

We can also do the opposite. That is, we can define a quantum operation which *adds* a state to the system. For instance, suppose we have a system S and we want to add an environment ancilla E in a state $|e_0\rangle$. Then we can define the Kraus operators

$$M_0 = 1_S \otimes |e_0\rangle_E. \quad (4.36)$$

The corresponding quantum operation will then be

$$M_0 \rho M_0^\dagger = \rho \otimes |e_0\rangle\langle e_0|. \quad (4.37)$$

Moreover,

$$M_0^\dagger M_0 = 1_S.$$

Of course, if we want to add an ancilla in a more general state, all we need to do is construct a larger set of Kraus operators. With these ideas we can actually cover all types of quantum operations. That is, any map can always be described by quantum operations mapping the same Hilbert space, combined with partial traces and adding ancillas.

4.3 Lindblad master equations

We have seen that a quantum operation is the most general map taking density matrices to density matrices. But sometimes maps are not so useful and it is better to have a differential equation for $\rho(t)$. That is, something like

$$\frac{d\rho}{dt} = \mathcal{L}(\rho), \quad (4.38)$$

where $\mathcal{L}(\rho)$ is some linear superoperator (a superoperator is just an operator acting on an operator). It is also customary to call $\mathcal{L}(\rho)$ the **Liouvillian** because of the analogy between Eq. (4.38) and the Liouville equation appearing in classical mechanics. An equation of the form (4.38) is also historically known as a **master equation**, a name which was first introduced in a completely different problem,⁴ but is supposed to mean an equation from which all other properties can be derived from.

We may then ask the following question: “Given an initial genuine density matrix $\rho(0)$, what is the general structure a Liouvillian \mathcal{L} must have in order to ensure that the solution $\rho(t)$ of Eq. (4.38) will also be a genuine density matrix at all times t ?” Putting it differently, suppose we happen to solve Eq. (4.38). Then the solution will be given by some linear map of the form

$$\rho(t) = \mathcal{V}_t(\rho(0)). \quad (4.39)$$

where \mathcal{V}_t is some superoperator. What we then really want is for \mathcal{V}_t to be a quantum operation at all times t . If that is the case we say the master equation is CPTP (because the map it generates is CPTP).

⁴A. Nordsieck, W. E. Lamb and G. T. Uhlenbeck, *Physica*, **7**, 344 (1940).

Eq. (4.38) has the form of a linear equation

$$\frac{dx}{dt} = Ax. \quad (4.40)$$

Equations of this form always have the property of being *divisible*. That is, the solution from $t = 0$ to $t = t_2$ can always be split into a solution from $t = 0$ to $t = t_1$ and then a solution from $t = t_1$ to $t = t_2$. Consequently, this implies that \mathcal{V}_t must satisfy the *semigroup property*:⁵

$$\mathcal{V}_{t_2} \mathcal{V}_{t_1} = \mathcal{V}_{t_2+t_1}. \quad (4.41)$$

Semigroup is therefore implied by the structure of Eq. (4.38). We can then ask, when can a semigroup map be CPTP? Quite remarkably, just by imposing these two properties one can determine a very specific structure for the Liouvillian ρ . This is the content of **Lindblad's theorem**:⁶ *The generator of any quantum operation satisfying the semigroup property must have the form:*

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k \gamma_k \left[L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right], \quad (4.42)$$

where H is a Hermitian operator, L_k are arbitrary operators and $\gamma_k \geq 0$. Master equations having this structure are then called Lindblad equations or, more generally, Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equations. If you have any equation satisfying this structure, then the corresponding evolution is guaranteed to be CPTP (i.e., physical). Conversely, any CPTP and divisible map is guaranteed to have to this form. Of course, this does not say anything about how to *derive* such an equation. That is a hard question, which we will start to tackle in the next section. But this result gives us an idea of what kind of structure we should look for and that is already remarkably useful.

Now here is our battle plan for this section: first, we will discuss some examples. Then we will prove Lindblad's theorem. Finally I will show you some tricks of the trade for dealing with these equations, specially from a numerical point of view.

