Chapter 5

Applications of open quantum systems

5.1 A crash course on Gaussian systems

In Sec. 3.5 we introduced the Husimi-Q function $Q(\alpha, \alpha^*) = \langle\alpha|\rho|\alpha\rangle/\pi$, describing the quantum phase space of continuous variable systems. In that context, a quantum state is called **Gaussian** if its Husimi-Q function is a Gaussian function of the coherent state variables. For instance, the thermal state of a bosonic mode [Eq. (3.111)]

$$Q(\alpha^*, \alpha) = \frac{1}{\pi(\bar{n}+1)} \exp\left\{-\frac{|\alpha|^2}{\bar{n}+1}\right\},$$

is Gaussian. On the other hand,the Schrödinger cat state [Eq. (3.108)]

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} e^{-|\alpha-\mu|^2}\left\{1 + \frac{e^{-2\mu^*\alpha} + e^{-2\mu\alpha^*}}{2}\right\},$$

is not.

A **Gaussian preserving map**, on the other hand, is a map that takes Gaussian states to Gaussian states. For a system of an arbitrary number of bosonic modes a_i, the most general such map corresponds to:

- The Hamiltonian being at most *quadratic* in the a_i and $a_i^\dagger$. Thus, the most general Gaussian preserving Hamiltonian has the form

  $$H = \sum_{i,j}\left\{A_{ij}a_i^\dagger a_j + \frac{1}{2}(B_{ij}a_i^\dagger a_j^\dagger + B_{ij}^* a_i a_j)\right\} + \sum_i (f_i a_i^\dagger + f_j a_j^\dagger),$$

  where A_{ij}, B_{ij} and f_i are coefficients (the factor of 1/2 is placed only for convenience). In order for H to be Hermitian we must have $A^\dagger = A$ and $B^{\mathrm{T}} = B$.

- The Lindblad generators L_α of a master equation being at most *linear* in the a_i and $a_i^\dagger$. Thus, the thermal bath generator

$$D(\rho) = \gamma\bar{n}\Big[a^\dagger\rho a - \frac{1}{2}\{aa^\dagger,\rho\}\Big] + \gamma(\bar{n}+1)\Big[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a,\rho\}\Big],$$

  is Gaussian preserving, whereas the bosonic dephasing model

$$D(\rho) = \lambda\Big[a^\dagger a\rho a^\dagger a - \frac{1}{2}\{(a^\dagger a)^2,\rho\}\Big],$$

  is not.

Gaussian states and Gaussian preserving maps are extremely useful since they simplify dramatically a potentially unsolvable problem. When dealing with continuous variables it is common to encounter models that have no analytical solutions. For instance, adding a term such as $a^\dagger a^\dagger aa$ to a Hamiltonian usually makes it unsolvable. If your problem involves only a single bosonic mode, than you can probably still deal with it numerically. But if you have a multi-mode system with these kinds of terms, then not even numerics will save you. Gaussian maps, on the other hand, can *always* be dealt with analytically, irrespective of the number of modes we have.

The reason is that for a Gaussian map the equations for the first and second moments are closed. By first moments I mean averages such as $\langle a_i\rangle$, whereas by second moments I mean covariances such as $\langle a_i^\dagger a_j\rangle - \langle a_i^\dagger\rangle\langle a_j\rangle$. In the non-Gaussian scenario, the equation for these guys will depend also on higher order moments, leading to an infinite hierarchy of coupled equations. But for Gaussian maps the equations are closed so that first moments only depend on first moments and second moments only depend on second moments. In the same spirit, just like in classical probability theory, a Gaussian state is fully determined by the first and second moments. So we don't ever need to work with ρ directly; it suffices to work with the moments. We therefore reduce the problem of dealing with an infinite dimensional Hilbert space, to that of only a few expectation values.

Gaussian systems play an important role in quantum information. One of the reasons is that many physical implementations involving quantum optics, mechanical vibrations and even collective atomic excitations, can be described in terms of Gaussian states. Unfortunately, however, Gaussian states cannot be used for universal quantum computing: even though most basic circuit operations can be implemented using Gaussian gates, for some operations non-Gaussian gates are necessary.

If you are interested in a more detailed source of information, I recommend the excellent book by Alessio Serafini entitled "Quantum Continuous Variables".

Algebraic structure

Consider a system of N bosonic modes $a_1,\ldots,a_N$ satisfying the usual algebra

$$[a_i, a_j^\dagger] = \delta_{i,j}, \qquad [a_i, a_j] = 0. \tag{5.1}$$

Alternatively, we may prefer to work with *quadratures*

$$q_i = \frac{1}{\sqrt{2}}(a_i^\dagger + a_i), \qquad p_i = \frac{i}{\sqrt{2}}(a_i^\dagger - a_i). \tag{5.2}$$

These are then Hermitian and satisfy the algebra

$$[q_i, p_j] = i\delta_{i,j}, \qquad [q_i, q_j] = [p_i, p_j] = 0. \tag{5.3}$$

Next define a vector of operators

$$\boldsymbol{X} = (a_1, a_1^\dagger, \ldots, a_N, a_N^\dagger), \qquad \boldsymbol{Y} = (q_1, p_1, \ldots, q_N, p_N). \tag{5.4}$$

In terms of these vectors, the algebras (5.1) and (5.3) become:

$$[X_i, X_j^\dagger] = \Sigma_{i,j}, \qquad [Y_i, Y_j] = i\Omega_{i,j}, \tag{5.5}$$

where Σ and Ω are called **symplectic forms** and are defined as

$$\Sigma = \bigoplus_{i=1}^{N} \sigma_z, \qquad \Omega = \bigoplus_{i=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \bigoplus_{i=1}^{N} (i\sigma_y). \tag{5.6}$$

The symbol $\oplus$ here means *direct sum* and stands for the block-wise composition of matrices. For instance, if we have $N = 2$ then the matrix Ω would read

$$\Omega = (i\sigma_y) \oplus (i\sigma_y) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which is just two blocks joined together.

Eq. (5.5) is somewhat important because it establishes the algebra of the group of operators. All other properties follow from this algebra. It turns out that there is a deep connection between this algebraic structure and the so-called **symplectic group** in classical mechanics. If you are interested in this topic, I recommend the papers by R. Simon (e.g. `arXiv:quant-ph/9509002v3`).

