5.2 Optomechanics

The name optomechanics refers, as you probably guessed, to the combined interaction of an optical mode and mechanical vibrations. The two most typical configurations are shown in Fig. 5.1. For simplicity, the problem is usually approximated to that of a single radiation mode interacting with a single harmonic oscillator. However, the interaction between the two is either cubic or quartic, so that Gaussianity is not preserved. Much of our mathematical work will then be on an approximation method which is used to re-Gaussianize the theory.

The radiation mode is a standing mode of a cavity, of frequency ω_c, which is pumped by a laser at frequency ω_p through a semi-transparent mirror. In the configuration of Fig. 5.1(a) the other mirror is allowed to vibrate slightly from its equilibrium position and this vibration is modeled as a harmonic oscillator. In (b), on the other hand, both mirrors are fixed, but a semi-transparent membrane is placed inside the cavity and allowed to vibrate.

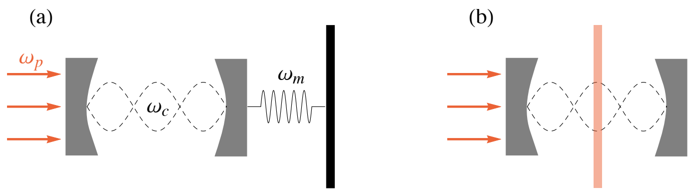

Figure 5.1: Schematic representation of the two most widely used optomechanical configurations. In both cases an optical cavity of frequency ω_c is pumped with a laser at frequency ω_p through a semi-transparent mirror. In (a) one of the mirrors is allowed to vibrate with a frequency ω_m. In (b), on the other hand, the mechanical vibration is that of a semi-transparent membrane placed inside the cavity.

When dealing with physical implementations, such as this one, it is always recommended that you start by establishing the Hamiltonian and the dissipation channels. I will call this **awesome advice # 1**. In the end, we want to start with a master equation of the form

$$\frac{d\rho}{dt} = -i[H, \rho] + D(\rho),$$

for some Hamiltonian H and some dissipator $D(\rho)$. Let us start with the cavity mode, which we associate with an annihilation operator a. Its Hamiltonian was discussed in Sec. 3.2 and reads

$$H_c = \hbar\omega_c a^\dagger a + \hbar\epsilon a^\dagger e^{-i\omega_p t} + \hbar\epsilon^* a e^{i\omega_p t}. \tag{5.45}$$

I have reintroduced $\hbar$ for now, just for completeness. But I will get rid of it very soon. Recall also that ϵ is the pump intensity and can be written as $|\epsilon|^2 = 2\kappa P/\hbar\omega_p$ where κ is the loss rate [that also appears in $D(\rho)$] and P is the laser pump power. Moreover, the loss of photons through the cavity is described by the dissipator

$$D_c(\rho) = 2\kappa\Big[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\Big], \tag{5.46}$$

which, as I probably mentioned before, is absolutely standard in all descriptions of lossy cavities.

Next we turn to the mechanical mode. We assume it is a single harmonic oscillator with position Q and momentum P satisfying $[Q, P] = i\hbar$. Its free Hamiltonian will then be

$$H_m = \frac{P^2}{2m} + \frac{1}{2}m\omega_m^2 Q^2 = \hbar\omega_m(b^\dagger b + 1/2), \tag{5.47}$$

where m is the mass, ω_m is the mechanical frequency and

$$b = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{m\omega_m}{\hbar}}Q + \frac{iP}{\sqrt{m\hbar\omega_m}}\right\}, \tag{5.48}$$

is the annihilation operator for the mechanical mode.

A much harder question concerns the choice of dissipator for the mechanical mode. The mechanical mode is of course dissipative because it is connected to your sample so the bath in this case are the phonons; i.e., the mechanical vibrations of the material which makes up both the vibrating mirror and its surroundings. Consequently, they will cause the oscillator to thermalize at the temperature of your experimental setup. But is not well modeled by a Lindblad equation, since Lindblad assumes a rotating-wave approximation, which is usually not good for mechanical frequencies. In fact, more than that, as shown recently in `arXiv 1305.6942`, the dynamics can actually be highly non-Markovian, so not even that is guaranteed. Traditionally, one normally uses quantum Brownian motion, in which the degree of non-Markovianity can be taken into account. However, this makes the entire treatment quite difficult.

