Bell and Leggett - Garg inequalities

these notes are based on

N. Brunner, et. al., "Bell nonlocality", Rev. Mod. Phys (2014)

C. Emary, et. al., " Leggett - Garg inequalities", Rep. Prog. Phys (2014)

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In these notes we shall discuss the idea of realism:

Realism: the idea that objects have properties which exist independent of observation.

A realist view would then mean that a measurement only reveals a pre-assigned value.

Of course, as we know, that is not the case in quantum mechanics. A wave function that is spread through space does not have a well defined position. Similarly, a spin 1/2 particle in general does not have a well defined spin component in the z direction. these properties are only established due to the act of measurement.

Abandoning a realist view of the world is not easy since the macroworld is realist. Quoting one of Einstein's famous sentences, "the moon is still there even if we don't look at it". Thus, for macroscopic systems, realism should appear as an emergent property. this is the idea of objective reality: for macroscopic objects different observers can agree on a given property of a system. One of the mechanisms to explain this is Zurek's Quantum Darwinism.

Abandoning a realist view also leads to other complications, the most famous one being the notion of locality: if properties are only established due to the act of measurement, then when a measurement is performed on one of two entangled particles, this should affect also the other particle, irrespective of how far from each other they are. this would thus seem to violate causality.

The above argument is originally due to Einstein, Podolski and Rosen (EPR), in 1935. And for 30 years it remained more of a philosophical question since there were no experiments that could test this

Then, in 1964 John Bell proposed a test to verify the locality hypothesis. He proposed an inequality which should be true for any local theory. A violation of this inequality would then imply that QM must indeed be a non-local theory.

Later, in 1985, Leggett and Garg considered another set of inequalities which did not involve locality. Instead, it involved the notion of:

Noninvasive measurability (NIM): the idea that measurements can be performed without disturbing the system.

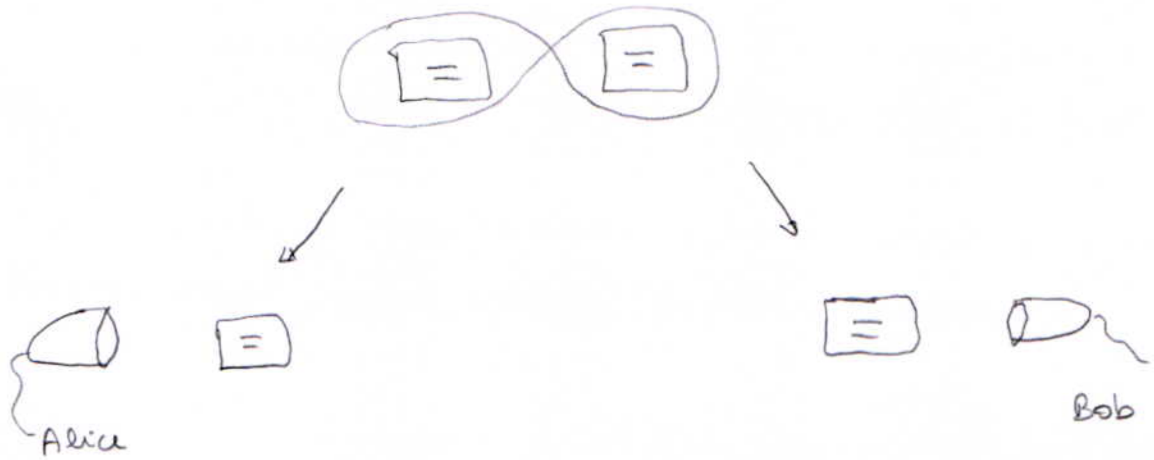
Note how this is intimately related to realism. After all, if a measurement should only reveal a pre-assigned property, then NIM complements the realist view. That is, it could be that properties are pre-assigned, but measurements can never be non-invasive so we can never know the pre-assigned value.

The Leggett-Garg inequality deals with measurements performed on different times (insted of different positions, as in the Bell case). Thus, they are sometimes called temporal Bell inequalities.

Violations of both Bell and Leggett-Garg inequalities have been observed experimentally in a variety of systems, starting in the 1980's with Aspect and Grangier and continuing until today. In fact, this is one of the most active topics of research in quantum information, with amazing new results appearing every week.