I should also mention that the two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are connected by

$$\boldsymbol{Y} = \Lambda \boldsymbol{X}, \qquad \Lambda = \bigoplus_{i=1}^{N} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}, \tag{5.7}$$

which is simply a different way of writing the linear transformation (5.3). The matrix Λ is unitary so the inverse transformation is simply

$$\boldsymbol{X} = \Lambda^\dagger \boldsymbol{Y}, \qquad \Lambda^\dagger = \bigoplus_{i=1}^{N} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}.$$

Covariance matrix

Given now the vector of operators, either $\boldsymbol{X}$ or $\boldsymbol{Y}$, we define their first moments as simply $x_i = \langle X_i \rangle$ and $y_i \langle Y_i \rangle$, which we shall sometimes also group to form a vector $\boldsymbol{x}$

(or $\boldsymbol{y}$). More interestingly, we define the **covariance matrix** (CM) as[1]

$$\Theta = \frac{1}{2}\langle\{X_i, X_j^\dagger\}\rangle - \langle X_i\rangle\langle X_j^\dagger\rangle = \frac{1}{2}\langle\{\delta X_i, \delta X_j^\dagger\}\rangle, \tag{5.8}$$

$$\sigma = \frac{1}{2}\langle\{Y_i, Y_j\}\rangle - \langle Y_i\rangle\langle Y_j\rangle = \frac{1}{2}\langle\{\delta Y_i, \delta Y_j\}\rangle. \tag{5.9}$$

In the second equality on each line I defined an operator $\delta X_i = X_i - \langle X_i\rangle$. This is the fluctuation operator, which means only the quantum fluctuations around the average value. I will leave for you as an exercise to check that the definition using δX_i coincides with the other one.

The covariance matrices are constructed in this way in order to have nice properties. In particular, we always use the symmetrized version $\{X_i, X_j^\dagger\} = X_i X_j^\dagger + X_j^\dagger X_i$. Consequently, by construction we have that Θ is Hermitian, $\Theta^\dagger = \Theta$, whereas σ is real and symmetric, $\sigma^{\mathrm{T}} = \sigma$. For example, if $N = 1$ we have

$$\Theta = \begin{pmatrix} \langle \delta a^\dagger \delta a\rangle + 1/2 & \langle \delta a \delta a\rangle \\ \langle \delta a^\dagger \delta a^\dagger\rangle & \langle \delta a^\dagger \delta a\rangle + 1/2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \langle \delta q^2\rangle & \frac{1}{2}\langle\{\delta q, \delta p\}\rangle \\ \frac{1}{2}\langle\{\delta q, \delta p\}\rangle & \langle \delta p^2\rangle \end{pmatrix}.$$

Let me also show you how they look like for $N = 2$. In this case, in order to make things clearer I will assume $\langle a_i\rangle = 0$ so that we don't need to distinguish between δa_i and a_i. But please remember that in general there should be δ's everywhere. For $N = 2$ the CMs look like:

$$\Theta = \begin{pmatrix} \langle a_1^\dagger a_1\rangle + 1/2 & \langle a_1 a_1\rangle & \langle a_1 a_2^\dagger\rangle & \langle a_1 a_2\rangle \\ \langle a_1^\dagger a_1^\dagger\rangle & \langle a_1^\dagger a_1\rangle + 1/2 & \langle a_1^\dagger a_2^\dagger\rangle & \langle a_1^\dagger a_2\rangle \\ \langle a_1^\dagger a_2\rangle & \langle a_1 a_2\rangle & \langle a_2^\dagger a_2\rangle + 1/2 & \langle a_2 a_2\rangle \\ \langle a_1^\dagger a_2^\dagger\rangle & \langle a_1 a_2^\dagger\rangle & \langle a_2^\dagger a_2^\dagger\rangle & \langle a_2^\dagger a_2\rangle + 1/2 \end{pmatrix},$$

and

$$\sigma = \begin{pmatrix} \langle q_1^2\rangle & \frac{1}{2}\langle\{q_1, p_1\}\rangle & \langle q_1 q_2\rangle & \langle q_1 p_2\rangle \\ \frac{1}{2}\langle\{q_1, p_1\}\rangle & \langle p_1^2\rangle & \langle p_1 q_2\rangle & \langle p_1 p_2\rangle \\ \langle q_2 q_1\rangle & \langle q_2 p_1\rangle & \langle q_2^2\rangle & \frac{1}{2}\langle\{q_2, p_2\}\rangle \\ \langle p_2 q_1\rangle & \langle p_2 p_1\rangle & \frac{1}{2}\langle\{q_2, p_2\}\rangle & \langle p_2^2\rangle \end{pmatrix}$$

Notice how the matrix is structured in blocks. The diagonal parts represent the CMs of modes 1 and 2, whereas the off-diagonal blocks represent their *correlations*. From one CM we can obtain the other using the same transformation as Eq. (5.7). Namely,

$$\sigma = \Lambda \Theta \Lambda^\dagger. \tag{5.10}$$

[1] Some authors define σ with a 2 in front, so that their covariance matrix is twice as ours. Please be careful!.

For instance, if $N = 1$ this is essentially a compact form of writing

$$\begin{aligned} q^2 &= \frac{1}{2}(2a^\dagger a + 1 + aa + a^\dagger a^\dagger), \\ p^2 &= \frac{1}{2}(2a^\dagger a + 1 - aa - a^\dagger a^\dagger), \\ \frac{1}{2}\{q, p\} &= \frac{i}{2}(a^\dagger a^\dagger - aa) \end{aligned} \tag{5.11}$$

Generalized uncertainty relations

Consider the operator

$$Z = \sum_{i=1}^{N} \delta X_i \, z_i,$$

where z_i are arbitrary complex numbers. It then follows by construction that $\langle ZZ^\dagger \rangle \geq 0$ since $ZZ^\dagger$ is a positive semi-definite operator. However, we also have that

$$\langle ZZ^\dagger \rangle = \sum_{i,j} z_i z_j^* \langle \delta X_i \delta X_j^\dagger \rangle.$$

But using the general algebraic structure in Eq. (5.5), which also holds for the fluctuation operators δX_i, we get

$$\{\delta X_i, \delta X_j^\dagger\} = 2\delta X_i \delta X_j^\dagger - \Sigma_{i,j}.$$

Thus,

$$\langle ZZ^\dagger \rangle = \sum_{i,j} z_i z_j^* \Big[\Theta_{i,j} + \Sigma_{i,j}/2 \Big] \geq 0.$$

This sum is now a quadratic form with respect to the matrix $\Theta + \Sigma/2$. It is a general theorem in linear algebra that the condition for a quadratic form to be non-negative, for any choice of numbers z_i, is that the matrix in question must be positive semi-definite. Hence, we conclude that the covariance matrix must satisfy what is usually called a *bona fide* (in good faith) relation:

$$\boxed{\Theta + \frac{\Sigma}{2} \geq 0, \qquad \sigma + \frac{i\Omega}{2} \geq 0.} \tag{5.12}$$

Here I also included the same result for σ, which is obtained by simply applying Eq. (5.10) to the first equation.

