are still in the interaction picture, so we may want to go back to the Shcrödinger picture. For practical purposes it is also convenient to open the commutators and rearrange this as follows:

$$\boxed{\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = \int_0^\infty \mathrm{d}s \;\; \mathrm{tr}_E \Big\{ V(t)\rho_S(t)\rho_E V(t-s) - \rho_S(t)\rho_E V(t-s)V(t) \Big\} + \text{h.c.},} \tag{4.129}$$

where h.c. stands for Hermitian Conjugate and simply means we should add the dagger of whatever we find in the first term. This is convenient because then out of the four terms, we only need to compute two.

As we will see below, it turns out that sometimes Eq. (4.129) is still not in Lindblad form. In these cases we will have to make a third approximation, which is the rotating-wave approximation (RWA) we discussed in the context of the Rabi model (Sec. 3.3). That is, *every now and then we will have to throw away some rapidly oscillatory terms*. The reason why this may be necessary is related to an argument about *time-scales and coarse graining*. The point worth remembering is that **bath-stuff are fast and system-stuff are slow**. This is again similar to classical Brownian motion: if the bath is a bear, trying to score some honey, then the bath is a swarm of bees desperately fighting to save their home.[10] During the time scale over which the bear takes a few steps, the bees have already lived half their lifetime. Due to this reason, a master equation is only *resolved* over time-scales much larger than the bath scales. If we ever encounter rapidly oscillatory terms, then they mean we are trying to model something in a time scale which we are not resolved. That is why it is justified to throw them away. So, in a sense, the RWA here is us trying to fix up the mess we ourselves created.

4.5 Open quantum harmonic oscillator

Now that we have a general recipe for finding Lindblad equations, we need to learn how to apply it. Let's do that in the context of a quantum harmonic oscillator coupled to a bath of harmonic oscillators. The total Hamiltonian $H = H_S + H_E + V$, Eq. (4.102), will be taken to be

$$H = \omega a^\dagger a + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k (a^\dagger b_k + b_k^\dagger a). \tag{4.130}$$

This is the simplest model possible of an open quantum system. Just so you can have an idea of where we want to arrive at, the dissipator that will come out of all of this is

$$D(\rho_S) = \gamma(\bar{n}+1)\Big[a\rho_S a^\dagger - \frac{1}{2}\{a^\dagger a, \rho_S\}\Big] + \gamma\bar{n}\Big[a^\dagger \rho_S a - \frac{1}{2}\{aa^\dagger, \rho_S\}\Big], \tag{4.131}$$

where $\gamma > 0$ and $\bar{n} = (e^{\beta\omega} - 1)^{-1}$ is the Bose-Einstein distribution with $\beta = 1/T$ and T being the bath temperature (I always set the Boltzmann constant to $k_B = 1$). I should

[10] A. A. Milne, *Winnie-the-Pooh*. New York, NY: Penguin Group, 1954.

also mention that this model can be solved exactly, which we will in fact do later on. Doing so is a nice way of checking the validity of all approximations we did in deriving the master equation.

Another quite similar model is that of a qubit interacting with a bath of oscillators. In this case the Hamiltonian is almost identical, we just replace a with σ_-:

$$H = \frac{\omega}{2}\sigma_z + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k(\sigma_+ b_k + b_k^\dagger \sigma_-). \tag{4.132}$$

Here we will work with Eq. (4.130) and I will leave it for you to repeat the steps for the Hamiltonian (4.132).

What we need to do is essentially apply Eq. (4.129). And this involves three steps:

1. Compute $V(t)$.
2. Compute the trace over E.
3. Compute the s integral.

For the first step, using Eq. (3.32) we get

$$V(t) = e^{iH_0 t} V e^{-iH_0 t} = \sum_k \lambda_k \left\{ e^{i\Delta_k t} a^\dagger b_k + e^{-i\Delta_k t} b_k^\dagger a \right\}, \tag{4.133}$$

where $\Delta_k = \omega - \Omega_k$ is the detuning between the system oscillator and each bath mode. Now we plug this into the first term in Eq. (4.129). This gives a messy combination of four term:

$$\begin{aligned} V(t)\rho_S\rho_E V(t-s) = \sum_{k,q} \lambda_k \lambda_q \Big\{ & e^{i(\Delta_k+\Delta_q)t} e^{-i\Delta_q s}\ a^\dagger b_k \rho_S \rho_E a^\dagger b_q \\ & + e^{i(\Delta_k-\Delta_q)t} e^{i\Delta_q s}\ a^\dagger b_k \rho_S \rho_E a b_q^\dagger \\ & + e^{-i(\Delta_k-\Delta_q)t} e^{-i\Delta_q s}\ a b_k^\dagger \rho_S \rho_E a^\dagger b_q \\ & + e^{-i(\Delta_k+\Delta_q)t} e^{i\Delta_q s}\ a b_k^\dagger \rho_S \rho_E a b_q^\dagger \Big\}. \end{aligned}$$

This is still just the first term Eq. (4.129). However, once we learn how to deal with it, it will be easy to repeat the procedure for the other terms.