Bell scenario

We shall consider here the following typical scenario. Two qubits are prepared in some arbitrary (usually entangled) states, for instance by allowing them to interact. The qubits are then sent to 2 distant labs where Alice and Bob can perform measurements



We assume that Alice and Bob can independently measure the spin component of their qubits so that each will obtain an outcome

$$a \in \{+1, -1\}$$
$$b \in \{+1, -1\}$$

(outcomes of Alice's and Bob's measurements)

However, we also assume that Alice and Bob have the flexibility to choose the direction of the spin component they want to measure

Recall that a direction on the Bloch sphere of a qubit can be specified by the unit vector

$$\vec{x} = (\sin\theta \cos\phi, \ \sin\theta \sin\phi, \ \cos\theta) \qquad (1)$$

The spin component in the direction $\vec{x}$ is then $\vec{x} \cdot \vec{\sigma}$.

We assume that Alice has a knob allowing her to choose an arbitrary direction $\vec{x}$ to measure. Similarly, Bob can choose another direction $\vec{y}$

The above experiment is then repeated many times, from which one constructs a table of joint probability distributions

$p(a,b|x,y)$ = prob. that Alice measures $a \in \{+1,-1\}$ and Bob measures $b \in \{+1,-1\}$ <u>given</u> <u>that</u> Alice measured in the direction $\vec{x}$ and Bob measured in $\vec{y}$.

From these probabilities we will in general find that

$$p(a,b|x,y) \neq p(a|x)\, p(b|y) \qquad (2)$$

That is, the outcomes obtained by Alice are not statistically independent of those obtained by Bob

This statistical dependence, however, does not have to be at all mysterious. They may simply stem from some relation that was established in the past, when they interacted. That is the idea behind a local theory

That is, in a local theory it should be possible to identify a set of past factors, described by some variables λ, which had an influence on the outcomes of both measurements. If we knew λ perfectly well then the outcomes a and b should be independent

$$p(a,b|x,y,\lambda) = p(a|x,\lambda)\, p(b|y,\lambda) \tag{3}$$

Thus, in this case, the correlations between a and b have a joint causal source, λ.

In general, however, λ may have some random fluctuations due to some source of noise that we cannot control. Thus, there will be a certain probability distribution $q(\lambda)$ for the possible outcomes of λ. In this case, we should then get

$$\boxed{p(a,b|x,y) = \int d\lambda \; p(a|x,\lambda)\, p(b|y,\lambda)\, q(\lambda)} \tag{4}$$

This is the condition that any local theory should satisfy. The fun thing to do is then to show that certain quantum mechanical experiments do not admit a decomposition of the form (4)

But before we do so, let me just make a few quick remarks. First, from the joint distribution $p(a,b|x,y)$ we can also compute the marginal distributions

$$p(a|x,y) = \sum_b p(a,b|x,y) \tag{5}$$

This is simply the probability that Alice measures a, irrespective of what Bob gets. It would then be very crazy if this were to depend on y, because it would then imply that the mere conscious choice of Bob as to what he wishes to measure would affect Alice's experiment (No! Quantum Healing is not a real thing! Don't even think about it random internet person!)

Thus, physically we expect that

$$\boxed{p(a|x,y) = p(a|x)} \tag{6}$$

This is called the no-signalling condition. And it can be shown that any local measurements satisfy it.

From the prob. distribution we can of course compute some expectation values. For instance

$$\langle A_x \rangle = \sum_{a,b} a\, p(a,b|x,y) = \sum_a a\, p(a|x) \tag{7}$$

this is simply the average spin component in the direction $\vec{x}$. Or we can consider joint expectation values

$$\langle A_x B_y \rangle = \sum_{a,b} a b\, p(a,b|x,y) \tag{8}$$

With these expectation values we can now consider the most famous of all Bell inequalities, known as the <u>CSCH inequality</u> after Clauser, Horne, Shimony and Holt (1969).

Suppose that x and y can take on two values and label them as $x \in \{0,1\}$ and $y \in \{0,1\}$, just for simplicity. Then consider the quantity

$$S = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \tag{9}$$

We will now show that the locality condition (4) implies that

$$\boxed{S \leqslant 2} \tag{10}$$

Then we will consider a quantum problem which can have $S > 2$, thus violating the inequality.