Eq. (5.12) is actually a stronger statement, or a type of generalization, of Heisenberg's uncertainty relation. To see that, take as an example a single mode. Then

$$\sigma + \frac{i\Omega}{2} = \begin{pmatrix} \langle \delta q^2 \rangle & \frac{1}{2}\langle \{\delta q, \delta p\} \rangle + i/2 \\ \frac{1}{2}\langle \{\delta q, \delta p\} \rangle - i/2 & \langle \delta p^2 \rangle \end{pmatrix}.$$

For this matrix to be positive semi-definite both of its eigenvalues must be non-negative. Or, what is equivalent, both its trace and determinant must be non-negative. The trace is clearly non-negative. As for the determinant, we get

$$|\sigma + \frac{i\Omega}{2}| = \langle \delta q^2 \rangle \langle \delta p^2 \rangle - \frac{1}{4} - \frac{1}{4}\langle \{\delta q, \delta p\} \rangle^2 \geq 0$$

This therefore leads to

$$\boxed{\langle \delta q^2 \rangle \langle \delta p^2 \rangle \geq \frac{1}{4} + \frac{1}{4}\langle \{\delta q, \delta p\} \rangle^2,} \tag{5.13}$$

In the literature this is usually called the **Robertson-Schrödinger uncertainty relation**. Note how it is stronger than the usual Heisenberg relation, which is contained only in the first term.

Another way in which I like to think about Eq. (5.12) is in comparison to classical probability theory. In this case, the condition on the covariance matrix of a classical Gaussian distribution is simply $\sigma \geq 0$ or $\Theta \geq 0$. Thus, a term like $\Sigma/2$ in Eq. (5.12) represents a *quantum correction*, which imposes a stronger bound due to quantum fluctuations. In fact, the uncertainty bound is found for the vacuum state, for which

$$\Theta = \sigma = \frac{\mathbb{I}_2}{2}.$$

Thus, we see that *the covariance matrix is never zero*. Even in the vacuum some fluctuations remain. That is in stark contrast with classical probability theory where zero fluctuations are perfectly allowed (the variables are then deterministic). On the other hand, if your fluctuations are really really large than the extra terms in Eq. (5.12) don't really matter so that $\Theta + \Sigma/2 \geq 0$ is practically the same as $\Theta \geq 0$.

Example: single-mode squeezing

To give a non-trivial example, consider a single-mode system prepared in the **squeezed thermal state**:

$$\rho = S_z \frac{e^{-\beta\omega a^\dagger a}}{Z} S_z^\dagger, \qquad S_z = \exp\left\{\frac{1}{2}(z a^\dagger a^\dagger - z^* a a)\right\}, \tag{5.14}$$

where $Z = (1 - e^{-\beta\omega})$ and z is a complex number that we parametrize as $z = re^{i\theta}$. This contemplates, as particular cases, the thermal state ($z = 0$) and the **squeezed vacuum**,

$$\rho = S_z |0\rangle\langle 0| S_z^\dagger,$$

which is obtained by taking $T = (1/\beta) \to 0$. In the squeezed thermal state the first moments are zero, whereas the second moments are given by

$$\Theta = \begin{pmatrix} \langle \delta a^\dagger \delta a \rangle + 1/2 & \langle \delta a \delta a \rangle \\ \langle \delta a^\dagger \delta a^\dagger \rangle & \langle \delta a^\dagger \delta a \rangle + 1/2 \end{pmatrix} = \begin{pmatrix} (\bar{n} + 1/2)\cosh(2r) & (\bar{n} + 1/2)e^{i\theta}\sinh(2r) \\ (\bar{n} + 1/2)e^{-i\theta}\sinh(2r) & (\bar{n} + 1/2)\cosh(2r) \end{pmatrix} \tag{5.15}$$

where $\bar{n} = (e^{\beta\omega} - 1)^{-1}$ is the Bose-Einstein thermal occupation.

In terms of the quadratures, using Eq. (5.11) we get

$$\langle \delta q^2 \rangle = (\bar{n} + 1/2)\Big[\cosh(2r) + \sinh(2r)\cos(\theta)\Big],$$

$$\langle \delta p^2 \rangle = (\bar{n} + 1/2)\Big[\cosh(2r) - \sinh(2r)\cos(\theta)\Big],$$

$$\frac{1}{2}\langle\{\delta q, \delta p\}\rangle = (\bar{n} + 1/2)\sinh(2r)\sin(\theta).$$

From these results it becomes easier to understand the physical meaning of $\bar{n}$, r and θ. First, suppose that $\theta = 0$. Then these simplify to

$$\langle \delta q^2 \rangle = (\bar{n} + 1/2)e^{2r},$$

$$\langle \delta p^2 \rangle = (\bar{n} + 1/2)e^{-2r},$$

$$\frac{1}{2}\langle\{\delta q, \delta p\}\rangle = 0.$$

Thus, $\bar{n}$ gives the overall width of the position and momentum fluctuations, whereas r (as the name already implies) gives the degree of *squeezing* of each quadrature. We see that if we squeeze in one direction, we must expand in the other. Notwithstanding, the uncertainty product (5.13) continues to be dictated by the thermal fluctuations

$$\langle \delta q^2 \rangle \langle \delta p^2 \rangle = (\bar{n} + 1/2)^2 \geq \frac{1}{4}.$$

This attributes a clear meaning to $\bar{n}$ vs. $1/2$. The former represents the overall width of the distribution, whereas the latter represents the width of the quantum fluctuations. At high temperatures $\bar{n} + 1/2 \simeq \bar{n}$ and we recover a classical harmonic oscillator.

