos(gt) & (1-p)\cos^2(gt) \end{pmatrix}. \tag{4.12}$$

Comparing this with the amplitude damping result (4.5) we see that this is also a quantum operation, again with the same Kraus operators (4.4), but with

$$\lambda = \sin^2(gt). \tag{4.13}$$

Thus, the evolution of two qubits, when viewed from the perspective of only one of them, will behave like a quantum operation. But unlike in the master equation example above, here the amplitude damping parameter λ will not increase monotonically, but will rather oscillate in time. If you happen to interrupt the evolution when gt is an integer multiple of π then it will look like a complete damping. But if we wait a bit longer it will seem that less damping occurred. This is what happens when the environment is small (in this case it is only one qubit). If your environment had 10^{23} qubits, which is what Eq. (4.7) tries to model, you would not observe these revivals.

Amplitude damping and spontaneous emission

The amplitude damping process is also what happens if you have an atom in the excited state interacting with the electromagnetic vacuum. In this case, the atom may fall down to the ground-state and emit a photon, a process we call spontaneous emission. To have a toy model to describe this, suppose that the atom only interacts with one mode of the electromagnetic field, whose frequency ω matches that of the atom Ω. In that case the Hamiltonian reduces to the resonant Jaynes-Cummings model [cf. Eq. (3.23)].

$$H = \Omega a^\dagger a + \frac{\Omega}{2}\sigma_z + g(a^\dagger \sigma_- + a\sigma_+). \tag{4.14}$$

In the resonant case we can move to the interaction picture and still get a time-independent Hamiltonian

$$\tilde{H} = g(a^\dagger \sigma_- + a\sigma_+). \tag{4.15}$$

Suppose now that the electromagnetic mode starts in the vacuum, $|0\rangle_E$, whereas the atom starts in an arbitrary state ρ_S. In principle, this Hamiltonian will act on the full Hilbert space, which is spanned by $|0, n\rangle_{SE}$ and $|1, n\rangle_{SE}$, where $n = 0, 1, 2, \ldots$ is the number of photons in the mode a. But since the Jaynes-Cummings Hamiltonian preserves the total number of quanta [Eq. (3.24)] and since the electromagnetic mode started in the vacuum, at any time there will be either 0 or 1 photons in the mode. Thus, the only basis elements that will matter to us are $|0, 0\rangle_{SE}$, $|0, 1\rangle_{SE}$ and $|1, 0\rangle_{SE}$.

The matrix elements of $\tilde{H}$ in these states are

$$\tilde{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g \\ 0 & g & 0 \end{pmatrix}.$$

Hence, the time-evolution operator will be

$$U = e^{-i\tilde{H}t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(gt) & -i\sin(gt) \\ 0 & -i\sin(gt) & \cos(gt) \end{pmatrix}. \tag{4.16}$$

I wrote down this result just so you could have a look at it. But the truth is we don't need it. Since we are restricting the dynamics to this sub-space, the problem is exactly identical to that generated by the Hamiltonian (4.10) (except for a phase factor, which makes no difference). Indeed, if you now repeat the steps of computing ρ'_{SE} and then ρ'_S, you will find as a result *exactly* the state (4.12).

This example serves to show that many Hamiltonians may lead to the same quantum operation. The quantum operation describes a dynamical evolution from the perspective of the system's density matrix and has *no information* on what exactly generated that evolution. It could have been one qubit, one electromagnetic mode, 10^{23} water molecules in a bucket of water or a swarm of killer bees armed with machine guns. From the perspective of the *map*, they may all lead to the same result.

The above paragraph is a common source of confusion. You may immediately protest and say "How can a one qubit environment lead to the same evolution as a 10^{23}-atom environment?". They don't! They lead to the same *map*, not the same

evolution. That's the point. If we analyze the evolution as a function of time, both will be completely different. But if we are only interested in the map that takes you from one state to another, then this map can be engineered by a single qubit or by 10^{23} of them.