If we assume (4) is true then

$$\langle A_x B_y \rangle = \sum_{a,b} a\, b\, p(a,b|x,y)$$

$$= \sum_{a,b} \int d\lambda\ a\, b\ p(a|x,\lambda)\, p(b|y,\lambda)\, q(\lambda)$$

If we define

$$\langle A_x \rangle_\lambda = \sum_a a\, p(a|x,\lambda) \tag{11}$$

$$\langle B_y \rangle_\lambda = \sum_b b\, p(b|y,\lambda)$$

Then

$$\langle A_x B_y \rangle = \int d\lambda\ q(\lambda) \langle A_x \rangle_\lambda \langle B_y \rangle_\lambda \tag{12}$$

Thus, S in (9) may be written as

$$S = \int d\lambda\, q(\lambda)\ S_\lambda \tag{13}$$

where

$$S_\lambda = \langle A_0 \rangle_\lambda \langle B_0 \rangle_\lambda + \langle A_0 \rangle_\lambda \langle B_1 \rangle_\lambda + \langle A_1 \rangle_\lambda \langle B_0 \rangle_\lambda - \langle A_1 \rangle_\lambda \langle B_1 \rangle_\lambda \tag{14}$$

Now we use the fact that, since these are spin components, $\langle A_x \rangle_\lambda \in [-1, 1]$ and $\langle B_y \rangle_\lambda \in [-1, 1]$. Then

$$S_\lambda = \langle A_0 \rangle_\lambda \left[\langle B_0 \rangle_\lambda + \langle B_1 \rangle_\lambda \right] + \langle A_1 \rangle_\lambda \left[\langle B_0 \rangle_\lambda - \langle B_1 \rangle_\lambda \right]$$

Every number here now lies between $[-1, 1]$ so that this can never exceed 2. Thus $S_\lambda \leq 2$ and consequently, the same is true for S in Eq (13) since the RHS is a convex sum. We therefore reach Eq (10)

$$\boxed{S = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2} \tag{14}$$

which is the CSCH inequality.

Now let's consider a quantum experiment and show that it can violate (14). Suppose the two qubits are prepared in the Bell state

$$|\psi\rangle = \frac{|0,1\rangle - |1,0\rangle}{\sqrt{2}} \tag{15}$$

(where $|0\rangle, |1\rangle$ is the usual σ_z computational basis).

A measurement on a direction

$$\vec{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{16}$$

is characterized by the projection operators

$$P_{a|x} = |x_a\rangle\langle x_a| \qquad a = \pm 1 \tag{17}$$

where

$$|x_+\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \qquad |x_-\rangle = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \tag{18}$$

Thus, the joint distribution will be given by

$$p(a,b|x,y) = \mathrm{tr}\left\{\rho \, P_{a|x} \otimes P_{b|y}\right\} \tag{19}$$

If we assume for simplicity that $\phi_x = \phi_y = 0$ then the probs (19) can be written quite simply as

$$\boxed{p(a,b|x,y) = \frac{1}{4}\left\{1 - a\, b \cos(\theta_x - \theta_y)\right\}} \tag{20}$$

This means, for instance, that

$$p(a|x) = p(b|y) = 1/2 \qquad \forall a,b. \tag{21}$$

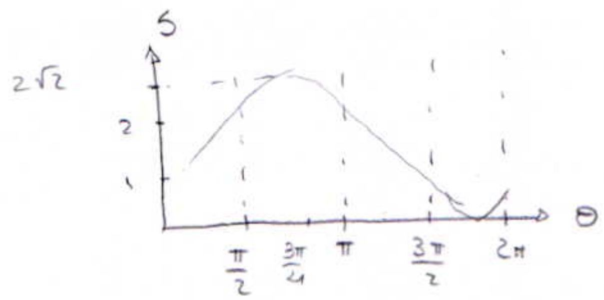
Now choose the two directions for x and y as

$$\begin{aligned} \theta_x &\in \{0, \pi/2\} \\ \theta_y &\in \{-\tfrac{3\pi}{4}, \theta\} \end{aligned} \tag{22}$$

where θ is a free angle. Then the quantity S in (14) becomes

$$S = \sqrt{2} + \sin\theta - \cos\theta \tag{23}$$

which looks like



It has a maximum at $\theta = 3\pi/4$ where

$$\boxed{S = 2\sqrt{2} > 2} \tag{24}$$

Thus, we see that quantum theory predicts a violation of the CSCH inequality. But all that was used to derive the inequality was the idea of locality. Thus, quantum theory is <u>not</u> a local theory.

If one makes an experiment in which S can be found to violate the inequality, then that would give experimental proof that nature cannot be described by a local theory. And that is exactly what has been found. Thus, as strange as it may sound, nature is non-local and non-realist. Measurements do not simply reveal some pre-assigned property. Instead, the property itself is determined by the act of measurement.

Leggett - Garg inequalities

In 1985 Leggett and Garg considered the idea of realism in a different context. Namely, concerning the following question:

"If properties exist independent of observation, is it possible to perform measurements which do not disturb the system?"

More precisely, they considered the predictions of a theory satisfying

1) Realism: properties exist independent of observation
2) Non-invasive measurability (NIM): it is possible, in principle, to determine the state of the system with arbitrarily small perturbations on its dynamics.