Amplitude damping at finite temperature

We have already discussed the amplitude damping master equation in Sec. 4.1. But in that case the equation described a zero temperature effect. For instance, as illustrated in Eq. (4.6), the steady-state of the equation was the pure state $|0\rangle\langle 0|$. The generalization to finite temperatures is captured by a dissipator of the form

$$\mathcal{D}(\rho) = \gamma(1-f) \left[\sigma_- \rho \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho \} \right] + \gamma f \left[\sigma_+ \rho \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+, \rho \} \right], \quad (4.43)$$

⁵It is a semigroup because this looks like the composition property of a group, but the inverse is not necessarily a member of the group.

⁶G. Lindblad, *Comm. Math. Phys.*, **48**, 119 (1976).

where $\gamma > 0$ and $f \in [0, 1]$. After a while we get tired of writing these equations explicitly, so it is more convenient to break them in blocks. Define

$$D[L] = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}. \quad (4.44)$$

Then we can rewrite Eq. (4.43) as

$$\mathcal{D}(\rho) = \gamma(1-f)D[\sigma_-] + \gamma f D[\sigma_+]. \quad (4.45)$$

To know what a dissipator such as this is doing, we look at the fixed points. That is, the density matrix satisfying $\mathcal{D}(\rho^*) = 0$. Of course, we also need to include the Hamiltonian part, which we will do so below. But for now let's just forget about H for a second. In this case you can check that the steady-state of \mathcal{D} is

$$\rho^* = \begin{pmatrix} f & 0 \\ 0 & 1-f \end{pmatrix}. \quad (4.46)$$

Thus, the constant f appearing in (4.45) represent the populations in the computational basis. If $f = 1$ the system will relax all the way to the north pole $|0\rangle$. If $f = 0$ it will relax to the south pole $|1\rangle$. For intermediate f , it will relax somewhere in the middle of the z axis, having $\langle\sigma_z\rangle_* = 2f - 1$.

After looking at the steady-state, the next nice thing is to look at the relaxation towards the steady-state. In this case, if we let $p = \langle 0|\rho|0\rangle$ be the population in the north pole and $q = \langle 0|\rho|1\rangle$, then the evolution under Eq. (4.45) will lead to the equations

$$\frac{dp}{dt} = \gamma(f - p), \quad (4.47)$$

$$\frac{dq}{dt} = -\frac{\gamma}{2}q. \quad (4.48)$$

The solutions are simply

$$p(t) = p(0)e^{-\gamma t} + f(1 - e^{-\gamma t}), \quad (4.49)$$

$$q(t) = q(0)e^{-\gamma t/2}. \quad (4.50)$$

Thus, the population $p(t)$ will relax exponentially towards the ‘‘bath-imposed’’ population f , whereas the coherence will relax towards zero. It is interesting to note that $q(t)$ always relaxes to zero, irrespective of what is the value of f .

Competition

I like to view master equations such as (4.42) as a competition between different terms. Each ingredient in the equation is trying to push the system toward some direction and the steady-state will be a kind of compromise between the relative strengths

of each term. This is already clear in the dissipator (4.44): the first term pushes to the south pole and the second term to the north pole. As a result, the system eventually settles down in the state (4.46), which is somewhere in the middle.

Unitary terms also contribute to the competition and this mixture of unitary and dissipative elements lead to interesting effects. To have an idea of what can happen, consider the Liouvillian

$$\mathcal{L}(\rho) = -i\frac{\Omega}{2}[\sigma_z, \rho] + \gamma(1-f)D[\sigma_-] + \gamma f D[\sigma_+]. \quad (4.51)$$

This is just like Eq. (4.45), except that now we added a Hamiltonian term corresponding to a qubit in the σ_z basis. The action of this unitary term turns out to be quite simple. All it will do is change the evolution of $q(t)$ to $q(t) = q(0)e^{-i(\Omega+\gamma/2)t}$. Thus, $q(t)$ will also oscillate a bit while relaxing. However, the steady-state remains the same, being given simply by Eq. (4.46).