Proof of the operator-sum representation

After this warm-up, we are now ready to prove Eq. (4.3). But let us be very precise on what we want to prove. We define $\mathcal{E}(\rho)$ as a map satisfying

1. Linearity: $\mathcal{E}(\alpha\rho_1 + \beta\rho_2) = \alpha\mathcal{E}(\rho_1) + \beta\mathcal{E}(\rho_2)$.
2. Trace preserving: $\mathrm{tr}[\mathcal{E}(\rho)] = \mathrm{tr}(\rho)$.
3. Completely positive: if $\rho \geq 0$ then $\mathcal{E}(\rho) \geq 0$.

There is a subtle difference between a map that is positive and a map that is *completely* positive. Completely positive means $\mathcal{E}(\rho) \geq 0$ even if ρ is a density matrix living in a larger space than the one $\mathcal{E}$ acts on. For instance, suppose $\mathcal{E}$ acts on the space of a qubit. But the ρ it is acting on could mean the density matrix of 2 entangled qubits, even though the map acts on only one of them. If even in this case the resulting ρ' is positive semi-definite, we say it is completely positive.[1] A map satisfying properties 1, 2 and 3 above is called a **Completely Positive Trace Preserving (CPTP) map**.

Our goal is now to show that any CPTP map can be written as an operator-sum representation [Eq. (4.3)] for some set of operators $\{M_k\}$. The proof of this claim is usually based on a powerful, yet abstract, idea related to what is called the *Choi-Jamiolkowski isomorphism*. Let S denote the space where our map $\mathcal{E}$ acts and define an auxiliary space R which is an exact copy of S. Define also the (unnormalized) Bell state

$$|\Omega\rangle = \sum_i |i\rangle_R \otimes |i\rangle_S, \tag{4.17}$$

where $|i\rangle$ is an arbitrary basis and from now on I will always write the R space in the left and the S space in the right. We now construct the following operator:

$$\boxed{\Lambda_{\mathcal{E}} = (\mathcal{I}_R \otimes \mathcal{E}_S)(|\Omega\rangle\langle\Omega|).} \tag{4.18}$$

This is called the **Choi matrix** of the map $\mathcal{E}$. Note how it is like a density operator. It is the outcome of applying the map $\mathcal{E}_S$ on one side of the maximally entangled Bell state of R+S.

The most surprising thing about the Choi matrix is that it completely determines the map $\mathcal{E}$. That is, if we somehow learn how our map $\mathcal{E}$ acts on $|\Omega\rangle\langle\Omega|$ we have completely determined how it will act on *any* other density matrix. This is summarized by the following formula:

$$\mathcal{E}(\rho) = \mathrm{tr}_R\left\{(\rho^{\mathrm{T}} \otimes I_S)\Lambda_{\mathcal{E}}\right\}. \tag{4.19}$$

[1] There aren't many examples of maps that are positive but not completely positive. The only example I know is the partial trace (see, for instance, Box 8.2 of Nielsen and Chuang).

I know what you are thinking: this is really weird! Yes, it is. But it is true. Note that here ρ^{T} is placed on the auxiliary space R in which the trace is being taken. Consequently, the result on the left-hand side is still an operator living on S. To verify that Eq. (4.19) is true we first rewrite (4.18) as

$$\Lambda_E = \sum_{i,j} |i\rangle_R\langle j| \otimes \mathcal{E}(|i\rangle\langle j|). \tag{4.20}$$

Then we get

$$\begin{aligned}
\mathrm{tr}_R\Big\{(\rho^{\mathrm{T}} \otimes I_S)\Lambda_{\mathcal{E}}\Big\} &= \sum_{i,j} \mathrm{tr}_R\Big\{(\rho^{\mathrm{T}} \otimes I_S)\Big[|i\rangle\langle j| \otimes \mathcal{E}(|i\rangle\langle j|)\Big]\Big\} \\
&= \sum_{i,j} \langle j|\rho^{\mathrm{T}}|i\rangle \mathcal{E}(|i\rangle\langle j|) \\
&= \mathcal{E}\Big(\sum_{i,j} \rho_{i,j}|i\rangle\langle j|\Big) \\
&= \mathcal{E}(\rho).
\end{aligned}$$

Here I used the fact that $\langle j|\rho^{\mathrm{T}}|i\rangle = \langle i|\rho|j\rangle = \rho_{i,j}$. Moreover, I used our assumption that $\mathcal{E}$ is a linear map.