So we now arrive at **awesome advice # 2**: *never* start with very realistic descriptions of your model. Realistic descriptions are always too complicated and always contain an enormous number of parameters whose values you usually don't know very well. This will then completely mask the physics of the problem. Instead, the advice is to always start with the simplest description possible, containing only a small amount of parameters. *Even if that description is not very good.* Then, after your learned everything you can from this simplified picture, you start to add ingredients and see how they affect your toy-model results. Even though this may at first seem like extra work, it turns out it is not: if you start with a complicated realistic model, it will take you forever to obtain answers. But if you start with a simple model, then each ingredient you add will only change the calculations by a small bit and therefore they will not be so hard.

Concerning the dissipative channel of the mechanical mode, the simplification I will adopt is to use a Lindblad equation to model $D_m(\rho)$. This is definitely a rough approximation, but will allow us to extract the physics more clearly. Thus, we will assume that

$$D_m(\rho) = \gamma(\bar{n}+1)\left[b\rho b^\dagger - \frac{1}{2}\{b^\dagger b, \rho\}\right] + \gamma\bar{n}\left[b^\dagger\rho b - \frac{1}{2}\{bb^\dagger, \rho\}\right]. \tag{5.49}$$

where γ is the coupling constant of the mechanical mode to its bath and $\bar{n} = (e^{\omega_m/T} - 1)^{-1}$ is the Bose-Einstein distribution, with T being the temperature of the mechanical mode.

Finally, we reach que most important question, which concerns the **optomechanical interaction**. Here we shall focus on the setup in Fig. 5.1(a). In this case the coupling comes from the fact that the cavity frequency ω_c actually depends on the position of the mirror. In fact, from electromagnetism[3] one can show that the dependence is of the form $\omega_c(L) = A/L$ where L is the size of the cavity and A is a constant. When the mirror is allowed to vibrate we should then replace L by $L + Q$. Assuming that Q is small compared to L we can then get

$$\omega_c(L+Q) \simeq \frac{A}{L}\Big(1 - \frac{Q}{L}\Big) = \omega_c - \frac{\omega_c}{L}Q,$$

where $\omega_c = \omega_c(L)$ is the equilibrium frequency of the cavity. Consequently, we see that the Hamiltonian $\omega_c a^\dagger a$ is to be transformed into

$$\omega_c a^\dagger a \to \omega_c a^\dagger a - \frac{\omega_c}{L} a^\dagger a Q.$$

We therefore now have a coupling between $a^\dagger a$ and Q. This is called the **radiation pressure** coupling. And if you think about it, it makes all the sense in the world: A term such as $-fQ$ in a Hamiltonian means a force f pushing the coordinate Q. This is exactly what we have here, except that now the force actually depends on the number of photons $a^\dagger a$ inside the cavity. The more photons we have, the more we push the mirror. Makes sense!

Collecting everything, our Hamiltonian can then be written as

$$H = \hbar\omega_c a^\dagger a + \hbar\omega_m b^\dagger b - \frac{\hbar\omega_c}{L} a^\dagger a Q + \hbar\epsilon a^\dagger e^{-i\omega_p t} + \hbar\epsilon^* a e^{i\omega_p t}.$$

To make it a little bit cleaner, we substitute $Q = \sqrt{\frac{\hbar}{2m\omega_m}}(b + b^\dagger)$ and then write this as

$$\boxed{H = \hbar\omega_c a^\dagger a + \hbar\omega_m b^\dagger b - \hbar g_0 a^\dagger a(b + b^\dagger) + \hbar\epsilon a^\dagger e^{-i\omega_p t} + \hbar\epsilon^* a e^{i\omega_p t},} \tag{5.50}$$

where $g_0 = \frac{\omega_c}{L}\sqrt{\frac{\hbar}{2m\omega_m}}$. This is the so-called *radiation pressure optomechanical coupling*. You will find it in most papers on optomechanics. Note also that this is *not* a Gaussian Hamiltonian since the interaction term is cubic in the creation and annihilation operators. Thus, it cannot be solved exactly and we will therefore have to resort to some approximations.