Now let's consider a tiny variation of Eq. (4.51), where the Hamiltonian is modified from σ_z to σ_x :

$$\mathcal{L}(\rho) = -i\frac{\Omega}{2}[\sigma_x, \rho] + \gamma(1-f)D[\sigma_-] + \gamma f D[\sigma_+]. \quad (4.52)$$

The steady-state of this equation is now completely different, being given by

$$p^* = \frac{f\gamma^2 + \Omega^2}{\gamma^2 + 2\Omega^2}, \quad (4.53)$$

$$q^* = i(2f-1)\frac{\gamma\Omega}{\gamma^2 + 2\Omega^2} \quad (4.54)$$

This is now a weird mixed state lying somewhere in the yz plane. If $\gamma \gg \Omega$ then we recover back the state (4.46). However, if $\Omega \gg \gamma$ then the state actually tends to the maximally mixed state $\rho^* = \mathbb{I}/2$. This is interesting because we could naively think the system would tend to the x axis. But it doesn't because unitary and dissipative contributions behave differently. Dissipative terms push you to places, whereas unitaries like to oscillate around.

A harmonic oscillator subject to a finite temperature bath

In Sec. 3.7 we discussed the idea of a lossy cavity, which is described by a Lindblad dissipator that pumps energy away from the system. A similar idea applies to a continuous variable mode subject to a finite temperature bath. But in this case energy is not only drained out, but some may also come in. The dissipator describing this type of process is

$$\begin{aligned} \mathcal{D}(\rho) &= \gamma(\bar{n}+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\} \right] + \gamma\bar{n}\left[a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\} \right] \\ &= \gamma(\bar{n}+1)D[a] + \gamma\bar{n}D[a^\dagger], \end{aligned} \quad (4.55)$$

where $\gamma > 0$ and \bar{n} is the Bose-Einstein distribution

$$\bar{n} = \frac{1}{e^{\beta\omega} - 1}, \quad (4.56)$$

with ω being the oscillator's frequency and $\beta = 1/T$ the inverse temperature. If $T = 0$ then $\bar{n} = 0$ and we recover the lossy cavity dissipator of Sec. 3.7.

Let us first ask what is the steady-state of (4.55). A honest guess would be a thermal state at temperature β . Indeed, I will leave for you the exercise of verifying that

$$\mathcal{D}(e^{-\beta\omega a^\dagger a}) = 0, \quad (4.57)$$

which works *only* if the β here is the same β appearing in Eq. (4.56). Thus, the steady-state is a thermal state with the same temperature as that imposed by the bath.

Dealing with these infinite dimensional master equations can sometimes be cumbersome. What I usually do is to always look first at expectation values of operators. And in this case it is useful to generalize a bit some of the tricks discussed in Sec. 3.7. Let us write our master equation as

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{D}(\rho). \quad (4.58)$$

Now we compute the expectation value of some operator O , which reads

$$\frac{d\langle O \rangle}{dt} = i\langle [H, O] \rangle + \text{tr} \left\{ O \mathcal{D}(\rho) \right\}. \quad (4.59)$$

The first term is simply the Heisenberg equation. It is useful to write down the second term in a similar way, as the expectation value of something on the state ρ .

Suppose we have a dissipator of the form $\mathcal{D}[L]$ in Eq. (4.44). Then, using the cyclic property of the trace, we can write

$$\text{tr} \left\{ O \left[L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\} \right] \right\} = \langle L^\dagger O L - \frac{1}{2}\{L^\dagger L, O\} \rangle. \quad (4.60)$$