We are now in the position to prove our claim. As I mentioned, the Choi matrix looks like a density matrix on R+S. In fact, we are assuming that our map $\mathcal{E}$ is CPTP. Thus, since $|\Omega\rangle\langle\Omega|$ is a positive semi-definite operator, then so will $\Lambda_{\mathcal{E}}$ (although it will not be normalized). We may then diagonalize $\Lambda_{\mathcal{E}}$ as

$$\Lambda_{\mathcal{E}} = \sum_k \lambda_k |\lambda_k\rangle\langle\lambda_k|,$$

where $|\lambda_k\rangle$ are vectors living in the big R+S space and $\lambda_k \geq 0$. For the purpose of what we are going to do next, it is convenient to absorb the eigenvalues into the eigenvectors (which will no longer be normalized) and define

$$\Lambda_{\mathcal{E}} = \sum_k |m_k\rangle\langle m_k|, \qquad |m_k\rangle = \sqrt{\lambda_k}|\lambda_k\rangle, \tag{4.21}$$

Note that here CPTP is crucial because it implies that $\lambda_k \geq 0$ so that $\langle m_k| = \langle\lambda_k|\sqrt{\lambda_k}$. To finish the proof we insert this into Eq. (4.19) to get

$$\mathcal{E}(\rho) = \sum_k \mathrm{tr}_R\Big\{(\rho^{\mathrm{T}} \otimes I_S)|m_k\rangle\langle m_k|\Big\}. \tag{4.22}$$

The right-hand side will still be an operator living in S, since we only traced over R. All we are left to do is convince ourselves that this will have the shape of the operator-sum representation in Eq. (4.3).

To do that things will get a little nasty. The trick is to connect the states $|m_k\rangle$ of the Choi matrix $\Lambda_{\mathcal{E}}$ with the Kraus operators M_k appearing in the operator-sum representation (4.3):

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger.$$

This is done by noting that since $|m_k\rangle$ lives on the R+S space, it can be decomposed as

$$|m_k\rangle = \sum_{i,j} (M_k)_{j,i} |i\rangle_R \otimes |j\rangle_S , \tag{4.23}$$

where $(M_k)_{j,i}$ are a set of coefficients which we can interpret as a matrix M_k. To establish this connection we first manipulate (4.22) to read

$$\mathcal{E}(\rho) = \sum_k \sum_{i,j} {}_R\langle i|\rho^{\mathrm{T}}|j\rangle_R \, {}_R\langle j|m_k\rangle\langle m_k|i\rangle_R .$$

Then we insert Eq. (4.23) to find

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_k \sum_{i,j} \sum_{i',j'} \rho_{j,i} (M_k)_{j',j} (M_k^*)_{i',i} |j'\rangle\langle i'| \\
&= \sum_k \sum_{i,j} \sum_{i',j'} |j'\rangle\langle j'|M_k|j\rangle\langle j|\rho|i\rangle\langle i|M_k^\dagger|i'\rangle\langle i'| . \\
&= \sum_k M_k \rho M_k^\dagger ,
\end{aligned}$$

and voilá!

In conclusion, we have seen that any map which is linear and CPTP can be described by an operator-sum representation, Eq. (4.3). I like this a lot because we are not asking for much: linearity and CPTP is just the *basic* things we expect from a physical map. Linearity should be there because everything in quantum mechanics is linear and CPTP must be there because the evolution must map a physical state into a physical state. When we first arrived at the idea of a unitary, we were also very relaxed because all we required was the conservation of ket probabilities. The spirit here is the same. For this reason, the quantum operation is really just a very natural and simplistic generalization of the evolution of quantum systems, using density matrices instead of kets.

4.2 Stinespring dilations

In the previous section we defined quantum operations based on the idea of a general map that takes density matrices to density matrices. We also showed that these maps may arise in different circumstances, such as from a master equation or from the unitary interaction of a qubit with a one-qubit environment. This last idea is very powerful and is related to the concept of a **dilation**. That is, the representation of a quantum operation as larger unitary between our system and some environment, as illustrated in Fig. 4.1. It turns out that this dilation idea is always possible and it works in both ways:

- Given a S+E unitary, the corresponding map in terms of S will be given by a quantum operation.
- Given a quantum operation, we can always find a global S+E unitary representing it (in fact, there is an infinite number of such unitaries, as we will see).