In particular, Leggett and Garg were originally more interested in macroscopic realism. That is, they were thinking about whether or not the above ideas could persist even in the classical limit.

The NIM assumption is somewhat tricky because the term "non-invasive" actually refers to a realist theory (which QM is not). To take an example, suppose we are measuring a dichotomic variable $A = \pm 1$, like a spin component. Then suppose we arrange our detector such that it interacts with the system only if $A = +1$

Then the absence of a click allows us to infer that $A=-1$, even though our detector did not interact with the system. If we were to keep only the data for these events, then they would constitute a non-invasive measurement from the point of view of a realist theory, since the apparatus never interacts with the system. But in QM that is not the case and even the absence of interactions still can collapse the wave-function.

To see that, suppose that

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle \tag{25}$$

and consider the generalized measurement operators

$$M_0 = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 0 \end{pmatrix} \qquad M_1 = \begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix} \qquad \lambda \in [0,1] \tag{26}$$

we can think of M_0 as a click and M_1 as a no-click. If $\lambda \ll 1$ then we have a weak measurement since most of the time nothing happens. But even when nothing happens, the state of the system still changes to

$$M_1|\psi\rangle = \sqrt{1-\lambda}\cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$

(up to normalization). Thus, even when nothing happens, the wave function still collapses

The Leggett-Garg inequality imposes a bound on any theory satisfying properties 1 (realism) or 2 (NIM). A violation of the LGI then implies that either realism or NIM must be rejected. We cannot exclude the possibility that realism remains intact and only NIM is rejected, although QM predicts both should be rejected. In fact, a similar problem also exists with Bell's inequalities, in which realism is tested in conjunction with locality. Thus we are always testing realism with something else.

Let's now prove the LGI. The LGIs refer to measurements performed on different times. Let $A = \pm 1$ be a dichotomic variable and suppose we measure it first at t_i then again at t_j. From this we construct a 2-time correlator

$$C_{ij} = \langle A(t_i)\, A(t_j) \rangle \tag{27}$$

In QM we know that performing measurements collapses the state. But if we assume NIM is possible, then constructing such objects should not be a problem since we can simply build up a table of probabilities

$$P(A_i, t_i\, ;\, A_j, t_j) = \text{Prob. of observing } A_i \text{ at time } t_i \text{ and then } A_j \text{ at time } t_j \tag{28}$$

then

$$C_{ij} = \langle A(t_i)\, A(t_j) \rangle = \sum_{A_i, A_j} A_i A_j \, P(A_i, t_i; A_j, t_j) \tag{29}$$

If we now assume a realist theory, then as in the Bell case we attribute an outcome of the first measurement to some pre-assignment λ that exists with some prob. $q(\lambda)$. then outcome A_i will be obtained with prob. $P_i(A_i | \lambda)$. If we further assume NIM then the measurement at time t_j will not have been affected by the first one and will therefore be sampled from the same pre-assignment λ with prob $P_j(A_j | \lambda)$. Thus, just like in the Bell case, we should have

$$\boxed{P(A_i, t_i; A_j, t_j) = \int d\lambda \; q(\lambda)\, P_i(A_i|\lambda)\, P_j(A_j|\lambda)} \tag{30}$$

then

$$C_{ij} = \int d\lambda \; q(\lambda) \, \langle A(t_i) \rangle_\lambda \, \langle A(t_j) \rangle_\lambda \tag{31}$$

Next consider 3 different times t_1, t_2 and t_3, and construct the quantity

$$K_3 = C_{21} + C_{32} - C_{31} \tag{32}$$

Using (31) it then follows that

$$K_3 = \int d\lambda \, q(\lambda) \left[\langle A_2 \rangle \langle A_1 \rangle + \langle A_3 \rangle \langle A_2 \rangle - \langle A_3 \rangle \langle A_1 \rangle \right] \tag{33}$$

But each $\langle A_i \rangle \in [-1, 1]$ so the quantity inside brakets is bounded between -3 and 1. Hence, the same must be true for K_3. We therefore reach the Leggett - Garg inequality

$$\boxed{K_3 = C_{21} + C_{32} - C_{31} \leqslant 1} \tag{34}$$

We can also use a similar reasoning to construct more general inequalities, such as

$$C_{21} + C_{32} + C_{43} - C_{41} \leqslant 2 \tag{35}$$

$$C_{21} + C_{32} + C_{43} + C_{54} - C_{51} \leqslant 3$$

and so on. In all cases we are considering sequential times (t_2, t_1), (t_3, t_2), ..., plus a more exotic combination like (t_4, t_1).