To summarize, the model in Fig. 5.1(a) can be described, to a first approximation, as

$$\boxed{\frac{d\rho}{dt} = -i[H,\rho] + D_c(\rho) + D_m(\rho),} \tag{5.51}$$

where H is given in (5.50), $D_c(\rho)$ is given in (5.46) and $D_m(\rho)$ is given in (5.49). As discussed above, the weakest link here is the choice of D_m, which is in general a bit

[3] The standard reference on this is C. Law, *Phys. Rev. A.*, **51**, 2537-2541 (1995).

Table 5.1: Typical parameters for an optomechanical setup, all given in Hz. Based on arXiv 1602.06958. Typical temperatures are of the order of 1 K, which give $\bar{n} = (e^{\hbar\omega_m/k_BT} - 1)^{-1} \sim 10^3$.

Parameter	ω_c	ω_m	κ	γ	g_0	ϵ
Order of magnitude (Hz)	10^{14}	10^6	10^7	10	10^3	10^{12}

drastic. All other ingredients are, in general, quite well justified. Typical values of the parameters for an experiment that I participated a few years ago (`arXiv 1602.06958`) are shown in Table 5.1. But, of course, part of the experimental game is to really have flexibility in changing these parameters.

Before we delve deeper into Eq. (5.51), let me comment on the configuration in Fig. 5.1(b). I will not try to derive the Hamiltonian in this case. But I want to simply point out that it *definitely* cannot be the same as (5.50) due to its symmetry. The Hamiltonian (5.50) is linear in Q precisely because it pushes the mirror in one specific direction. In the case of Fig. 5.1(b) there is no preferred direction. Thus, from such an argument we expect that the radiation pressure interaction in this case should, to lowest order in Q, be *quadratic*. That is, something like

$$g_0^{(2)} a^\dagger a (b + b^\dagger)^2,$$

for some constant $g_0^{(2)}$. Indeed, that is what is found from a more careful derivation.

Pump it up!

The first step in dealing with the Hamiltonian (5.50) is to move to a rotating frame with respect to the pump frequency, exactly as was done in Sec. 3.3. That is, the unitary transformation is taken to be $e^{i\omega_p t a^\dagger a}$, while nothing is done on the mechanical part. The dissipative part does not change, whereas the Hamiltonian simplifies to

$$H = \Delta' a^\dagger a + \omega_m b^\dagger b - g_0 a^\dagger a (b + b^\dagger) + \epsilon a^\dagger + \epsilon^* a, \tag{5.52}$$

where $\Delta' = \omega_c - \omega_p$ is the cavity detuning (I'm using Δ' instead of Δ because below we will come across another quantity that I will want to call Δ). As promised, here I already set $\hbar = 1$.

This Hamiltonian is still non-linear (higher than quadratic) and therefore cannot be solved analytically. However, in this case, and in many other problems involving cavities, there is a trick to obtain very good approximations, which is related to the pump intensity. Roughly speaking $\langle a \rangle$ will try to follow the intensity ϵ. So if the pump is sufficiently large the first moments $\langle a \rangle$ and $\langle b \rangle$ will tend to be much larger than the fluctuations (i.e., the second moments such as $\langle \delta a^\dagger \delta a \rangle$). This then allows us to *linearize* our equations and Hamiltonians and therefore obtain solvable models. I call this the **pump trick**. In statistical mechanics they would call it a mean-field approximation.

To see how it works, let us consider the evolution equations for the first moments

$\alpha = \langle a \rangle$ and $\beta = \langle b \rangle$. Following the usual procedure, they read

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -(\kappa + i\Delta')\alpha - i\epsilon + ig_0\langle a(b + b^\dagger)\rangle,$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = -(\frac{\gamma}{2} + i\omega_m)\beta + ig_0\langle a^\dagger a\rangle.$$

Thus, as promised, since the Hamiltonian is non-Gaussian, the evolution of the first moments actually depend on second moments. And if we were to try to compute the evolution of the second moments, they would depend on third moments and so on.