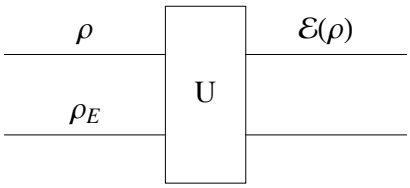

Figure 4.1: Idea behind a Stinespring dilation: a quantum operation $\mathcal{E}(\rho)$ can always be constructed by evolving the system together with an environment, with a global unitary U, and then discarding the environment.

More precisely, a dilation is described as follows. Our quantum system, with density matrix ρ, is put to interact via a global unitary U with an environment (which can be of any size) having an initial density matrix ρ_E. After the interaction we throw away the environment. The result, from the perspective of the system, is a quantum operation. This can be summarized by the expression:

$$\boxed{\mathcal{E}(\rho) = \mathrm{tr}_E \left\{ U(\rho \otimes \rho_E) U^\dagger \right\}.} \tag{4.24}$$

We will now demonstrate that this is indeed a quantum operation.

Top-down, easy case

Let $|e_k\rangle$ denote a basis for the enviroment. To warm up assume the initial state of the environment is pure, $\rho_E = |e_0\rangle\langle e_0|$. Then Eq. (4.24) becomes

$$\mathcal{E}(\rho) = \sum_k \langle e_k | U \rho | e_0 \rangle \langle e_0 | U^\dagger | e_k \rangle,$$

which is similar to a calculation we did in Sec. 2.10. Since ρ and $|e_0\rangle$ live on different Hilbert spaces, we may define[2]

$$\boxed{M_k = \langle e_k | U | e_0 \rangle,} \tag{4.25}$$

with which we arrive at the usual formula for a quantum operation.

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger. \tag{4.26}$$

[2] Remember that what this formula really means is

$$M_k = \Big[1_S \otimes \langle e_k | \Big] U \Big[1_S \otimes |e_0\rangle \Big].$$

We can also check that the M_k in Eq. (4.25) form a valid set of Kraus operators:

$$\sum_k M_k^\dagger M_k = \sum_k \langle e_0|U^\dagger|e_k\rangle\langle e_k|U^\dagger|e_0\rangle = 1$$

Each term in this sum cancelled sequentially: first a completeness relation of the $|e_k\rangle$, then the unitarity of U, then $\langle e_0|e_0\rangle = 1$. The result is still an identity on the space of S.

Top-down, general case

It turns out that the assumption that the environment started in a pure state is not at all restrictive. After all, we can always *purify* the mixed state ρ_E. That is, we can always say the environment actually lives on a larger Hilbert space in which its state is pure. Notwithstanding, it is still useful, from a practical point of view, to generalize (4.25) for general mixed states. In this case the trick is to choose the environment basis $|e_k\rangle$ as the eigenbasis of ρ_E. That is,

$$\rho_E = \sum_k p_k|e_k\rangle\langle e_k|.$$

We now write Eq. (4.24) as

$$\mathcal{E}(\rho) = \sum_{k,q}\langle e_k|U\rho p_q|e_q\rangle\langle e_q|U^\dagger|e_k\rangle.$$

And, instead of (4.25), we define the Kraus operators as

$$M_{k,q} = \sqrt{p_q}\langle e_k|U|e_q\rangle. \tag{4.27}$$

Then the map becomes

$$\mathcal{E}(\rho) = \sum_{k,q} M_{k,q}\rho M_{k,q}^\dagger. \tag{4.28}$$

At first it seems we are cheating a bit because we have two indices. But if we think about (k, q) as a collective index α, then we go back to the usual structure of the quantum operation.

Bottom-up

Now let's go the other way around. Suppose we are given a quantum operation of the form (4.26), with a given set of Kraus operators $\{M_k\}$. We then ask how to construct a global S+E unitary with some environment E, such as to reproduce this quantum operation. That turns out to be quite simple.