Next we take the trace over the environment. This is a nice exercise on how to move things around. For instance,

$$\mathrm{tr}_E \left\{ a^\dagger b_k \rho_S \rho_E a b_q^\dagger \right\} = a^\dagger \rho_S a\, \mathrm{tr}_E \left\{ b_k \rho_E b_q^\dagger \right\} = a^\dagger \rho_S a \langle b_q^\dagger b_k \rangle.$$

Thus, we get

$$\begin{aligned}\mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} = \sum_{k,q}\lambda_k\lambda_q\Big\{&e^{i(\Delta_k+\Delta_q)t}e^{-i\Delta_q s}\; a^\dagger\rho_S a^\dagger\langle b_q b_k\rangle\\ &+e^{i(\Delta_k-\Delta_q)t}e^{i\Delta_q s}\; a^\dagger\rho_S a\langle b_q^\dagger b_k\rangle\\ &+e^{-i(\Delta_k-\Delta_q)t}e^{-i\Delta_q s}\; a\rho_S a^\dagger\langle b_q b_k^\dagger\rangle\\ &+e^{-i(\Delta_k+\Delta_q)t}e^{i\Delta_q s}\; a\rho_S a\langle b_q^\dagger b_k^\dagger\rangle\Big\}. \end{aligned} \tag{4.134}$$

So far nothing has been said about the initial state of the bath. But from now on some assumption must be made in order to compute the bath correlation functions $\langle b_q^\dagger b_k\rangle$ and $\langle b_q b_k\rangle$. If we assume that the bath is in the thermal Gibbs state (4.106), we get

$$\langle b_q b_k\rangle = 0, \tag{4.135}$$

$$\langle b_q^\dagger b_k\rangle = \delta_{q,k}\bar{n}(\Omega_k), \tag{4.136}$$

$$\langle b_k b_q^\dagger\rangle = \langle b_q^\dagger b_k\rangle + \delta_{k,q} = \delta_{k,q}(\bar{n}(\Omega_k)+1), \tag{4.137}$$

where

$$\bar{n}(x) = \frac{1}{e^{\beta x}-1}. \tag{4.138}$$

Plugging everything together we then get

$$\mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} = \sum_k \lambda_k^2\Big\{e^{i(\omega-\Omega_k)s}\;\bar{n}(\Omega_k)\; a^\dagger\rho_S a + e^{-i(\omega-\Omega_k)s}[\bar{n}(\Omega_k)+1]\; a\rho_S a^\dagger\Big\}. \tag{4.139}$$

Note how the Lindblad structure of Eq. (4.131) is starting to appear.

This is now a good point to introduce an important quantity known as the **spectral density of the bath**. It is defined as

$$J(\Omega) = 2\pi\sum_k \lambda_k^2\;\delta(\Omega-\Omega_k). \tag{4.140}$$

This is a function of a continuous variable Ω and, in general, this function will look like a series of delta-peaks whenever Ω equals one of the Ω_k. However, for a macroscopically large bath, the frequencies Ω_k will change smoothly with k and therefore $J(\Omega)$ will become a smooth function. In terms of the spectral density, Eq. (4.139) can be written as an integral

$$\mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} = \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi}\; J(\Omega)\Big\{e^{i(\omega-\Omega)s}\bar{n}(\Omega)a^\dagger\rho_S a + e^{-i(\omega-\Omega)s}(\bar{n}(\Omega)+1)a\rho_S a^\dagger\Big\}. \tag{4.141}$$

Here we are also implicitly assuming that the frequencies Ω_k vary smoothly between 0 and ∞ (they cannot be negative since a mode with negative frequency would be

unstable). It is really nice to notice that all properties of the system-bath interaction are summarized in the spectral density $J(\Omega)$. This means that the tiny details of λ_k do not matter. All that matter is their combined effect.