The pump trick is now to write $a = \alpha + \delta a$ and $b = \beta + \delta b$. Exploiting the fact that $\langle \delta a \rangle = \langle \delta b \rangle = 0$, by construction, we can then write, for instance,

$$\langle ab \rangle = \langle (\alpha + \delta a)(\beta + \delta b) \rangle = \alpha\beta + \langle \delta a \delta b \rangle.$$

So far this is exact. The approximation is now to assume that the second term is much smaller than the first, so that it may be neglected. A similar idea holds for all other terms.

With this trick the equations for α and β become *closed, but non-linear*:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -(\kappa + i\Delta')\alpha - i\epsilon + ig_0\alpha(\beta + \beta^*), \tag{5.53}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = -(\frac{\gamma}{2} + i\omega_m)\beta + ig_0|\alpha|^2. \tag{5.54}$$

We are interested in the steady-states of these equations, obtained by setting $\mathrm{d}\alpha/\,\mathrm{d}t = \mathrm{d}\beta/\,\mathrm{d}t = 0$. From the second equation we get

$$\beta = \frac{ig_0|\alpha|^2}{\gamma/2 + i\omega_m}. \tag{5.55}$$

This result highlights some of the weirdness of using a Lindblad description for the mechanical mode. What we are talking about here is really the *equilibrium configuration* of the mirror and $\mathrm{Re}(\beta)$ is proportional the displacement $\langle Q \rangle$, whereas $\mathrm{Im}(\beta)$ is related to $\langle P \rangle$. Of course, since we are talking about a mechanical dude, equilibrium should mean $\langle P \rangle = 0$, but this is not what happens in Eq. (5.55). So Lindblad predicts an equilibrium with a finite momentum, which doesn't make much sense. As I said, in this case the rotating wave approximation is a bit rough. However, lucky for us, the value of γ is usually really small (see Table 5.1) so that this imaginary part is almost negligible. In fact, if we discard it we get something that makes quite some sense, which is a displacement $\langle Q \rangle = \mathrm{Re}(\beta)$ proportional to the number of photons $|\alpha|^2$.

Substituting (5.55) into (5.53) then yields the equation

$$\left\{\kappa + i\Delta' - \frac{2ig_0^2\omega_m|\alpha|^2}{\frac{\gamma^2}{4} + \omega_m^2}\right\}\alpha = -i\epsilon.$$

This is now a non-linear equation for α, which has to be solved numerically. It is convenient to define an effective detuning

$$\Delta = \Delta' - g_0(\beta + \beta^*) = \Delta' - \frac{2g_0^2\omega_m|\alpha|^2}{\frac{\gamma^2}{4} + \omega_m^2}, \tag{5.56}$$

so that we can rewrite the equation above as

$$\alpha = \frac{-i\epsilon}{\kappa + i\Delta}. \tag{5.57}$$

Of course, Δ is still a function α so this is an implicit equation. But we can just assume that we have solved this equation numerically and therefore found the numerical value of Δ.

Another useful trick is to adjust the relative phase of ϵ in order to make α real. The phase of the pump is arbitrary so we can also tune it in this way. And, of course, the final result will not depend on this, so it is just a way to make the calculations a bit simpler. Hence, from now on we will assume that $\alpha \in \mathbb{R}$.

Fluctuations around the average

The next step is to rewrite the master equation (5.51) in terms of the fluctuation operators $\delta a = a - \alpha$ and $\delta b = b - \beta$. Note that these are still bosonic operators, the only difference is that they now have zero mean and therefore describe only fluctuations around the average. We start with the Hamiltonian (5.50) and then express each term as something like:

$$a^\dagger a = |\alpha|^2 + \alpha\delta a^\dagger + \alpha^*\delta a + \delta a^\dagger \delta a.$$