First let us ask what should be the dimension of the environment. If we had a vector of dimension d, we all now that the most general linear operation would be given by a $d \times d$ matrix. In our case our system has dimensions d, but we want operations on a density matrix, which is already a $d \times d$ matrix. However, recall that matrices also form a vector space, so the quantum operation can be thought of as an operation on a vector with d^2 entries. The only point is that this vector is displaced like a matrix, so

things become messy because we have to multiply it on both sides. Notwithstanding, we can infer from this argument that we need at most d^2 Kraus operators M_k in order to fully describe a quantum operation. But we have already seen from Eq. (4.25) that the number of k values is related to the number of basis elements $|e_k\rangle$ of the environment. Hence, we conclude that *any quantum operation on a d-dimensional system may be reproduced by a dilation with an environment of dimension d^2*. This fact is quite remarkable. In many cases we are interested in what happens when a system S interacts with a very large environment E. But this argument shows that, as far as the map is concerned, we can always reproduce it with an environment that is only d^2.

Suppose now that the environment starts in some state $|e_0\rangle$. We then construct a unitary U such as to obtain the Kraus operators in Eq. (4.25). This unitary is more easily mocked up if we consider the Hilbert space structure $\mathcal{H}_E \otimes \mathcal{H}_S$ (that is, the environment on the left). Then the unitary that does the job can be written in Block form as

$$U = \begin{pmatrix} M_0 & \dots & \dots & \dots \\ M_1 & \dots & \dots & \dots \\ M_2 & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots \\ M_{d^2-1} & \dots & \dots & \dots \end{pmatrix}. \tag{4.29}$$

where the remainder of the matrix should be filled with whatever it needs to make U an actual unitary. The reason why this works is actually related all the way back to the matrix definition of the Kronecker product, Eq. (2.43). The operator M_k is just the matrix element $U_{k,0}$ in the basis of the environment.

As an example, consider the unitary in the two-qubit example (4.11). In this case the left blocks are

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos(gt) \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & -i\sin(gt) \\ 0 & 0 \end{pmatrix}.$$

This is the same as the amplitude damping Kraus operators in Eq. (4.4), with $\lambda = \sin^2(gt)$ [Eq. (4.13)]. There is an extra weird factor of i, but that doesn't matter because it vanishes when we do $M_1 \rho M_1^\dagger$.

Interpretation in terms of measurements

There is a nice way to picture a quantum operation within this Stinespring dilation setting. You of course noticed that what we are doing here is somewhat similar to the generalized measurement scenario discussed in Sec. 2.10. In fact, there we said that a generalized measurement was also described by a set of Kraus operators $\{M_k\}$ and was such that the probability of obtaining measurement outcome k was

$$p_k = \mathrm{tr}(M_k \rho M_k^\dagger).$$

Moreover, if outcome k was obtained, the state would collapse to

$$\rho \to \rho_k = \frac{M_k \rho M_k^\dagger}{p_k}.$$

We can therefore interpret a quantum operation of the form (4.26) as

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger = \sum_k p_k \rho_k.$$

That is, we can view it as just a random sampling of states ρ_k with probability p_k. The total effect of the quantum operation is a convex combinations of the possible outcomes with different probability weights. Of course, we don't really need to do a measurement. Is just how the system behaves from the eyes of S.

Freedom in the operator-sum representation

There is a reason why we distinguish between the terms "quantum operation" and "operator-sum representation". As the name of the latter implies, when we write a quantum operation in terms of the Kraus operators, like in Eq. (4.26), we are really introducing a *representation* for the map. And the point I wish to make now is that this representation is not unique: there is a freedom in how we choose the Kraus operators which lead to the same quantum operation. The same happens for unitaries: two unitaries U and $U' = e^{i\theta}U$ are physically equivalent so multiplying by a global phase changes nothing. For quantum operations the freedom is even larger.