This motivates us to define the **adjoint dissipator**

$$\bar{\mathcal{D}}[L](O) = L^\dagger O L - \frac{1}{2}\{L^\dagger L, O\}. \quad (4.61)$$

which is a superoperator acting on observables O , instead of density matrices. It is nice to have a look at the structure of $\bar{\mathcal{D}}$. In the original dissipator (4.44) the first term has $L \bullet L^\dagger$ but the second term has $L^\dagger L$. In the adjoint dissipator, on the other hand, everything is in the same order, with L^\dagger always in the left. What is more, because of this more symmetric structure, we can actually factor the adjoint dissipator as

$$\bar{\mathcal{D}}[L](O) = \frac{1}{2}L^\dagger [O, L] + \frac{1}{2}[L^\dagger, O]L. \quad (4.62)$$

With this structure it is now extremely easy to compute expectation values of observables since it amounts only to the computation of a commutator. And just to summarize, can now write Eq. (4.59) as

$$\boxed{\frac{d\langle O \rangle}{dt} = i\langle [H, O] \rangle + \langle \bar{\mathcal{D}}(O) \rangle.} \quad (4.63)$$

Going back to the harmonic oscillator dissipator (4.55), the corresponding adjoint dissipator will be

$$\begin{aligned} \bar{\mathcal{D}}(O) &= \gamma(\bar{n} + 1) \left[a^\dagger O a - \frac{1}{2} \{a^\dagger a, O\} \right] + \gamma\bar{n} \left[a O a^\dagger - \frac{1}{2} \{a a^\dagger, O\} \right] \\ &= \frac{\gamma}{2}(\bar{n} + 1) \left\{ a^\dagger [O, a] + [a^\dagger, O] a \right\} + \frac{\gamma}{2}\bar{n} \left\{ a [O, a^\dagger] + [a, O] a^\dagger \right\}. \end{aligned} \quad (4.64)$$

Please take a second to notice what I did. In the first line I just used the shape of the original dissipator (4.55) and changed the order $L\rho L^\dagger \rightarrow L^\dagger O L$. In the second line I just used the structure of Eq. (4.62) to rewrite this in terms of commutators.

Let us now look at some examples, starting with $O = a$. Inserting this in Eq. (4.64) leads to

$$\bar{\mathcal{D}}(a) = \gamma(\bar{n} + 1) \left\{ -\frac{a}{2} \right\} + \gamma\bar{n} \left\{ \frac{a}{2} \right\} = -\frac{\gamma a}{2}. \quad (4.65)$$

For concreteness, let us also suppose $H = \omega a^\dagger a$. Then the equation for $\langle a \rangle$ will be simply

$$\frac{da}{dt} = -(i\omega + \gamma/2)\langle a \rangle. \quad (4.66)$$

Interestingly, this is the same equation as for the zero temperature case. Thus, thermal fluctuations turn out not to affect the first moment $\langle a \rangle$.

Next we turn to $O = a^\dagger a$. Eq. (4.64) then gives

$$\bar{\mathcal{D}}(a^\dagger a) = \gamma(\bar{n} + 1) \left\{ -a^\dagger a \right\} + \gamma\bar{n} \left\{ a a^\dagger \right\} = -\gamma a^\dagger a + \gamma\bar{n} \quad (4.67)$$

The evolution of $\langle a^\dagger a \rangle$ will then be given by

$$\frac{d\langle a^\dagger a \rangle}{dt} = \gamma(\bar{n} - \langle a^\dagger a \rangle). \quad (4.68)$$

This will therefore be an exponential relaxation, from the initial occupation $\langle a^\dagger a \rangle_0$ to the bath-imposed occupation \bar{n} . It is interesting to note how the right-hand side of Eq. (4.68) can be viewed as the **current of quanta**, in the sense of a *continuity equation*:

$$\frac{d\langle a^\dagger a \rangle}{dt} := J. \quad (4.69)$$

That is, the rate at which the number of quanta changes is related to the *flux* of quanta from the system to the environment. If at any given time $\langle a^\dagger a \rangle > \bar{n}$ then the current will be negative, meaning quanta is flowing from the system to the environment. Conversely, if $\langle a^\dagger a \rangle < \bar{n}$ the current becomes positive, meaning that quanta is flowing from the environment to the system.