Finally, still with Eq. (4.129) in mind, we compute the integral in s:

$$\int_0^\infty \mathrm{d}s\, \mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} = \int_0^\infty \mathrm{d}s \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi}\, J(\Omega)\Big\{e^{i(\omega-\Omega)s}\bar{n}(\Omega)a^\dagger\rho_S a + e^{-i(\omega-\Omega)s}(\bar{n}(\Omega)+1)a\rho_S a^\dagger\Big\}. \tag{4.142}$$

To continue, the best thing to do is to take the s integral first. And, of course, what we would love to do is use the delta-function identity

$$\int_{-\infty}^\infty \frac{\mathrm{d}s}{2\pi}\, e^{i(\omega-\Omega)s} = \delta(\omega-\Omega). \tag{4.143}$$

But there is one tiny problem: in Eq. (4.142) the lower limit of integration is 0 instead of $-\infty$. The correct identity is then

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, e^{i(\omega-\Omega)s} = \frac{1}{2}\delta(\omega-\Omega) - \frac{i}{2}P\frac{1}{\omega-\Omega}, \tag{4.144}$$

where P denotes the Cauchy principal value. It can be shown that this last term only causes a tiny rescaling of the oscillator frequency ω called a **Lamb shift**. (i.e., it leads to a unitary contribution, instead of a dissipative one). Computing the Lamb shift is actually a difficult task. However, since in practice all it does is rescale ω, we can simply pretend it doesn't exist and focus only on the first term.

Neglecting the lamb-shift we then get

$$\begin{aligned}\int_0^\infty \mathrm{d}s\, \mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} &= \int_0^\infty \mathrm{d}\Omega\, \frac{J(\Omega)}{2}\delta(\omega-\Omega)\Big\{\bar{n}(\Omega)a^\dagger\rho_S a + (\bar{n}(\Omega)+1)a\rho_S a^\dagger\Big\} \\ &= \frac{J(\omega)}{2}\Big\{\bar{n}(\omega)a^\dagger\rho_S a + (\bar{n}(\omega)+1)a\rho_S a^\dagger\Big\}.\end{aligned} \tag{4.145}$$

We did it! We started all the way with Eq. (4.129) and we computed the first term!

Computing the second term is now quite easy: Going back to Eq. (4.134), we now compute

$$\begin{aligned}\mathrm{tr}_E\left\{\rho_S\rho_E V(t-s)V(t)\right\} = \sum_{k,q}\lambda_k\lambda_q\Big\{&e^{i(\Delta_k+\Delta_q)t}e^{-i\Delta_q s}\,\rho_S a^\dagger a^\dagger\langle b_q b_k\rangle \\ &+e^{i(\Delta_k-\Delta_q)t}e^{i\Delta_q s}\,aa^\dagger\rho_S\langle b_q^\dagger b_k\rangle \\ &+e^{-i(\Delta_k-\Delta_q)t}e^{-i\Delta_q s}\,\rho_S a^\dagger a\langle b_q b_k^\dagger\rangle \\ &+e^{-i(\Delta_k+\Delta_q)t}e^{i\Delta_q s}\,\rho_S aa\langle b_q^\dagger b_k^\dagger\rangle\Big\}.\end{aligned} \tag{4.146}$$

And we are done! If you just think about this for a second you will notice that this has the *exact* same structure as (4.134), except that the order of the operators is exchanged. For instance, in the second line of (4.134) we had $a^\dagger \rho_S a^\dagger$. Now we have $\rho_S a a^\dagger$.

The entire procedure we did for the first term is now repeated identically for this new guy, so in the end we will arrive at the same form as (4.145), but with the operators exchanged:

$$\mathrm{tr}_E\left\{\rho_S \rho_E V(t-s)V(t)\right\} = \frac{J(\omega)}{2}\Big\{\bar{n}(\omega)\rho_S a a^\dagger + (\bar{n}(\omega)+1)\rho_S a^\dagger a\Big\}. \tag{4.147}$$

For simplicity, I will now change notations to

$$J(\omega) := \gamma, \qquad \bar{n}(\omega) := \bar{n} = \frac{1}{e^{\beta\omega}-1}. \tag{4.148}$$

Then, combining (4.145) and (4.147) into Eq. (4.129) we get

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = \frac{\gamma}{2}\bar{n}\Big\{a^\dagger \rho_S a - a a^\dagger \rho_S\Big\} + \frac{\gamma}{2}(\bar{n}+1)\Big\{a\rho_S a^\dagger - a^\dagger a \rho_S\Big\} + \text{h.c.}.$$

We cannot forget the h.c. Plugging it back, we then finally get

$$\boxed{\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = \gamma\bar{n}\Big[a^\dagger \rho_S a - \frac{1}{2}\{aa^\dagger, \rho_S\}\Big] + \gamma(\bar{n}+1)\Big[a\rho_S a^\dagger - \frac{1}{2}\{a^\dagger a, \rho_S\}\Big],} \tag{4.149}$$

which is Eq. (4.131) and we are done.