Doing this for every term allow us to write

$$H = \text{const} + H_1 + H_2 + H_3,$$

where "const" refers to a unimportant constant and

$$H_1 = \Delta'(\alpha\delta a^\dagger + \alpha^*\delta a) + \omega_m(\beta\delta b^\dagger + \beta^*\delta b) + \epsilon\delta a^\dagger + \epsilon^*\delta a \tag{5.58}$$

$$-g_0\Big\{|\alpha|^2(\delta b + \delta b^\dagger) + (\beta + \beta^*)(\alpha\delta a^\dagger + \alpha^*\delta a)\Big\}, \tag{5.59}$$

$$H_2 = \Delta'\delta a^\dagger\delta a + \omega_m\delta b^\dagger\delta b - g_0(\alpha\delta a^\dagger + \alpha^*\delta a)(\delta b + \delta b^\dagger) \tag{5.60}$$

$$-g_0(\beta + \beta^*)\delta a^\dagger\delta a, \tag{5.61}$$

$$H_3 = -g_0\delta a^\dagger\delta(\delta b + \delta b^\dagger). \tag{5.62}$$

Yeah. I know its messy. But don't panic. There is nothing conceptually difficult. It is just a large number of terms that we have to be patiently organized.

The key difficulty lies with the term H_3, which is cubic in the creation and annihilation operators. But note also that this is the *only* term which is not multiplied by either α or β. This is the spirit behind the pump trick: we are assume the pump is large so α and β are large. Consequently, the cubic term H_3 will be much smaller than the other terms and we may then neglect it. If we do so, the resulting theory is quadratic and therefore Gaussianity is restored.

Next let us do the same expansion for the dissipators. It is useful to write down the following formulas, which I will leave for you as an exercise to check:

$$D[a] = -\frac{1}{2}[\alpha\delta a^\dagger - \alpha^*\delta a, \rho] + D[\delta a], \tag{5.63}$$

$$D[a^\dagger] = \frac{1}{2}[\alpha\delta a^\dagger - \alpha^*\delta a, \rho] + D[\delta a^\dagger] \tag{5.64}$$

It is interesting to realize that the linear contribution in this expansion actually looks like a unitary term. Of course, these formulas hold for any operator a, or b, expanded around its average. Thus, for instance, the dissipator $D_m(\rho)$ of the mechanical part, Eq. (5.49), becomes

$$D_m(\rho) = -\frac{\gamma}{2}[\beta\delta b^\dagger - \beta^*\delta b, \rho] + \gamma(\bar{n}+1)D[\delta b] + \gamma\bar{n}D[\delta b^\dagger].$$

If we now plug all these results into the master equation (5.51) we shall get, already neglecting H_3,

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_1 - i\kappa(\alpha\delta a^\dagger - \alpha^*\delta a) - i\frac{\gamma}{2}(\beta\delta b^\dagger - \beta^*\delta b), \rho]$$
$$-i[H_2, \rho] + 2\kappa D[\delta a] + \gamma(\bar{n}+1)D[\delta b] + \gamma\bar{n}D[\delta b^\dagger].$$

The first line in this expression contains only linear terms, whereas the second line contains quadratic terms. Let me call the term inside the commutator in the first line as $H_{1,\text{eff}}$. Organizing it a bit, we may write it as

$$H_{1,\text{eff}} = i\delta a^\dagger\Big\{ - (\kappa + i\Delta')\alpha + ig_0\alpha(\beta+\beta^*) - i\epsilon\Big\}$$
$$+ i\delta b^\dagger\Big\{ - (\frac{\gamma}{2} + i\omega_m)\beta + ig_0|\alpha|^2\Big\} + \text{h.c.}.$$

I wrote it in this clever/naughty way because I already have Eqs. (5.53) and (5.54) in mind: the terms multiplying each operator are just the steady-state of these equations. Thus, if we are only interested in the fluctuations around the average, then $H_{1,\text{eff}} = 0$. It should be noted, however, that in practice we don't actually need to worry about this. When a Hamiltonian is Gaussian, the linear terms do not interfere with the evolution of the covariance matrix. So we don't even need to care about the linear terms. All that is going to matter for us is the quadratic part.