Proof of Lindblad's theorem

Let us now prove Lindblad's theorem. That is, we will show that any quantum operation which also satisfies the semigroup property can be written in the Lindblad form (4.42). If the dynamics is to satisfy the semigroup property (i.e., if it is divisible) then we must be able to write the evolution over an infinitesimal time Δt as

$$\rho(t + \Delta t) = \sum_k M_k(\Delta t) \rho(t) M_k^\dagger(\Delta t), \quad (4.70)$$

where the Kraus operators $M_k(\Delta t)$ cannot depend on the time t . Let us then ask what we want for the $M_k(\Delta t)$. We are after a differential equation for $\rho(t)$, of the form (4.38). This means that for small Δt we want something like

$$\rho(t + \Delta t) \simeq \rho(t) + \Delta t \mathcal{L}(\rho(t)). \quad (4.71)$$

In general, since the first correction is of the order Δt , we will then need to have $M_k(\Delta t) = \sqrt{\Delta t} L_k$, where L_k is some operator. This is so because then $M_k \rho M_k^\dagger \sim \Delta t$. But we also have the additional property that, if $\Delta t = 0$, then nothing should happen: $\sum_k M_k(0) \rho M_k(0) = \rho$. One way to introduce this would be to take one Kraus operator, for instance $k = 0$, to be $M_k = I$. However, as we will see, this will give us trouble with the normalization of the Kraus operators.

The correct way to fix it is by defining

$$\boxed{M_0 = I + G\Delta t, \quad M_k = \sqrt{\Delta t} L_k, \quad k \neq 0} \quad (4.72)$$

where G and L_k are arbitrary operators. The normalization condition for the Kraus operators then leads to

$$\begin{aligned} 1 &= \sum_k M_k^\dagger M_k = M_0^\dagger M_0 + \sum_{k \neq 0} M_k^\dagger M_k \\ &= (I + G^\dagger \Delta t)(I + G\Delta t) + \Delta t \sum_{k \neq 0} L_k^\dagger L_k \\ &= I + (G + G^\dagger) \Delta t + \Delta t \sum_{k \neq 0} L_k^\dagger L_k + \mathcal{O}(\Delta t^2). \end{aligned}$$

This shows why we need this G guy. Otherwise, we would never be able to normalize the Kraus operators. Since G is arbitrary, we may parametrize it as

$$G = K - iH, \quad (4.73)$$

where K and H are both Hermitian. It then follows from the normalization condition that

$$K = -\frac{1}{2} \sum_{k \neq 0} L_k^\dagger L_k, \quad (4.74)$$

whereas nothing can be said about H .

With this at hand, we can finally substitute our results in Eq. (4.70). We then get

$$\begin{aligned}
\rho(t + \Delta t) &= (I + G\Delta t)\rho(I + G^\dagger\Delta t) + \Delta t \sum_{k \neq 0} L_k \rho L_k^\dagger \\
&= \rho(t) + \Delta t(G\rho + \rho G^\dagger) + \Delta t \sum_{k \neq 0} L_k \rho L_k^\dagger \\
&= \rho(t) - i\Delta t[H, \rho] + \Delta t \sum_{k \neq 0} \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]
\end{aligned}$$

Rearranging and taking the limit $\Delta t \rightarrow 0$ we then finally obtain

$$\frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} \simeq \frac{d\rho}{dt} = -i[H, \rho] + \sum_{k \neq 0} \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right], \quad (4.75)$$

which is Lindblad's equation (4.42). Woo-hoo! We did it! The only tiny difference is that in Eq. (4.42) there are also some coefficients γ_k . But you can just think that we redefine $\sqrt{\gamma_k}L_k \rightarrow L_k$, so they are both really the same thing.

In summary, we have seen that if we combine the semigroup property and the structure of a quantum operation, the corresponding differential equation must have Lindblad's form. As I mentioned before, we still have no idea of what the operators H and L_k should be. That will be the topic of next section. But it is great that we can already tell what the general structure should be.