What is cool about this is that not only can we derive a Lindblad master equation from a microscopic theory of system-environment interactions, but we can also attribute a clear physical meaning to the damping rate γ: it is simply given by the spectral density $J(\Omega)$ evaluated at the system's frequency ω. The spectral density is defined in Eq. (4.140) and represents the intensity of the S-E interaction λ_k^2. The damping rate γ is therefore found to be related to the coupling strength between the system and those oscillators whose frequency Ω_k are in the vicinity of ω; i.e., this is a resonant effect.

We should also never forget that we are working in the interaction picture so that ρ_S in Eq. (4.149) is actually $\tilde{\rho}_S$, which was defined as

$$\tilde{\rho}_S(t) = e^{iH_S t}\rho_S e^{-iH_S t}.$$

Going back turns out to be quite easy however because the dissipative part (4.149) is invariant under the rotating frame transformation. The reason is that $a \to a e^{i\omega t}$ and $a^\dagger \to a^\dagger e^{-i\omega t}$. But since a and $a^\dagger$ always appear in pairs in the dissipator, these exponentials cancel. For this reason, going back to the Schrödinger picture amounts to simply reintroducing the system Hamiltonian H_S:

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -i[H_S, \rho] + D(\rho). \tag{4.150}$$

And voilá, we are done.

Non-RWA interaction

Before we move on to another model, I want to quickly explore what happens if we use, instead of the initial Hamiltonian (4.130), a Hamiltonian of the form

$$H = \omega a^\dagger a + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k (a^\dagger + a)(b_k^\dagger + b_k). \tag{4.151}$$

The difference is only in the last term. This is the same idea as in the Jaynes-Cummings vs. Rabi discussion. Such a term no longer preserves the number of quanta in the system, making everything more complicated. But we study it because it appears often in nature. One context in which it appears is in the dissipative properties of mechanical systems.

Consider a mechanical harmonic oscillator with coordinate and momenta q and p, coupled to an infinite number of harmonic oscillators modeled by coordinate and momenta Q_k and P_k. The total Hamiltonian may look like

$$H = \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right) + \left(\sum_k \frac{P_k^2}{2M_k} + \frac{1}{2}M_k\Omega_k^2 Q_k^2\right) + \frac{1}{2}\sum_k c_k (q - Q_k)^2. \tag{4.152}$$

This is the typical *harmonic coupling*: the interaction potential depends on the relative displacement $(q - Q_k)^2$. If we now expand the square in the last term we get $q^2 + Q_k^2 - 2qQ_k$. The first two terms can actually be reabsorbed into renormalized system and bath frequencies:

$$\frac{1}{2}m\omega^2 q^2 + \frac{1}{2}\sum_k c_k q^2 := \frac{1}{2}m\tilde{\omega}^2 q^2,$$

$$\frac{1}{2}M_k\Omega_k^2 Q_k^2 + \frac{c_k}{2}Q_k^2 := \frac{1}{2}M_k\tilde{\Omega}_k Q_k^2$$

We now define creation and annihilation operators for the system and bath as

$$q = \sqrt{\frac{\hbar}{m\tilde{\omega}}}(a + a^\dagger), \qquad Q_k = \sqrt{\frac{\hbar}{M_k\tilde{\Omega}_k}}(b_k + b_k^\dagger).$$

With these choices (and similar definitions for p and P_k) the mechanical Hamiltonian (4.152) becomes

$$H = \tilde{\omega} a^\dagger a + \sum_k \tilde{\Omega}_k b_k^\dagger b_k - \sum_k \frac{\hbar c_k}{\sqrt{m\tilde{\omega} M_k \tilde{\Omega}_k}}(a + a^\dagger)(b_k + b_k^\dagger), \tag{4.153}$$

which is of the form of Eq. (4.151) with a certain redefinition of the λ_k.

To analyze a Hamiltonian such as (4.151), the recipe we developed may not be the most ideal. It all depends on whether the coupling strength λ_k are much smaller than the frequency ω or not. If they are, then we will see that a rotating-wave approximation will give us the same Lindblad equation as in (4.149). This is usually the case in quantum optical systems. But it is often violated in the case of mechanical vibrations, for which a more detailed theory must be developed. This theory is usually called

Quantum Brownian Motion. Two papers which I particularly like about QBM are `arXiv 0904.0950` (theory) and `arXiv 1305.6942` (experiment).