But, in any case, summarizing, we find that after *linearizing* the system around the fluctuations, we end up with the master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[H_2, \rho] + 2\kappa D[\delta a] + \gamma(\bar{n}+1)D[\delta b] + \gamma\bar{n}D[\delta b^\dagger]. \tag{5.65}$$

which is now a quadratic and Gaussian equation for the new operators δa and δb. Let us also work a bit more on H_2 in Eq. (5.60). The term multiplying $\delta a^\dagger\delta a$ is actually $\Delta' - g_0(\beta+\beta^*)$, which is nothing but the quantity Δ in Eq. (5.56). Thus,

$$H_2 = \Delta\delta a^\dagger\delta a + \omega_m\delta b^\dagger\delta b - g_0(\alpha\delta a^\dagger + \alpha^*\delta a)(\delta b + \delta b^\dagger).$$

This Hamiltonian is Gaussian so we could in principle just keep going. However, the final result will appear rather ugly, so it is convenient to do here another approximation. Namely, we shall do a rotating-wave approximation and neglect the counter-rotating terms $\delta a\delta b$ and $\delta a^\dagger\delta b^\dagger$. With this approximation our Gaussian Hamiltonian simplifies further to

$$H_2 = \Delta\delta a^\dagger\delta a + \omega_m\delta b^\dagger\delta b - g(\delta a^\dagger\delta b + \delta a\delta b^\dagger), \tag{5.66}$$

where $g = g_0\alpha$ and only now did I assume that α was real. After we are done, it is a good idea to come back and redo the calculations without the RWA, which I will leave for you as an exercise.

Lyapunov equation

We are now ready to set up our Lyapunov equation for the covariance matrix using the tools we developed in the previous section. In this case the covariance matrix Θ, defined in Eq. (5.8), has the form

$$\Theta = \begin{pmatrix} \langle\delta a^\dagger\delta a\rangle + 1/2 & \langle\delta a\delta a\rangle & \langle\delta a\delta b^\dagger\rangle & \langle\delta a\delta b\rangle \\ \langle\delta a^\dagger\delta a^\dagger\rangle & \langle\delta a^\dagger\delta a\rangle + 1/2 & \langle\delta a^\dagger\delta b^\dagger\rangle & \langle\delta a^\dagger\delta b\rangle \\ \langle\delta a^\dagger\delta b\rangle & \langle\delta a\delta b\rangle & \langle\delta b^\dagger\delta b\rangle + 1/2 & \langle\delta b\delta b\rangle \\ \langle\delta a^\dagger\delta b^\dagger\rangle & \langle\delta a\delta b^\dagger\rangle & \langle\delta b^\dagger\delta b^\dagger\rangle & \langle\delta b^\dagger\delta b\rangle + 1/2 \end{pmatrix},$$

and it will satisfy the Lyapunov equation (5.20):

$$\frac{d\Theta}{dt} = W\Theta + \Theta W^\dagger + F.$$

The matrices W and F can be found using the tricks discussed in the previous section. I will simply state the result. The matrix F has two diagonal blocks containing the contributions from each dissipative channel:

$$F = \begin{pmatrix} \kappa\,\mathbb{I}_2 & 0 \\ 0 & \gamma(\bar{n} + 1/2)\,\mathbb{I}_2 \end{pmatrix}.$$

The matrix W, on the other hand, has both a dissipative and a unitary contribution. In fact, the unitary contribution is identical to Eq. (5.27) since our final Hamiltonian H_2 in Eq. (5.66) is structurally identical to the Hamiltonian (5.26). Thus,

$$W = \begin{pmatrix} -i\Delta - \kappa & 0 & ig & 0 \\ 0 & i\Delta - \kappa & 0 & -ig \\ ig & 0 & -i\omega_m - \gamma/2 & 0 \\ 0 & -ig & 0 & i\omega_m - \gamma/2 \end{pmatrix}.$$

It is now a matter of asking the friendly electrons living in our computer to solve for the steady-state:

$$W\Theta + \Theta W^\dagger = -F.$$

As a result we find a CM with the following structure

$$\Theta = \begin{pmatrix} \langle\delta a^\dagger\delta a\rangle + 1/2 & 0 & \langle\delta a\delta b^\dagger\rangle & 0 \\ 0 & \langle\delta a^\dagger\delta a\rangle + 1/2 & 0 & \langle\delta a^\dagger\delta b\rangle \\ \langle\delta a^\dagger\delta b\rangle & 0 & \langle\delta b^\dagger\delta b\rangle + 1/2 & 0 \\ 0 & \langle\delta a\delta b^\dagger\rangle & 0 & \langle\delta b^\dagger\delta b\rangle + 1/2 \end{pmatrix},$$