Vectorization/Choi-Jamiolkowski isomorphism

Master equations and quantum operations can be annoying because we always have to multiply ρ on both sides. But if you remember your linear algebra course, you will recall that matrices also form a vector space. Hence, we can think of superoperators (such as the Liouvillian) as just a big matrices multiplying a big vector ρ . This idea can be made more formal using the Choi-Jamiolkowski isomorphism, or vectorization. It is neatly captured by the following relation:

$$|i\rangle\langle j| \rightarrow |j\rangle \otimes |i\rangle. \quad (4.76)$$

That is, we can think about an outer product (which has two indices) as being just a vector (with one index) in a doubled dimension. In this way, when we have an arbitrary density matrix

$$\rho = \sum_{i,j} \rho_{i,j} |i\rangle\langle j|, \quad (4.77)$$

we can write its vectorized form as

$$\text{vec}(\rho) = \sum_{i,j} \rho_{i,j} |j\rangle \otimes |i\rangle. \quad (4.78)$$

From a matrix point of view, this operation is the same as *stacking columns of a matrix*

$$\text{vec} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}. \quad (4.79)$$

This vectorization trick is very useful, in particular due to two main properties. The first is related to the *Hilbert-Schmidt inner product*, defined as

$$(A, B) := \text{tr}(A^\dagger B). \quad (4.80)$$

This quantity satisfies all properties of an inner product and is therefore the operator analog of $\langle \psi | \phi \rangle$. And, in terms of the vectorized operators (4.79), it becomes exactly what one would intuitively guess:

$$\text{tr}(A^\dagger B) = \text{vec}(A)^\dagger \text{vec}(B). \quad (4.81)$$

That is, just the inner product between the two vectors.

A particularly important state in this sense is the vectorized version of the identity operator:

$$I = \sum_i |i\rangle\langle i| \quad \rightarrow \quad \text{vec}(I) = \sum_i |i\rangle \otimes |i\rangle. \quad (4.82)$$

We therefore see that the identity vectorizes to the (unnormalized) maximally entangled Bell state. One of the reasons why the identity is so important is in connection with the normalization of a density matrix:

$$\text{tr}(\rho) = \text{vec}(I)^\dagger \text{vec}(\rho) = 1. \quad (4.83)$$

The second useful property of the vectorization is as follows. Suppose we vectorize the product of three matrices ABC . It then turns out that

$$\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B). \quad (4.84)$$

(Please note that what appears here is not the dagger, but the transpose). This is certainly not an intuitive property. The best way I know of convincing ourselves that it works is to simply write it out in the ugliest way possible:

$$ABC = \sum_{i,j,k,\ell,m,n} (A_{i,j}|i\rangle\langle j|)(B_{k,\ell}|k\rangle\langle \ell|)(C_{m,n}|m\rangle\langle n|) = \sum_{i,j,m,n} A_{i,j} B_{j,m} C_{m,n} |i\rangle\langle n|.$$

Then

$$\text{vec}(ABC) = \sum_{i,j,m,n} A_{i,j} B_{j,m} C_{m,n} |n\rangle \otimes |i\rangle.$$

On the other hand

$$(C^T \otimes A) \text{vec}(B) = \left(\sum_{m,n,i,j} C_{m,n} A_{i,j} |n\rangle \langle m| \otimes |i\rangle \langle j| \right) \sum_{k,\ell} B_{k,\ell} |\ell\rangle \otimes |k\rangle = \sum_{m,n,i,j} C_{m,n} A_{i,j} B_{j,m} |n\rangle \otimes |i\rangle,$$

which is the same thing.