But we also see that one quadrature may also go below the uncertainty bound, at the expense of the other going up. That will happen when $(\bar{n} + 1/2)e^{-2r} \leq 1/2$. This therefore defines a critical squeezing

$$r_c = \frac{1}{2}\ln(2\bar{n} + 1). \tag{5.16}$$

If $r > r_c$ then one quadrature has surpassed the uncertainty bound. This is also related to a concept known as **P representability** introduced by C. T. Lee in *PRA* **41** 2775 (1991), which is a famous paper in the quantum optics community. Essentially, the argument behind **P representability** is that if $r > r_c$ then the state cannot be represented as being simply a superposition of coherent states.

Husimi-Q function of a single-mode Gaussian state

The most general Gaussian state of a single mode turns out to be displaced squeezed thermal state

$$\rho = D(\alpha)S_z\frac{e^{-\beta\omega a^\dagger a}}{Z}S_z^\dagger D^\dagger(\alpha), \qquad S_z = \exp\left\{\frac{1}{2}(za^\dagger a^\dagger - z^* aa)\right\},$$

where $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$. This state has $\langle a \rangle = \alpha$ and a covariance matrix Θ whose entries are exactly (5.15). Although I will not demonstrate this here, I wanted to write down the Husimi-Q function for this state. It reads

$$Q(\alpha, \alpha^*) = \frac{1}{\pi \sqrt{|\tilde{\Theta}|}} \exp\Big\{ -\frac{1}{2} \boldsymbol{\alpha}^\dagger \tilde{\Theta}^{-1} \boldsymbol{\alpha} \Big\}, \qquad \tilde{\Theta} = \Theta + \mathbb{I}_2/2. \tag{5.17}$$

That is, what appears in the argument is a quadratic form over the vector $\boldsymbol{\alpha} = (\alpha, \alpha^*)$, but not with the covariance matrix itself, but rather $\Theta + \mathbb{I}_2/2$.

As I probably mentioned before, the Husimi-Q function is not the only way of representing quantum phase space. Notably, two other important representations are the Wigner function and the Glauber-Sudarshan P function. Both have a similar structure for Gaussian states. In the Wigner function the quadratic form is with Θ itself, whereas in the P function it is with $\Theta - \mathbb{I}_2/2$.

Dynamics of Gaussian systems: the Lyapunov equation

We now turn to the dynamical evolution of Gaussian systems subject to a Lindblad master equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H, \rho] + D(\rho), \tag{5.18}$$

where H is some Gaussian Hamiltonian and $D(\rho)$ is a Gaussian preserving Lindblad dissipator (in the spirit of what was discussed in the beginning of the section). As already mentioned, in this case the equations describing the evolution of the averages and the covariance matrix will be completely decoupled from each other. Here I want to convince you that these equations have the following form. First, the vector of averages $\boldsymbol{x} = \langle \boldsymbol{X} \rangle$ will evolve according to

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = W\boldsymbol{x} - \boldsymbol{f}, \tag{5.19}$$

where the matrix W and the vector $\boldsymbol{f}$ depend on the choice of Hamiltonian and dissipators. Second, the covariance matrix Θ evolves according to the **Lyapunov equation**

$$\boxed{\frac{\mathrm{d}\Theta}{\mathrm{d}t} = W\Theta + \Theta W^\dagger + F,} \tag{5.20}$$

where F is a matrix that depends only on the dissipators, whereas the matrix W that appears here is *the same* as the one appearing in Eq. (5.19).

I will not try to convince you of this in the general case, but we will focus only on a single mode and then I will show you how this could be extended to multi-modes. Recall from Sec. 4.3 that given a master equation of the form (5.18), the evolution of any observable could be written as in Eq. (4.63):

$$\frac{\mathrm{d}\langle O \rangle}{\mathrm{d}t} = i\langle [H, O] \rangle + \langle \bar{\mathcal{D}}(O) \rangle, \tag{5.21}$$

where $\bar{D}$ is the adjoint dissipator, defined in Eq. (4.62). Here we assume to have a single mode subject to the Hamiltonian

$$H = \omega a^\dagger a + \frac{1}{2}(\lambda a^\dagger a^\dagger + \lambda^* aa) + (f a^\dagger + f^* a),$$

and the thermal dissipator

$$D(\rho) = \gamma\bar{n}\left[a^\dagger \rho a - \frac{1}{2}\{aa^\dagger, \rho\}\right] + \gamma(\bar{n}+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right].$$

First moments: we have

$$i[H, a] = -i\omega a - i\lambda a^\dagger - if$$

and, as already discussed in Sec. 4.3,

$$\langle \bar{D}(a)\rangle = -\frac{\gamma}{2}\langle a\rangle,$$

Hence, the equation for $x = \langle a\rangle$ will be

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -i\omega x - i\lambda x^* - \frac{\gamma}{2}x - if. \tag{5.22}$$

This can now be cast in the form (5.19) for the vector $\boldsymbol{x} = (x, x^*)$. We simply need to identify:

$$W = -i\begin{pmatrix} \omega & \lambda \\ -\lambda^* & -\omega \end{pmatrix} - \frac{\gamma}{2}\mathbb{I}_2, \qquad \boldsymbol{f} = -i\begin{pmatrix} f \\ -f^* \end{pmatrix}, \tag{5.23}$$

which is the desired result.