Or, more generally

$$\boxed{K_M = C_{21} + C_{32} + \cdots + C_{M,M-1} - C_{M,1} \leq M-2} \qquad (36)$$

Examples with a qubit

Consider a qubit evolving unitarily according to the Hamiltonian

$$H = \frac{\Omega}{2} \sigma_z \tag{37}$$

and suppose we measure the x component of the magnetization.

Initially the qubit is prepared in an arbitrary state $\rho(0)$. Then, at time t_i the state will be

$$\rho(t_i) = e^{-i\Omega t \sigma_z/2} \rho(0) e^{i\Omega t \sigma_z/2} = \begin{pmatrix} p & q e^{-i\Omega t} \\ q e^{i\Omega t} & 1-p \end{pmatrix} \tag{38}$$

At this stage we perform a projective measurement on the σ_x basis, defined by the projection operators

$$P_+ = |+\rangle\langle+| \qquad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \tag{39}$$

$$P_- = |-\rangle\langle-| \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Let $\sigma = \pm 1$ denote the eigenvalues of σ_x. Then, after this measurement, the system will collapse to the pure state $|\sigma\rangle\langle\sigma|$ with prob.

$$\begin{aligned} P_\sigma(t_i) &= \mathrm{tr}\{P_\sigma \rho(t_i) P_\sigma\} \\ &= \frac{1}{2} + q\sigma \cos(\Omega t_i) \end{aligned} \tag{40}$$

After this first measurement we evolve the system again up to a time t_j, at which point we do another projective measurement in the same basis. the probabilities of each outcome will then be

$$P_{\sigma'|\sigma}(t_j, t_i) = \mathrm{tr}\{P_{\sigma'}\, e^{-i\Omega(t_j-t_i)\sigma_z/2} |\sigma\rangle\langle\sigma| e^{i\Omega(t_j-t_i)\sigma_z/2} P_{\sigma'}\}$$

$$= \delta_{\sigma\sigma'} \cos^2 \frac{\Omega}{2}(t_j - t_i)$$

$$+ \delta_{\sigma,-\sigma'} \sin^2\left(\frac{\Omega(t_j-t_i)}{2}\right) \tag{41}$$

This is a conditional probability of observing σ' at the second measurement given that we obtained σ in the first.

The joint distribution of the two measurements is then

$$P(\sigma', t_j ; \sigma, t_i) = P_{\sigma'|\sigma}(t_j, t_i)\, P_\sigma(t_i) \tag{42}$$

From this we can now compute the 2-point function

$$C_{ij} = \langle \sigma(t_i)\sigma'(t_j)\rangle = \sum_{\sigma,\sigma'} \sigma\sigma' P(\sigma', t_i ; \sigma, t_j)$$

$$= (+1)^2 \cos^2 \frac{\Omega}{2}(t_j - t_i)\left(\frac{1}{2} + q\cos\Omega t_i\right)$$

$$+ (-1)^2 \cos^2\frac{\Omega}{2}(t_j - t_i)\left(\frac{1}{2} - q\cos\Omega t_i\right)$$

$$+ (+1)(-1) \sin^2\frac{\Omega}{2}(t_j - t_i)\left(\frac{1}{2} - q\cos\Omega t_i\right)$$

$$+ (-1)(+1) \sin^2\frac{\Omega}{2}(t_j - t_i)\left(\frac{1}{2} + q\cos\Omega t_i\right)$$

$$= \cos^2 \frac{\Omega}{2}(t_j - t_i) - \sin^2 \frac{\Omega}{2}(t_j - t_i)$$

Thus

$$\boxed{C_{ij} = \cos \Omega (t_j - t_i)} \qquad (43)$$

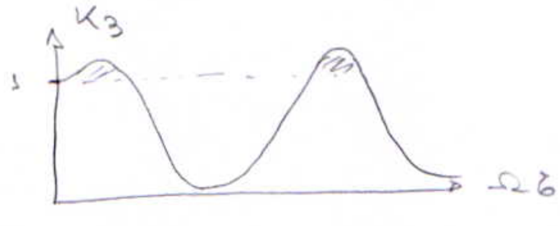
which is independent of the initial condition

Let's now assume equally spaced time intervals, $t_{m+1} - t_m = \tau$, so that

$$K_m = C_{21} + C_{32} + \dots + C_{m,m-1} - C_{m,1}$$

$$K_m = (m-1)\cos \Omega\tau - \cos[\Omega(m-1)\tau] \leq m-2$$

Violations can be observed for certain values of τ



In fact, similar results can be found for any n.