where

$$\langle\delta a^\dagger\delta a\rangle = \frac{2g^2\gamma\bar{n}(\gamma+2\kappa)}{2g^2(\gamma+2\kappa)^2 + \gamma\kappa[(\gamma+2\kappa)^2 + 4(\Delta-\omega_m)^2]},$$

$$\langle\delta b^\dagger\delta b\rangle = \bar{n} - \frac{4g^2\kappa\bar{n}(\gamma+2\kappa)}{2g^2(\gamma+2\kappa)^2 + \gamma\kappa[(\gamma+2\kappa)^2 + 4(\Delta-\omega_m)^2]},$$

$$\langle\delta a^\dagger\delta b\rangle = \frac{2g\gamma\kappa\bar{n}[2(\Delta-\omega_m) - i(\gamma+2\kappa)]}{2g^2(\gamma+2\kappa)^2 + \gamma\kappa[(\gamma+2\kappa)^2 + 4(\Delta-\omega_m)^2]}.$$

You see, even though we already did a bunch of approximations, we still end up with a rather ugly result.

To clarify the physics, it is useful to assume (as is often the case) that $\gamma \ll \kappa$. In this case the results are more neatly expressed in terms of a quantity called the **cooperativity**:

$$C = \frac{2g^2}{\kappa\gamma}. \tag{5.67}$$

We then get

$$\langle\delta a^\dagger\delta a\rangle = \frac{g^2\bar{n}}{\kappa^2(1+C) + (\Delta-\omega_m)^2}. \tag{5.68}$$

$$\langle\delta b^\dagger\delta b\rangle = \bar{n} - \frac{\bar{n}\kappa^2 C}{(1+C)\kappa^2 + (\Delta-\omega_m)^2}, \tag{5.69}$$

$$\langle\delta a^\dagger\delta b\rangle = \frac{g\bar{n}(\Delta-\omega_m - i\kappa)}{(1+C)\kappa^2 + (\Delta-\omega_m)^2}. \tag{5.70}$$

Now things are starting to look much better.

So let us extra the physics from Eqs. (5.68)-(5.70). We first look at a phenomenon called **sideband cooling**. Namely, we look at the thermal fluctuations of the mechanical mode, Eq. (5.69). As can seen, $\langle\delta b^\dagger\delta b\rangle$ is always *lower* than the sample temperature $\bar{n}$. And we can lower it more by two different paths. The first is by increasing the cooperativity C in Eq. (5.67). This makes sense since C is a type of competition between the coupling g and the damping mechanisms κ and γ. So the higher is the value of C the more strongly coupled are the optical and mechanical modes. Hence, by making the coupling stronger, we can cool the mechanical mode more.

However, making C large is not always an easy task. Instead, another efficient way to make the cooling effect stronger is by playing with $\Delta - \omega_m$. This is something that can be done rather easily since the Detuning Δ is something one usually has great control over. Thus, we see that cooling is maximized in the so-called **side-band cooling condition** $\Delta = \omega_m$. In this case Eqs. (5.68)-(5.70) can be simplified even further to

$$\langle \delta a^\dagger \delta a \rangle = \frac{g^2 \bar{n}}{\kappa^2 (1 + C)}. \tag{5.71}$$

$$\langle \delta b^\dagger \delta b \rangle = \frac{\bar{n}}{1 + C}, \tag{5.72}$$

$$\langle \delta a^\dagger \delta b \rangle = -\frac{ig\bar{n}}{\kappa(1 + C)}. \tag{5.73}$$

Another result that is also more transparent in this case is the fact that the steady-state photon fluctuations are proportional to $\bar{n}$. If the cavity was not coupled to the mechanical mode, the electromagnetic mode would be in a coherent state, which has $\langle \delta a^\dagger \delta a \rangle = 0$. Instead, due to the contact with the mechanical vibration, the occupation increases a bit by a term proportional to both the coupling strength, g^2 and the thermal fluctuations $\bar{n}$.