Second moments: we have

$$i[H, a^\dagger a] = -i\lambda a^\dagger a^\dagger + i\lambda^* aa - if a^\dagger + if^* a,$$

$$i[H, aa] = -2i\omega aa - 2i\lambda(a^\dagger a + 1/2) - 2if a$$

and, again as found in Sec. 4.3,,

$$\langle \bar{D}(a^\dagger a)\rangle = \gamma(\bar{n} - \langle a^\dagger a\rangle), \qquad \langle \bar{D}(aa)\rangle = -\gamma\langle aa\rangle.$$

Hence

$$\frac{\mathrm{d}\langle a^\dagger a\rangle}{\mathrm{d}t} = \gamma(\bar{n} - \langle a^\dagger a\rangle) - i\lambda\langle a^\dagger a^\dagger\rangle + i\lambda^*\langle aa\rangle - if\langle a^\dagger\rangle + if^*\langle a\rangle.$$

$$\frac{\mathrm{d}\langle aa\rangle}{\mathrm{d}t} = -(\gamma + 2i\omega)\langle aa\rangle) - 2i\lambda(\langle a^\dagger a\rangle + 1/2) - 2if\langle a\rangle.$$

As can be seen, not only are the equations a bit messy, but they also mix second moments with the first moments. However, we must never forget that the covariance

matrix depends on the fluctuation operators, so we should actually look for an equation for $\langle \delta a^\dagger \delta a \rangle + 1/2$ and $\langle \delta a \delta a \rangle$. Thus we have, for instance,

$$\frac{\mathrm{d}\Theta_{11}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\langle \delta a^\dagger \delta a \rangle + 1/2) = \frac{\mathrm{d}\langle \delta a^\dagger \delta a \rangle}{\mathrm{d}t} = \frac{\mathrm{d}\langle a^\dagger a \rangle}{\mathrm{d}t} - \frac{\mathrm{d}\langle a^\dagger \rangle}{\mathrm{d}t}\langle a \rangle - \langle a^\dagger \rangle \frac{\mathrm{d}\langle a \rangle}{\mathrm{d}t}$$

Substituting the equations for $\langle a^\dagger a \rangle$, $\langle a \rangle$ and $\langle a^\dagger \rangle$ we then get

$$\begin{aligned}\frac{\mathrm{d}\Theta_{11}}{\mathrm{d}t} &= \gamma(\bar{n} - \langle \delta a^\dagger \delta a \rangle) - i\lambda \langle \delta a^\dagger \delta a^\dagger \rangle + i\lambda^* \langle \delta a \delta a \rangle \\ &= \gamma(\bar{n} + 1/2 - \Theta_{11}) - i\lambda \Theta_{21} + i\lambda^* \Theta_{12}.\end{aligned}$$

Similarly, the equation for $\langle \delta a \delta a \rangle$ is

$$\frac{\mathrm{d}\Theta_{12}}{\mathrm{d}t} = -(\gamma + 2i\omega)\Theta_{12} - 2i\lambda\Theta_{11}.$$

What I want you to remember about this result is that the terms depending on f vanish identically, whereas all other remain completely intact. Hence, the second moments become fully decoupled from the first moments.

Now that we have these equations for the entries of Θ, we just need to play around with them a bit in order to write them in a more organized way. I will therefore leave for you to check that they can be written in the Lypaunov form (5.20), with the same matrix W as in Eq. (5.23) and a matrix F which reads

$$F = \gamma \begin{pmatrix} \bar{n} + 1/2 & 0 \\ 0 & \bar{n} + 1/2 \end{pmatrix}. \tag{5.24}$$

This is what we wanted to show.

A popular thing to study, in the context of Lyapunov equations, is the steady-state, which is the solution of

$$W\Theta + \Theta W^\dagger = -F. \tag{5.25}$$

This represents the state that the system will relax to in the long-time limit. Solving this by hand can become nasty quickly, but all numerical libraries have routines to do so. In Mathematica it is called `LyapunovSolve[W,-F]` and in Matlab it is called `lyap(W,-F)`.

For the problem in question, with W given in Eq. (5.23) and F in Eq. (5.25), we get (assuming $\lambda \in \mathbb{R}$ for simplicity)

$$\Theta = \frac{\bar{n} + 1/2}{\gamma^2 + 4(\omega^2 - \lambda^2)} \begin{pmatrix} \gamma^2 + 4\omega^2 & -2i\lambda(\gamma - 2i\omega) \\ 2i\lambda(\gamma + 2i\omega) & \gamma^2 + 4\omega^2 \end{pmatrix}.$$

If $\lambda = 0$ we get a thermal state, $\Theta = (\bar{n} + 1/2)\mathbb{I}_2$. But for $\lambda \neq 0$ we get a competition between the dissipative and the squeezing terms, which end up pushing the system towards a squeezed thermal state.

Application: transport of heat in a bosonic chain

In the next section we will start discussing some real applications of these techniques, in particular to optomechanics and optical parametric oscillators. For now, let me give you a simpler example. Suppose we have two bosonic modes, a_1 and a_2, each connected to a Lindblad thermal dissipator having its own coupling constant γ_i and its own temperature $\bar{n}_i$. That is, we take the total dissipator to have the form

$$D(\rho) = D_1(\rho) + D_2(\rho),$$

where

$$D_i(\rho) = \gamma_i \bar{n}_i \Big[a_i^\dagger \rho a_i - \frac{1}{2}\{a_i a_i^\dagger, \rho\} \Big] + \gamma_i(\bar{n}_i + 1) \Big[a_i \rho a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i, \rho\} \Big].$$

Moreover, suppose they interact according to the Hamiltonian

$$H = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g a_1^\dagger a_2 + g^* a_1 a_2^\dagger, \tag{5.26}$$

(I leave the parameters quite general so that we can keep track of them as we move along; in the end you can set $\omega_1 = \omega_2$, $g = g^*$ and so on).

To treat this problem let us start with the unitary part. We try to write an equation for the vector $\boldsymbol{x} = (\langle a_1 \rangle, \langle a_1^\dagger \rangle, \langle a_2 \rangle, \langle a_2^\dagger \rangle)$. We therefore list the commutators appearing in Eq. (5.21):

$$\begin{aligned}
i[H, a_1] &= -i\omega_1 a_1 - i g a_2 \\
i[H, a_1^\dagger] &= -i\omega_1 a_1^\dagger + i g^* a_2^\dagger \\
i[H, a_2] &= -i\omega_1 a_2 - i g^* a_1 \\
i[H, a_2^\dagger] &= -i\omega_1 a_2^\dagger i g a_1^\dagger
\end{aligned}$$

From this we can already read off the unitary contribution of the matrix W in Eq. (5.19):

$$W\Big|_{\text{unitary}} = \begin{pmatrix} -i\omega_1 & 0 & -ig & 0 \\ 0 & i\omega_1 & 0 & ig^* \\ -ig^* & 0 & -i\omega_2 & 0 \\ 0 & ig & 0 & i\omega_2 \end{pmatrix} \tag{5.27}$$

The point I want to emphasize is that, now that we have found this matrix for the first moments, it will also be the matrix appearing in the Lyapunov equation, so that we don't have to find it again.