Let's then see what happens if we apply our recipe to the Hamiltonian (4.151). First we compute the interaction picture potential

$$\begin{aligned} V(t) &= \sum_k \lambda_k (a^\dagger e^{i\omega t} + a e^{-i\omega t})(b_k^\dagger e^{i\Omega_k t} + b_k e^{-i\Omega_k t}) \\ &= \sum_k \lambda_k \Big\{ a^\dagger b_k e^{i\Delta_k t} + a b_k^\dagger e^{-i\Delta_k t} + a^\dagger b_k^\dagger e^{i(\omega+\Omega_k)t} + a b_k e^{-i(\omega+\Omega_k)t} \Big\}. \end{aligned}$$

Next we use this to compute terms such as $V(t)\rho_S\rho_E V(t-s)$. As you can probably see, this will become quite a mess. I will spare you the sordid details. All I want to focus on is the following result:

$$\begin{aligned} \int_0^\infty \mathrm{d}s\, \mathrm{tr}_E \Big\{ V(t)\rho_S(t)\rho_E V(t-s) \Big\} = \frac{J(\omega)}{2} \Big\{ &\bar{n}(\omega) a^\dagger \rho_S a + (\bar{n}(\omega)+1) a \rho_S a^\dagger \\ &+ \bar{n}(\omega) a \rho_S a\, e^{-2i\omega t} + (\bar{n}(\omega)+1) a^\dagger \rho_S a^\dagger\, e^{2i\omega t} \Big\}. \end{aligned}$$

The first line is exactly what we had before, Eq. (4.145). But now we see the appearance of two new terms in the second line. These terms are fundamentally different in the sense that they are rapidly oscillatory terms, even thought we are in the interaction picture, where things should oscillate less. Moreover, we see that this is not yet in Lindblad's form. This is where the rotating-wave approximation applies. If $\omega \gg \gamma J(\omega)$ then these new terms will induce fast oscillations on top of a slowly changing evolution. In this case it is therefore reasonable to neglect them. Otherwise, one must resort to quantum Brownian motion.

4.6 The spin-boson model

The name spin-boson refers to a single qubit interacting with an infinite number of bosonic modes. Just like with the British royal family, when we talk about this model, there is a good boy and a bad boy. They are:

$$H = \frac{\omega}{2}\sigma_z + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k\, \sigma_z (b_k + b_k^\dagger), \qquad \text{(good boy)}, \tag{4.154}$$

$$H = \frac{\omega}{2}\sigma_z + \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k\, \sigma_x (b_k + b_k^\dagger), \qquad \text{(bad boy)}. \tag{4.155}$$

The fundamental difference between the two models is that in the first the operator appearing in the S-E interaction (σ_z) is the same as the operator in H_S. Consequently, the model (4.154) cannot generate transitions between energy levels (population changes) and, consequently, the most that can happen is environment-induced decoherence. In Eq. (4.155), on the other hand, the operator σ_x is the spin flip and therefore causes

population changes. Consequently, it will give rise to an amplitude damping-type of dynamics.

In this section we will talk about the good-boy spin-boson model, Eq. (4.154). First we will try to find a master equation for the qubit, by simply plugging it into our recipe, Eq. (4.129). Then we will discuss an *exact* solution of it, which as we will see, has a much richer physics.

Approximate derivation of the master equation

In this section we follow once again the steps in our recipe (4.129). In the interaction picture the S-E interaction of Eq. (4.154) becomes

$$V(t) = \sum_k \lambda_k \, \sigma_z (b_k e^{-i\Omega_k t} + b_k^\dagger e^{i\Omega_k t}). \tag{4.156}$$

Then

$$V(t)\rho_S\rho_E V(t-s) = \sum_{k,q} \lambda_k\lambda_q (\sigma_z\rho_S\sigma_z)(b_k e^{-i\Omega_k t} + b_k^\dagger e^{i\Omega_k t})\rho_E (b_q e^{-i\Omega_q (t-s)} + b_q^\dagger e^{i\Omega_q (t-s)}),$$

where I already moved things around a bit in order to separate system operators and bath operators. Next:

$$\begin{aligned}\mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} &= \sum_k \lambda_k^2 \, \sigma_z\rho_S\sigma_z\Big\{\bar{n}(\Omega_k)e^{-i\Omega_k s} + (\bar{n}(\Omega_k)+1)e^{i\Omega_k s}\Big\} \\ &= \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi}\, J(\Omega)\, \sigma_z\rho_S\sigma_z\Big\{\bar{n}(\Omega)e^{-i\Omega s} + (\bar{n}(\Omega)+1)e^{i\Omega s}\Big\}.\end{aligned}$$

And finally,

$$\int_0^\infty \mathrm{d}s\, \mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} = \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi}\, J(\Omega)\, (2\bar{n}(\Omega)+1)\delta(\Omega)\, \sigma_z\rho_S\sigma_z.$$

We now reach a somewhat awkward place because the Dirac delta will push $J(\Omega)$ towards $J(0)$ and $\bar{n}(\Omega)$ towards $\bar{n}(0)$. But in this limit $\bar{n}(\Omega) = (e^{\beta\Omega}-1)^{-1}$ diverges, whereas $J(\Omega)$ tends to zero. This is one of those buggy features of trying to apply a general recipe for all types of open-system problems. Below when we compute the exact solution, no problem of this sort will appear.