The usefulness of Eq. (4.84) lies in the fact that it provides us with a recipe to write superoperator products such as $A\rho C$ in the form of a big matrix times $\text{vec}(\rho)$:

$$\text{vec}(A\rho C) = (C^T \otimes A) \text{vec}(\rho).$$

This also works for terms like

$$\text{vec}(H\rho) = \text{vec}(H\rho I) = (I \otimes H) \text{vec}(\rho).$$

In this way we can write the full Liouvillian as just a big big matrix:

$$\text{vec}\left(-i[H, \rho]\right) = -i\left[I \otimes H - H^T \otimes I\right] \text{vec}(\rho), \quad (4.85)$$

$$\text{vec}\left(L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}\right) = \left[L^* \otimes L - \frac{1}{2}I \otimes L^\dagger L - \frac{1}{2}(L^\dagger L)^T \otimes I\right] \text{vec}(\rho) \quad (4.86)$$

Taking the vec of the original master equation (4.38), we can now rewrite it as

$$\frac{d}{dt} \text{vec}(\rho) = \hat{\mathcal{L}} \text{vec}(\rho), \quad (4.87)$$

where $\hat{\mathcal{L}}$ is now a matrix (which is why I put a hat on it). For the general Liouvillian structure such as (4.42), this matrix will then read

$$\hat{\mathcal{L}} = -i(I \otimes H - H^T \otimes I) + \sum_k \gamma_k \left[L_k^* \otimes L_k - \frac{1}{2}I \otimes L_k^\dagger L_k - \frac{1}{2}(L_k^\dagger L_k)^T \otimes I \right] \quad (4.88)$$

Eq. (4.87) then nothing but a simple matrix-vector equation so that its properties can all be deduced from the properties of the matrix \mathcal{L} .

As an example of Eq. (4.88), the vectorized version of the amplitude damping dissipator (4.43) is

$$\hat{\mathcal{D}} = \begin{pmatrix} -\gamma(1-f) & 0 & 0 & \gamma f \\ 0 & -\gamma/2 & 0 & 0 \\ 0 & 0 & -\gamma/2 & 0 \\ \gamma(1-f) & 0 & 0 & -\gamma f \end{pmatrix}. \quad (4.89)$$

The matrix is not Hermitian. Notwithstanding, we will now see that it does satisfy a series of special properties.

Spectral properties of \mathcal{L}

As you may know from the theory of ordinary differential equations, the solution of Eq. (4.87) is simply

$$\text{vec}(\rho(t)) = e^{\hat{\mathcal{L}}t} \text{vec}(\rho(0)). \quad (4.90)$$

Hence, all properties of the solution are determined by this matrix exponential and hence by the spectral properties of $\hat{\mathcal{L}}$. In principle $\hat{\mathcal{L}}$ may not be diagonalizable. But let's assume it is. However, since it is not Hermitian, it will in general have different left and right eigenvectors

$$\hat{\mathcal{L}} \mathbf{x}_\alpha = \lambda_\alpha \mathbf{x}_\alpha, \quad (4.91)$$

$$\mathbf{y}_\alpha^\dagger \hat{\mathcal{L}} = \lambda_\alpha \mathbf{y}_\alpha^\dagger. \quad (4.92)$$

where λ_α are the eigenvalues and \mathbf{x}_α and \mathbf{y}_α are the corresponding right and left eigenvectors (they are both column vectors so $\mathbf{y}_\alpha^\dagger$ is a row vector). The diagonal decomposition of $\hat{\mathcal{L}}$ will then read

$$\hat{\mathcal{L}} = S \Lambda S^{-1}. \quad (4.93)$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots)$ is the diagonal matrix containing the eigenvalues and S is the matrix whose columns are the right eigenvectors \mathbf{x}_α , whereas S^{-1} is a matrix whose rows are $\mathbf{y}_\alpha^\dagger$. Hence we may also write the diagonal decomposition as

$$\hat{\mathcal{L}} = \sum_{\alpha} \lambda_{\alpha} \mathbf{x}_{\alpha} \mathbf{y}_{\alpha}^{\dagger}, \quad (4.94)$$

These decompositions are useful when we want to write the matrix exponential, which simply becomes

$$e^{\hat{\mathcal{L}}t} = S e^{\Lambda t} S^{-1} = \sum_{\alpha} e^{\lambda_{\alpha} t} \mathbf{x}_{\alpha} \mathbf{y}_{\alpha}^{\dagger}. \quad (4.95)$$

With this form, we can now finally ask what should the properties of the eigenvalues and eigenvectors be in order for the dynamics to be physically consistent.