Next, the dissipative part is also really easy because the dissipators act separately on each mode. Thus, their contributions will always appear in block form:

$$W\Big|_{\text{dissipative}} = \begin{pmatrix} -\frac{\gamma_1}{2}\mathbb{I}_2 & 0 \\ 0 & -\frac{\gamma_2}{2}\mathbb{I}_2 \end{pmatrix},$$

and

$$F = \begin{pmatrix} \gamma_1(\bar{n}_1 + 1/2)\mathbb{I}_2 & 0 \\ 0 & \gamma_2(\bar{n}_2 + 1/2)\mathbb{I}_2 \end{pmatrix}.$$

With these matrices, we now have all the ingredients to study the first and second moments. Since there is no pump term, the first moments will evolve according to

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = W\boldsymbol{x}.$$

The matrix W definitely has eigenvalues with a negative real part, so that the first moments will simply relax towards zero, $\boldsymbol{x}(t \to \infty) \to 0$.

Next we turn to the second moments and the Lyapunov equation (5.20). In particular, we focus on the steady-state, which is the solution of Eq. (5.25). For simplicity I will now assume that $\gamma_1 = \gamma_2 = \gamma$, $\omega_1 = \omega_2 = \omega$ and $g^* = g$. Dr. Mathematica then tells us that the solution is

$$\Theta = \begin{pmatrix} \langle a_1^\dagger a_1\rangle + 1/2 & 0 & \langle a_1 a_2^\dagger\rangle & 0 \\ 0 & \langle a_1^\dagger a_1\rangle + 1/2 & 0 & \langle a_1^\dagger a_2\rangle \\ \langle a_1^\dagger a_2\rangle & 0 & \langle a_2^\dagger a_2\rangle + 1/2 & 0 \\ 0 & \langle a_1 a_2^\dagger\rangle & 0 & \langle a_2^\dagger a_2\rangle + 1/2 \end{pmatrix},$$

where

$$\begin{aligned} \langle a_1^\dagger a_1\rangle &= \bar{n}_1 + \frac{2g^2}{4g^2+\gamma^2}(\bar{n}_2 - \bar{n}_1), \\ \langle a_2^\dagger a_2\rangle &= \bar{n}_2 - \frac{2g^2}{4g^2+\gamma^2}(\bar{n}_2 - \bar{n}_1), \\ \langle a_1^\dagger a_2\rangle &= \frac{ig\gamma}{4g^2+\gamma^2}(\bar{n}_2 - \bar{n}_1). \end{aligned} \tag{5.28}$$

I think these results are quite interesting. First we see that the populations of 1 and 2 are not exactly $\bar{n}_1$ and $\bar{n}_2$, which is what the Lindblad dissipators would want. Instead, it is modified by a term proportional to the interaction g between them. However, this term only exists if there is a "temperature gradient" between the two modes; that is, if $\bar{n}_1 \neq \bar{n}_2$. In fact, we also see that this gradient generates *correlation* between the two modes $\langle a_1^\dagger a_2\rangle$.

To understand the meaning of a term such as $\langle a_1^\dagger a_2\rangle$, it is helpful to look at the **current of quanta** between the two modes. First we write down the equation for $\langle a_1^\dagger a_1\rangle$:

$$\frac{\mathrm{d}\langle a_1^\dagger a_1\rangle}{\mathrm{d}t} = \gamma(\bar{n}_1 - \langle a_1^\dagger a_1\rangle) - ig(\langle a_1^\dagger a_2\rangle - \langle a_1 a_2^\dagger\rangle). \tag{5.29}$$

This can now be viewed as a *continuity equation*. It essentially says that the rate at which the number of quanta in mode 1 changes is due to a current of quanta entering

from the bath (the first term) and the current of quanta leaving towards mode 2. In the steady-state $\mathrm{d}\langle a_1^\dagger a_1\rangle/\,\mathrm{d}t = 0$ and the two currents therefore coincide:

$$\gamma(\bar{n}_1 - \langle a_1^\dagger a_1\rangle) = ig(\langle a_1^\dagger a_2\rangle - \langle a_1 a_2^\dagger\rangle) := J \tag{5.30}$$

We therefore see that the imaginary part of $\langle a_1^\dagger a_2\rangle$ is actually related to the current of quanta. This means that for energy to flow, the two modes must be correlated, which makes sense since a current implies that information is being transferred from one mode to the other.

The explicit formula for J is found using the results in Eq. (5.28) and reads:

$$J = \frac{2g^2\gamma}{4g^2+\gamma^2}(\bar{n}_2 - \bar{n}_1). \tag{5.31}$$

This result makes sense: the current is zero if $g = 0$ (we break the link between 1 and 2) or if $\gamma = 0$ (we break the link between 1,2 and their baths). Moreover, the current increases with the temperature gradient $\bar{n}_2 - \bar{n}_1$ and its *sign* depends on whether 1 is warmer than 2 or vive-versa. Thus, as intuitively expected, current always flows from hot to cold.

Of course, these ideas can be extended in an infinite number of ways and, in fact, that is a line of research which I really like. But in order for us to not get off track, I will stop with this for now.

Gaussian quantum information

Finally, I want to discuss some tricks for dealing with information-theoretic quantities of Gaussian states, such as measures of purity and correlations. The literature on this subject is quite vast. But here I would like to focus on the particularly recent result of `arXiv 1203.5116`, which bases the entire analysis on the Rényi-2 entropy.

In Sec. 2.9, when we talked about entropy, I mentioned the so-called *strong subadditivity inequality* of the von Neumann entropy: given an arbitrary tri-partite system, it reads

$$S(AB) + S(BC) \geq S(ABC) - S(B). \tag{5.32}$$

The strong subadditivity is, in a sense, an "approval seal" that an entropy should have in order to be employed as an information-theoretic quantity. And, in general, strong subadditivity is a unique feature of von Neumann's entropy and does not hold for the Rényi-α entropies. It is for this reason that in most of quantum information, the von Neumann reigns supreme, as the ultimate entropic quantifier.