If we want to continue, the best we can do is to assume that even though one diverges and the other tends to zero, their product tends to a finite value, which we define, just for convenience, as

$$\lim_{\Omega\to 0} J(\Omega)(2\bar{n}(\Omega)+1) := \frac{\lambda}{4} \tag{4.157}$$

Then we get

$$\int_0^\infty \mathrm{d}s\, \mathrm{tr}_E\left\{V(t)\rho_S\rho_E V(t-s)\right\} = \frac{\lambda}{4}\sigma_z\rho_S\sigma_z.$$

The other term in Eq. (4.129) is identical, except that the order of the operators changes to $\rho_S \sigma_z \sigma_z = \rho_S$. Thus we get, considering also the Hermitian conjugate (which in this case is the same as the operator)

$$D(\rho_S) = \frac{\lambda}{2}(\sigma_z \rho_S \sigma_z - \rho_S). \tag{4.158}$$

This is the dephasing dissipator that we studied all the way back in Eq. (2.22). There we saw that such a dissipator does not induce any changes in population, but only causes the coherence $q(t) = \langle 0|\rho_S(t)|1\rangle$ to decay exponentially as

$$q(t) = q(0)e^{-\lambda t}. \tag{4.159}$$

Exact solution

Now let's find the *exact* solution for $\rho_S(t)$. This is one of the few models for which exact solutions are available, so enjoy it! The starting point is von Neumann's equation (in the Schrödinger picture) for the total density matrix of S+E:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H, \rho], \tag{4.160}$$

where H is the total Hamiltonian (4.154). This is subject to the initial condition

$$\rho(0) = \rho_S(0)\rho_E(0), \qquad \rho_E(0) = \frac{e^{-\beta H_E}}{Z_E}. \tag{4.161}$$

However, now we are interested in exact dynamics so the bath will also evolve in time and the system and bath will become correlated (no Born and no Markov approximations).

The solution of Eq. (4.160) is

$$\rho(t) = e^{-iHt}\rho_S(0)\rho_E(0)e^{iHt}.$$

What we want is the partial trace over the environment

$$\rho_S(t) = \mathrm{tr}_E\left\{e^{-iHt}\rho_S(0)\rho_E(0)e^{iHt}\right\}.$$

Let us now divide the total Hamiltonian H as

$$H = H_S + H_0,$$

where $H_S = \frac{\omega}{2}\sigma_z$ and

$$H_0 = \sum_k \Omega_k b_k^\dagger b_k + \sum_k \lambda_k \sigma_z (b_k + b_k^\dagger)$$

The Hamiltonian H_S lives on the qubit space and therefore can be taken out of the partial trace:

$$\rho_S(t) = e^{-iH_S t}\,\mathrm{tr}_E\left\{e^{-iH_0 t}\rho_S(0)\rho_E(0)e^{iH_0 t}\right\}e^{iH_S t}.$$

In this way, we have separated the local unitary dynamics, described by H_S, to the dissipative dynamics described by everything inside the trace. In fact, if you think about it, this whole partial trace is a quantum operation in the spirit of Stinespring's theorem. So for now let us focus on this dissipative part defined by the map

$$\tilde{\rho}_S(t) = \mathrm{tr}_E\left\{e^{-iH_0 t}\rho_S(0)\rho_E(0)e^{iH_0 t}\right\}.$$

The easiest way to proceed from here is to actually look at the matrix elements of this map in the computational basis. The reason why this is useful is because H_0 is already diagonal in the qubit sector. In fact, we can define

$$H_0|0\rangle = H_0^+|0\rangle, \qquad H_0|1\rangle = H_0^-|1\rangle,$$

where

$$H_0^\pm = \sum_k \Omega_k b_k^\dagger b_k \pm \sum_k \lambda_k(b_k + b_k^\dagger)$$