First we look at the trace preserving property (4.83). Multiplying Eq. (4.87) by $\text{vec}(I)^\dagger$ we get

$$0 = \frac{d}{dt} \text{vec}(I)^\dagger \text{vec}(\rho) = \text{vec}(I)^\dagger \hat{\mathcal{L}} \text{vec}(\rho).$$

But this must be true for all density matrices. Hence, we must have

$$\text{vec}(I)^\dagger \hat{\mathcal{L}} = 0. \quad (4.96)$$

Comparing this with Eq. (4.92), we then conclude that the identity must *always* be a left eigenstate of $\hat{\mathcal{L}}$ with eigenvalue 0. Let us label this eigenvector $\alpha = 0$. Then $\lambda_0 = 0$ and $\mathbf{y}_0 = \text{vec}(I)$. But what about \mathbf{x}_0 ? Well, if we think about it, this will be nothing but the *steady-state* of the Liouvillian. That is,

$$\boxed{\mathbf{x}_0 = \text{vec}(\rho^*) \quad \text{where} \quad \mathcal{L}(\rho^*) = 0.} \quad (4.97)$$

This is a really powerful result: *any trace-preserving Liouvillian must have a zero eigenvalue. Its right eigenvector will be the steady-state of the equation, whereas the left eigenvector will be the identity.* Of course, a more subtle question is whether this steady-state is unique. That is, whether the eigenvalue 0 is degenerate or not. I would say quite often the steady-state is unique, but unfortunately this really depends on the problem in question.

Let us now return to the general solution (4.90). Using the diagonal decomposition (4.95) we get

$$\rho(t) = \sum_{\alpha} e^{\lambda_{\alpha} t} \mathbf{x}_{\alpha} \left[\mathbf{y}_{\alpha}^{\dagger} \text{vec}(\rho(0)) \right] = \sum_{\alpha} c_{\alpha} e^{\lambda_{\alpha} t} \mathbf{x}_{\alpha}, \quad (4.98)$$

where $c_{\alpha} = \mathbf{y}_{\alpha}^{\dagger} \text{vec}(\rho(0))$ are just coefficients related to the initial conditions (you may see a similarity here with the usual solution of Schrödinger's equation). From this result we also arrive at another important property of Liouvillians: *the eigenvalues must always have a non-positive real part.* That is to say, either $\lambda_{\alpha} = 0$ or $\text{Re}(\lambda_{\alpha}) < 0$. Otherwise, the exponentials would blow up, which would be unphysical.

As an example, the Liouvillian in Eq. (4.89) has eigenvalues

$$\text{eigs}(\hat{\mathcal{D}}) = \left\{ -\gamma, -\frac{\gamma}{2}, -\frac{\gamma}{2}, 0 \right\}. \quad (4.99)$$

In this case they turn out to be real. But if we also add a unitary term, then they will in general be complex. Notwithstanding their real part will always be non-positive.

Assume now that the zero eigenstate is unique. Then we can write Eq. (4.98) as

$$\rho(t) = c_0 \mathbf{x}_0 + \sum_{\alpha \neq 0} c_{\alpha} e^{\lambda_{\alpha} t} \mathbf{x}_{\alpha}. \quad (4.100)$$

I really like this result. First, note that in the first term $c_0 = \mathbf{y}_0^{\dagger} \text{vec}(\rho(0)) = 1$ by normalization. Secondly, in the second term all eigenvalues have negative real part so that, in the long-time limit, they will relax to zero. Consequently, we see that

$$\lim_{t \rightarrow \infty} \rho(t) = \mathbf{x}_0. \quad (4.101)$$

which, as expected, is the steady-state (4.97). Thus, we conclude that if the steady-state is unique, no matter where you start, the system will always eventually relax towards the steady-state. The real part of the eigenvalues λ_{α} therefore tell you about the *relaxation rate* of the different terms. That is, they give you information on the time-scale with which the relaxation will occur.