The key result of `arXiv 1203.5116` was to show that, **for Gaussian states, strong subadditivity holds for the Rényi-2**. And this is extremely useful because the Rényi-2 is very easy to compute since it is simply related to the purity of the state:

$$S_2(\rho) = -\ln \mathrm{tr}(\rho^2). \tag{5.33}$$

What is even more remarkable, for Gaussian states the purity actually turns out to be

$$\mathrm{tr}(\rho^2) = \frac{1}{2^N\sqrt{|\Theta|}}, \tag{5.34}$$

where N is the number of modes in question.[2] I will leave the demonstration of this result for you as an exercise (see problem set). Consequently, we find that the Rényi-2 entropy of a Gaussian state is

$$\boxed{S_2(\Theta) = \frac{1}{2}\ln|\Theta| + N\ln 2.} \tag{5.35}$$

Maybe I should have written $S_2(\rho)$, but I like to write it as $S_2(\Theta)$ to emphasize that for a Gaussian state all that matters is the CM. As far as Gaussian states are concerned, the Rényi-2 entropy (5.35) is therefore a perfectly valid entropic measure, so that everything that can be done with von Neumann's, can also be done with Rényi-2.

An obvious reason why Eq. (5.35) is easy to deal with is because computing a determinant is easy. But another, perhaps even stronger reason, is that given a density matrix of a multi-partite system, finding the **partial trace** is *trivial*. Suppose you have two modes, A and B. The joint covariance matrix of the two modes can then be written in block form as

$$\Theta_{AB} = \begin{pmatrix} \Theta_A & S_{AB} \\ S_{AB}^\dagger & \Theta_B \end{pmatrix}, \tag{5.36}$$

where Θ_A and Θ_B are the covariance matrices of A and B individually and S_{AB} represents their correlation. If we now wish to take the partial trace over B, for instance, then the reduced state of A will still be a Gaussian state. Consequently, it is fully characterized by its covariance matrix Θ_A. Hence, taking the partial trace over a system simply means throwing away the lines and columns in the matrix that you don't want anymore.

For instance, suppose we have a tripartite system ABC with a CM

$$\Theta_{ABC} = \begin{pmatrix} \Theta_A & S_{AB} & S_{AC} \\ S_{AB}^\dagger & \Theta_B & S_{BC} \\ S_{AC}^\dagger & S_{BC}^\dagger & \Theta_C \end{pmatrix}. \tag{5.37}$$

Now suppose we wish to take the partial trace over B. The reduced density matrix of AC will then still be a Gaussian state, with a CM:

$$\Theta_{AC} = \begin{pmatrix} \Theta_A & S_{AC} \\ S_{AC}^\dagger & \Theta_C \end{pmatrix}.$$

You see what I did there? I simply threw away the lines and columns corresponding to system B.

As a first application, consider a bipartite system AB and let us compute the mutual information

$$\boxed{\mathcal{I}_{AB} = S_A + S_B - S_{AB} = \frac{1}{2}\ln\left\{\frac{|\Theta_A||\Theta_B|}{|\Theta_{AB}|}\right\}.} \tag{5.38}$$

[2] Sanity check: for the vacuum state of N modes, $\Theta = \mathbb{I}_{2N}/2$ so that $|\Theta| = (1/2)^{2N}$ and hence $2^N\sqrt{|\Theta|} = 1$, so that the system is in a pure state, $\mathrm{tr}(\rho^2) = 1$.

Recall that the mutual information is a quantifier of the *total* correlations between two systems, irrespective of whether these correlations are quantum or classical. The proof that this quantity is non-negative (which is the as proving the sub-additivity inequality) can be done using something called the **Hadamard-Fisher inequality**.

Let M denote a positive semi-definite Hermitian matrix of size K and let α, β denote index sets of $\{1, \ldots, K\}$. For instance, $\alpha = \{1, 2, 3\}$ and $\beta = \{1, 5, 20\}$, or whatever. Moreover, given an index set α, let M_α denote the matrix M chopped up to contain only the rows and columns of the index set α . The Hadamard-Fisher inequality then says that

$$|M_{\alpha\cup\beta}||M_{\alpha\cap\beta}| \leq |M_\alpha||M_\beta|, \tag{5.39}$$

with the proviso that $|M_\varnothing| = 1$. In the case of Eq. (5.38) we take α to refer to the index set of modes A and β to refer to the index set of modes B, which then gives $|\Theta_{AB}| \leq |\Theta_A||\Theta_B|$. Hence $\mathcal{I}_{AB} \geq 0$.

If the composite AB system is in a pure state then all correlation must be entanglement. In this case we know that $S_{AB} = 0$. Moreover, as we have seen when we discussed the Schmidt decomposition in Sec. 2.8, we also have $S_A = S_B$. Hence

$$\mathcal{I}_{AB} = 2S(\Theta_A) = 2S(\Theta_B), \qquad \text{For a pure state AB.} \tag{5.40}$$

In this case the mutual information gives twice the entanglement entropy between the two sub-systems.

The inequality appearing in the strong subadditivity inequality (5.32) can be used to define a **conditional mutual information**

$$\boxed{\mathcal{I}(A:C|B) := S_{AB} + S_{BC} - S_{ABC} - S_C = \frac{1}{2}\ln\left\{\frac{|\Theta_{AB}||\Theta_{BC}|}{|\Theta_{ABC}||\Theta_C|}\right\}.} \tag{5.41}$$

This represents the amount of information shared between A and C, intermediated by B. The positivity of this quantity is again demonstrated using the Hadamard-Fisher inequality (5.39). One need only take α to denote the index set of AB and β to denote the index set of BC.

We can go further and also define measures of Rényi-2 entanglement and Rényi-2 quantum discord. I will not go through these guys right now, since they take some time to discuss. If you are interested, please have a look at `arXiv 1203.5116`.

Duan-Duan

To finish this section, I want to briefly discuss a criteria for determining whether two continuous variables are entangled or not when they are in a *mixed* state. If the state is pure, then correlation = entanglement. But if the state is mixed, part of the correlations may be quantum and part may be classical (recall that, by classical, we mean a correlation related to our lack of knowledge about the system). We haven't discussed a lot about this quantum-classical separation (sorry about that!) but I will try to compensate this a bit now. The main point is that this separation is not sharp, meaning there is no universal criteria for separating quantum and classical correlations.