We then have, for instance

$$\begin{aligned}
\langle 0|\tilde{\rho}_S(t)|0\rangle &= \langle 0|\,\mathrm{tr}_E\left\{e^{-iH_0 t}\rho_S(0)\rho_E(0)e^{iH_0 t}\right\}|0\rangle \\
&= \mathrm{tr}_E\left\{\langle 0|e^{-iH_0 t}\rho_S(0)\rho_E(0)e^{iH_0 t}|0\rangle\right\} \\
&= \mathrm{tr}_E\left\{e^{-iH_0^+ t}\langle 0|\rho_S(0)|0\rangle\rho_E(0)e^{iH_0^+ t}\right\} \\
&= \langle 0|\rho_S(0)|0\rangle\,\mathrm{tr}_E\left\{e^{-iH_0^+ t}\rho_E(0)e^{iH_0^+ t}\right\}
\end{aligned}$$

This set of steps is important and a bit confusing, so make sure you understand what I am doing. I push the system bra and ket $|0\rangle$ inside the partial trace. But then I know how H_0 acts on it. And after it has acted, H_0^+ will no longer have any components on the qubit space, so we can move $|0\rangle$ through it at will. Finally, when $\langle 0|$ and $|0\rangle$ encounter $\rho_S(0)$, they form a number, which can then be taken outside the partial trace.

But now comes the magic trick: $H_0^\pm$ is an operator that lives only on the environment's Hilbert space. Hence, we are now allowed to use the cyclic property of the trace. This is a useful trick to remember: *if an operator lives on a larger space, cyclic property is forbidden. But if acts only over the space you are tracing, then it becomes allowed again.* And if we do that the two exponentials cancel and we are left with

$$\langle 0|\tilde{\rho}_S(t)|0\rangle = \mathrm{tr}_E\left\{\langle 0|\rho_S(0)|0\rangle\rho_E(0)\right\} = \langle 0|\rho_S(0)|0\rangle. \tag{4.162}$$

Thus, as anticipated, we see that the action of the bath does not change the populations (diagonal elements) of ρ_S. A similar argument can of course be used for $\langle 1|\tilde{\rho}_S(t)|1\rangle$ but we don't need to do it because, if $\langle 0|\tilde{\rho}_S(t)|0\rangle$ doesn't change, then $\langle 1|\tilde{\rho}_S(t)|1\rangle$ cannot change also due to normalization.

Next we look at the off-diagonal element

$$\langle 0|\tilde{\rho}_S(t)|1\rangle = \langle 0|\rho_S(0)|1\rangle\,\mathrm{tr}_E\left\{e^{-iH_0^+ t}\rho_E(0)e^{iH_0^- t}\right\}. \tag{4.163}$$

We see now that the exponentials do not cancel, so the result of the trace will not be just $\mathrm{tr}_E\,\rho_E(0) = 1$. In fact, motivated by Eq. (4.159), let us define a general dephasing rate as

$$\boxed{e^{-\Lambda(t)} = \mathrm{tr}_E\left\{e^{-iH_0^+ t}\rho_E(0)e^{iH_0^- t}\right\}.} \tag{4.164}$$

Then Eq. (4.163) acquires the more familiar form

$$\langle 0|\tilde{\rho}_S(t)|1\rangle = \langle 0|\rho_S(0)|1\rangle e^{-\Lambda(t)t}. \tag{4.165}$$

Our task has now been reduced to the calculation of the decoherence rate $\Lambda(t)$.

Explicit calculation of the decoherence rate

To compute the trace in Eq. (4.164) we being by noticing that the calculation factors into a product of traces, one for each mode of the environment:

$$\begin{aligned} e^{-\Lambda(t)} &= \prod_k \mathrm{tr}\left\{e^{-it\left[\Omega_k b_k^\dagger b_k + \lambda_k(b_k + b_k^\dagger)\right]}\rho_k e^{it\left[\Omega_k b_k^\dagger b_k - \lambda_k(b_k + b_k^\dagger)\right]}\right\} \\ &= \langle e^{it\left[\Omega_k b_k^\dagger b_k - \lambda_k(b_k + b_k^\dagger)\right]}e^{-it\left[\Omega_k b_k^\dagger b_k + \lambda_k(b_k + b_k^\dagger)\right]}\rangle. \end{aligned}$$

where ρ_k is the initial state of mode k of the environment. If we assume the environment is in a thermal state then

$$\rho_k = (1 - e^{-\beta\Omega_k})e^{-\beta\Omega_k b_k^\dagger b_k}.$$

Since the calculations for all modes are equivalent, let us clean up the notation a bit and focus on the quantity

$$B = \langle e^{it\left[\Omega b^\dagger b - \lambda(b + b^\dagger)\right]}e^{-it\left[\Omega b^\dagger b + \lambda(b + b^\dagger)\right]}\rangle \tag{4.166}$$

Computing this is a good exercise on operator algebra.