Let $\{M_k\}$ be a set of Kraus operators and consider the quantum operation (4.26). Now define a new set of Kraus operators $\{N_\alpha\}$ as

$$\boxed{N_\alpha = \sum_k V_{\alpha,k} M_k, \qquad M_k = \sum_\alpha V^*_{\alpha,k} N_\alpha,} \tag{4.30}$$

where V is a unitary matrix.[3] Substituting (4.30) in (4.26) we find

$$\mathcal{E}(\rho) = \sum_{k,\alpha,\beta} V^*_{\alpha,k} V_{\beta,k} N_\alpha \rho N_\beta^\dagger.$$

The trick now is to do the sum over k first. Since V is unitary

$$\sum_k V^*_{\alpha,k} V_{\beta,k} = \sum_k V_{\beta,k} (V^\dagger)_{k,\alpha} = \delta_{\beta,\alpha}.$$

Hence we conclude that

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger = \sum_\alpha N_\alpha \rho N_\alpha^\dagger. \tag{4.31}$$

Thus, *two sets of Kraus operators connected by a unitary transformation lead to the same quantum operation*. It is cool that this even works when the two sets have a

[3] I know this can sound strange at first. Here M_k are operators (maybe there are 7 of them). But we can arrange them to form a list. What we are doing is writing each element in this list as a linear combination of another set of operators N_α. However, we are choosing the coefficients of this linear combinations $V_{k,\alpha}$ to form a unitary matrix, $VV^\dagger = V^\dagger V = 1$.

different number of elements. For instance, suppose $\{M_k\}$ has 5 elements, $M_0, \ldots, M_4$, and $\{N_\alpha\}$ has 3 elements, $N_0, \ldots, N_2$. Then we can add to the list $\{N_\alpha\}$ two zero elements $N_3 = 0$ and $N_4 = 0$. Now both have the same number of elements and we can construct a unitary connecting the two sets.

The next interesting question is what is the origin of this freedom. It turns out it is related to local operations on the environment. Recall that, as shown in Eq. (4.25), $M_k = \langle e_k|U|e_0\rangle$. Now suppose that before we finish the evolution, we perform a unitary $V_E \otimes 1_S$ on the environment. Then the new set of Kraus operators will be

$$N_\alpha = \langle e_\alpha|(V \otimes 1)U|e_0\rangle = \sum_k \langle e_\alpha|V|e_k\rangle\langle e_k|U|e_0\rangle = \sum_k V_{\alpha,k} M_k,$$

which is exactly Eq. (4.30). Thus, we can view this freedom of choice as a sort of "post-processing" on the environment, which has no effect on the system.

Partial trace as a quantum operation

So far we have considered quantum operations that map a given Hilbert space to the same space. However, the entire framework generalizes naturally to maps taking a density matrix in a given subspace $\mathcal{H}_1$ to another subspace $\mathcal{H}_2$. In this case all that changes is that the condition on Kraus operators become

$$\sum_k M_k^\dagger M_k = I_1 \tag{4.32}$$

That is, with the identity being on the space $\mathcal{H}_1$. An example of such an operation is the partial trace. Suppose our system S is actually a bipartite system AB. The partial trace over B is written, as we know, as

$$\mathrm{tr}_B(\rho) = \sum_k \langle b_k|\rho|b_k\rangle = \sum_k (1_A \otimes \langle b_k|)\rho(1_A \otimes |b_k\rangle). \tag{4.33}$$

If we define the Kraus operators

$$\boxed{M_k = 1_A \otimes \langle b_k|,} \tag{4.34}$$

then the partial trace can be identified with the quantum operation

$$\mathrm{tr}_B(\rho) = \sum_k M_k \rho M_k^\dagger. \tag{4.35}$$

Moreover we see that

$$\sum_k M_k^\dagger M_k = \sum_k 1_A \otimes |b_k\rangle\langle b_k| = 1_{AB}.$$

That is, the identity on the original space.

We can also do the opposite. That is, we can define a quantum operation which *adds* a state to the system. For instance, suppose we have a system S and we want to add an environment ancilla E in a state $|e_0\rangle$. Then we can define the Kraus operators

$$M_0 = 1_S \otimes |e_0\rangle_E. \tag{4.36}$$

The corresponding quantum operation will then be

$$M_0 \rho M_0^\dagger = \rho \otimes |e_0\rangle\langle e_0|. \tag{4.37}$$

Moreover,

$$M_0^\dagger M_0 = 1_S.$$

Of course, if we want to add an ancilla in a more general state, all we need to do is construct a larger set of Kraus operators. With these ideas we can actually cover all types of quantum operations. That is, any map can always be described by quantum operations mapping the same Hilbert space, combined with partial traces and adding ancillas.