Essentially, what one would hope is to be able to divide the the mutual information as $\mathcal{I} = \mathcal{I}_C + \mathcal{I}_Q$, where $\mathcal{I}_C$ quantifies the classical correlations and $\mathcal{I}_Q$ quantifies the quantum correlations. This is the approach of the so-called **quantum discord**, introduced by Henderson and Vedral in `arXiv quant-ph/0105028` and simultaneously by Ollivier and Zurek in `arXiv quant-ph/0105072`. But discord is not perfect and there are heated debates in the literature about it. Some people love it. Some people hate it. (As for me, I'm just too stupid to have a strong opinion about it).

What we *do* have, however, is some idea of when a state contains quantum features and when it does not. And this can lead us to the criteria of **separability**. It is fair to assume that a state such as $\rho_A \otimes \rho_B$ does not have any quantum correlations between A and B. Of course, inside ρ_A and ρ_B there can still be a bunch of quantum features. But as far as AB correlations are concerned, such a product state has none. Motivated by this, we define a **separable** state as a state of the form

$$\rho_{AB} = \sum_i p_i \, \rho_{A,i} \otimes \rho_{B,i}, \qquad p_i \in [0,1], \qquad \sum_i p_i = 1. \tag{5.42}$$

The logic here is that such a state is just a classical probabilistic combination of product states and, therefore, any correlations cannot come from entanglement, but must come from the classical probabilities p_i. For this reason, we can say that a separable state is not entangled.

Instead of trying to quantify the degree of entanglement, we can now take on a more soft approach and simply ask whether a certain state is separable or not. If it is separable than all correlations must be of classical origin, whereas if it is not separable, than some degree of quantum correlation is present (exactly how much we cannot know). A large number of criteria are available for both discrete and continuous variables. A comprehensive review can be found in a famous review by the Horodecki clan (`arXiv quant-ph/0702225`). Here I will focus on continuous variables and discuss a criteria developed in `arXiv quant-ph/9908056` by Duan, Giedke, Cirac and Zoller. For some reason, people forget about the other authors and simply call it the **Duan criteria**. The idea is as follows. Consider two bosonic modes with operators a_1 and a_2. Define the quadrature for the first, as usual:

$$q_1 = \frac{1}{\sqrt{2}}(a_1^\dagger + a_1), \qquad p_1 = \frac{i}{\sqrt{2}}(a_1^\dagger - a_1).$$

But for the second, define rotated quadrature operators

$$q_2 = \frac{1}{\sqrt{2}}(e^{i\phi} a_2^\dagger + e^{-i\phi} a_2), \qquad p_2 = \frac{i}{\sqrt{2}}(e^{i\phi} a_2^\dagger - e^{-i\phi} a_2),$$

where ϕ is an arbitrary angle. Note that we still have $[q_2, p_2] = i$. Finally, define

$$Q_+ = \frac{q_1 + q_2}{\sqrt{2}}, \qquad P_- = \frac{p_1 - p_2}{\sqrt{2}}. \tag{5.43}$$

According to Duan, Giedke, Cirac and Zoller, a sufficient criteria for a state to be separable is

$$\boxed{\langle \delta Q_+^2 \rangle + \langle \delta P_-^2 \rangle \geq 1,} \tag{5.44}$$

for all ϕ. This criteria holds even for non-Gaussian states. However, for Gaussian states, it turns out it is both sufficient *and* necessary. Thus, within the context of Gaussian states, if you find a angle ϕ such that $\langle \delta Q_+^2 \rangle + \langle \delta P_-^2 \rangle < 1$, then the state is definitely not separable.

5.2 Optomechanics

The name optomechanics refers, as you probably guessed, to the combined interaction of an optical mode and mechanical vibrations. The two most typical configurations are shown in Fig. 5.1. For simplicity, the problem is usually approximated to that of a single radiation mode interacting with a single harmonic oscillator. However, the interaction between the two is either cubic or quartic, so that Gaussianity is not preserved. Much of our mathematical work will then be on an approximation method which is used to re-Gaussianize the theory.

The radiation mode is a standing mode of a cavity, of frequency ω_c, which is pumped by a laser at frequency ω_p through a semi-transparent mirror. In the configuration of Fig. 5.1(a) the other mirror is allowed to vibrate slightly from its equilibrium position and this vibration is modeled as a harmonic oscillator. In (b), on the other hand, both mirrors are fixed, but a semi-transparent membrane is placed inside the cavity and allowed to vibrate.

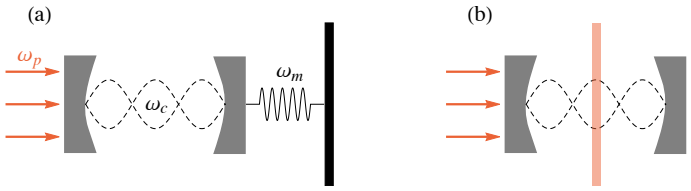

Figure 5.1: Schematic representation of the two most widely used optomechanical configurations. In both cases an optical cavity of frequency ω_c is pumped with a laser at frequency ω_p through a semi-transparent mirror. In (a) one of the mirrors is allowed to vibrate with a frequency ω_m. In (b), on the other hand, the mechanical vibration is that of a semi-transparent membrane placed inside the cavity.

When dealing with physical implementations, such as this one, it is always recommended that you start by establishing the Hamiltonian and the dissipation channels. I will call this **awesome advice # 1**. In the end, we want to start with a master equation of the form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H,\rho] + D(\rho),$$

for some Hamiltonian H and some dissipator $D(\rho)$. Let us start with the cavity mode, which we associate with an annihilation operator a. Its Hamiltonian was discussed in Sec. 3.2 and reads

$$H_c = \hbar\omega_c a^\dagger a + \hbar\epsilon a^\dagger e^{-i\omega_p t} + \hbar\epsilon^* a e^{i\omega_p t}. \tag{5.45}$$

I have reintroduced $\hbar$ for now, just for completeness. But I will get rid of it very soon. Recall also that ϵ is the pump intensity and can be written as $|\epsilon|^2 = 2\kappa P/\hbar\omega_p$ where κ is