We will need to recall some definitions of displacement operators $D(\alpha) = e^{\alpha b^\dagger - \alpha^* b}$, discussed in Sec. 3.4. Recall that $D^\dagger(\alpha)bD(\alpha) = b + \alpha$. We can then use this to write

$$\Omega b^\dagger b \pm \lambda(b + b^\dagger) = \Omega\, D^\dagger(\pm\lambda/\Omega)(b^\dagger b)D(\pm\lambda/\Omega) - \frac{\lambda^2}{\Omega}. \tag{4.167}$$

But the displacement operator is unitary, so it can enter or leave exponentials at will. Consequently

$$e^{-it\left[\Omega b^\dagger b + \lambda(b + b^\dagger)\right]} = e^{-it\lambda^2/\Omega}D^\dagger(\lambda/\Omega)e^{-i\Omega t b^\dagger b}D(\lambda/\Omega),$$

with a similar result for the other exponential. Eq. (4.166) then becomes

$$\begin{aligned} B &= \langle D^\dagger(-\lambda/\Omega)e^{i\Omega t b^\dagger b}D(-\lambda/\Omega)D^\dagger(\lambda/\Omega)e^{-i\Omega t b^\dagger b}D(\lambda/\Omega)\rangle \\ &= \langle D(\lambda/\Omega)e^{i\Omega t b^\dagger b}D^\dagger(2\lambda/\Omega)e^{-i\Omega t b^\dagger b}D(\lambda/\Omega)\rangle, \end{aligned}$$

where I used the fact that $D(-\alpha) = D^\dagger(\alpha)$ and that $D(\alpha)D(\alpha) = D(2\alpha)$ (all these properties are described in Sec. 3.4).

In the middle term we infiltrate the exponential inside $D^\dagger(2\lambda/\Omega)$:

$$\begin{aligned} e^{i\Omega t b^\dagger b} D^\dagger(2\lambda/\Omega) e^{-i\Omega t b^\dagger b} &= \exp\Big\{ -\frac{2\lambda}{\Omega} e^{i\Omega t b^\dagger b}(b^\dagger - b) e^{-i\Omega t b^\dagger b}\Big\} \\ &= \exp\Big\{ -\frac{2\lambda}{\Omega}(b^\dagger e^{i\Omega t} - b e^{-i\Omega t})\Big\} \\ &= D^\dagger(2\lambda e^{i\Omega t}/\Omega) \end{aligned}$$

We then arrive at the simpler result:

$$B = \langle D(\lambda/\Omega) D^\dagger(2\lambda e^{i\Omega t}/\Omega) D(\lambda/\Omega)\rangle$$

Finally, we combine the three displacement operators using $D(\alpha)D(\beta) = e^{(\beta^*\alpha - \alpha^*\beta)/2} D(\alpha+\beta)$ [c.f. Eq. (3.85)]. We then finally arrive at

$$\boxed{B = \langle D(\alpha_t)\rangle, \qquad \alpha_t := \frac{2\lambda}{\Omega}(1 - e^{i\Omega t}).} \tag{4.168}$$

This result is somewhat general since it holds for an arbitrary bath initial state.

Next let us specialize it for the case of a thermal state. In this case, it turns out that the trace of a displacement operator in a thermal state is:[11]

$$\boxed{B = \langle D(\alpha_t)\rangle = \exp\Big\{ -|\alpha_t|^2(\bar{n} + 1/2)\Big\}, \qquad \text{when } \rho = (1 - e^{-\beta\Omega}) e^{-\beta\Omega b^\dagger b},} \tag{4.169}$$

where $\bar{n} = (e^{\beta\Omega} - 1)^{-1}$.

Analysis of the decoherence rate

Going back now to Eq. (4.164) and reintroducing an index k everywhere, we get

$$e^{-\Lambda(t)} = \prod_k \exp\Big\{ -|\alpha_{k,t}|^2(\bar{n}(\Omega_k) + 1/2)\Big\}.$$

[11] One way to derive this result is to write it as

$$\langle D(\alpha)\rangle = (1 - e^{-\beta\Omega}) \sum_{n=0}^{\infty} e^{-\beta\Omega n} \langle n|D(\alpha)|n\rangle.$$

In K. Cahill and R. Glauber in *Phys. Rev.* **177**, 1857-1881 (1969), they show that $\langle n|D(\alpha)|n\rangle = e^{-|\alpha|^2/2} L_n(|\alpha|^2)$ where $L_n(x)$ are the Laguerre polynomials. The sum in n may then be related to the generating function of Laguerre polynomials:

$$\sum_{n=0}^{\infty} x^n L_n(y) = \frac{1}{1-x} \exp\Big\{ -\frac{yx}{1-x}\Big\}.$$

Using this yields, after some simplifications, the result in Eq. (4.169).