# Bell and Leggett-Garg inequalities

these notes are based on

N. Brunner, et. al., "Bell nonlocality", Rev. Mod. Phys (2014)

C. Emary, et. al., " Leggett-Garg inequalities", Rep. Prog. Phys (2014)

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In these notes we shall discuss the idea of realism:

Realism: the idea that objects have properties which exist independent of observation.

A realist view would then mean that a measurement only reveals a pre-assigned value.

Of course, as we know, that is not the case in quantum mechanics. A wave function that is spread through space does not have a well defined position. Similarly, a spin 1/2 particle in general does not have a well defined spin component in the z direction. these properties are only established due to the act of measurement.

Abandoning a realist view of the world is not easy since the macroworld is realist. Quoting one of Einstein's famous sentences, "the moon is still there even if we don't look at it". Thus, for macroscopic systems, realism should appear as an emergent property. This is the idea of objective reality: for macroscopic objects different observers can agree on a given property of a system. One of the mechanisms to explain this is Zurek's Quantum Darwinism.

Abandoning a realist view also leads to other complications, the most famous one being the notion of locality: if properties are only established due to the act of measurement, then when a measurement is performed on one of two entangled particles, this should affect also the other particle, irrespective of how far from each other they are. This would thus seem to violate causality.

The above argument is originally due to Einstein, Podolski and Rosen (EPR), in 1935. And for 30 years it remained more of a philosophical question since there were no experiments that could test this

Then, in 1964 John Bell proposed a test to verify the locality hypothesis. He proposed an inequality which should be true for any local theory. A violation of this inequality would then imply that QM must indeed be a non-local theory.

Later, in 1985, Leggett and Garg considered another set of inequalities which did not involve locality. Instead, it involved the notion of:

Noninvasive measurability (NIM): the idea that measurements can be performed without disturbing the system.

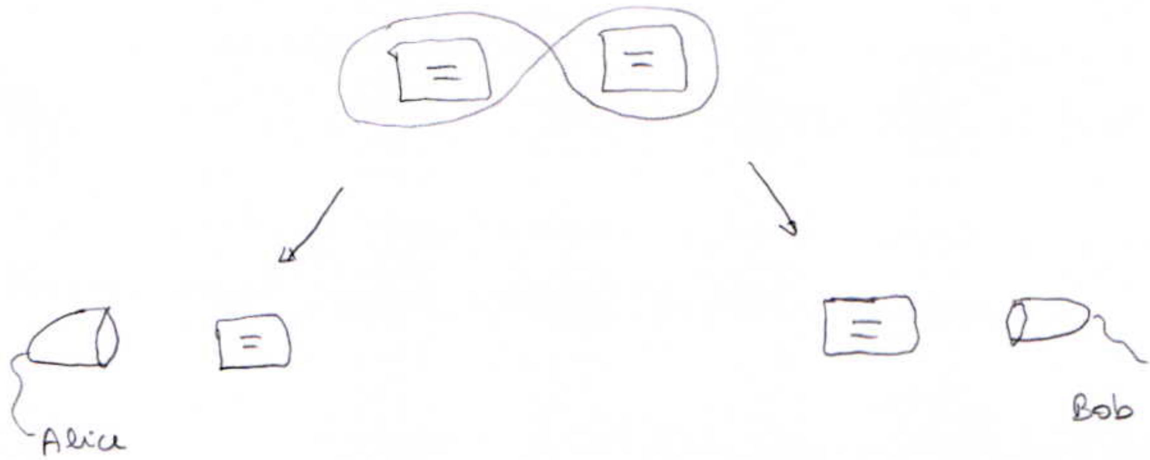
Note how this is intimately related to realism. After all, if a measurement should only reveal a pre-assigned property, then NIM complements the realist view, that is, it could be that properties are pre-assigned, but measurements can never be non-invasive so we can never know the pre-assigned value.

The Leggett-Garg inequality deals with measurements performed on different times (instead of different positions, as in the Bell case). Thus, they are sometimes called temporal Bell inequalities.

Violations of both Bell and Leggett-Garg inequalities have been observed experimentally in a variety of systems, starting in the 1980's with Aspect and Grangier and continuing until today. In fact, this is one of the most active topics of research in quantum information, with amazing new results appearing every week.

## Bell scenario

We shall consider here the following typical scenario. Two qubits are prepared in some arbitrary (usually entangled) state, for instance by allowing them to interact. The qubits are then sent to 2 distant labs where Alice and Bob can perform measurements




We assume that Alice and Bob can independently measure the spin component of their qubits so that each will obtain an outcome

$$a \in \{+1, -1\}$$
$$b \in \{+1, -1\}$$
(outcomes of Alice's and Bob's measurements)

However, we also assume that Alice and Bob have the flexibility to choose the direction of the spin component they want to measure

Recall that a direction on the Bloch sphere of a qubit can be specified by the unit vector

$$\vec{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad (1)$$

the spin component in the direction $\vec{x}$ is then $\vec{x}\cdot\vec{\sigma}$.

We assume that Alice has a knob allowing her to choose an arbitrary direction $\vec{x}$ to measure. Similarly, Bob can choose another direction $\vec{y}$

The above experiment is then repeated many times, from which one constructs a table of joint probability distributions

$p(a,b|x,y)$ = prob. that Alice measures $a \in \{+1,-1\}$ and Bob measures $b \in \{+1,-1\}$ <u>given that</u> Alice measured in the direction $\vec{x}$ and Bob measured in $\vec{y}$.

From these probabilities we will in general find that

$$p(a,b|x,y) \neq p(a|x)\, p(b|y) \qquad (2)$$

That is, the outcomes obtained by Alice are not statistically independent of those obtained by Bob

This statistical dependence, however, does not have to be at all mysterious. They may simply stem from some relation that was established in the past, when they interacted. That is the idea behind a local theory

That is, in a local theory it should be possible to identify a set of past factors, described by some variables $\lambda$, which had an influence on the outcomes of both measurements. If we knew $\lambda$ perfectly well then the outcomes $a$ and $b$ should be independent

$$p(a,b|x,y,\lambda) = p(a|x,\lambda)\,p(b|y,\lambda) \tag{3}$$

Thus, in this case, the correlations between $a$ and $b$ have a joint causal source, $\lambda$.

In general, however, $\lambda$ may have some random fluctuations due to some source of noise that we cannot control. Thus, there will be a certain probability distribution $q(\lambda)$ for the possible outcomes of $\lambda$. In this case, we should then get

$$\boxed{p(a,b|x,y) = \int d\lambda\; p(a|x,\lambda)\,p(b|y,\lambda)\,q(\lambda)} \tag{4}$$

This is the condition that any local theory should satisfy. The fun thing to do is then to show that certain quantum mechanical experiments do not admit a decomposition of the form (4)

But before we do so, let me just make a few quick remarks. First, from the joint distribution $p(a,b|x,y)$ we can also compute the marginal distributions

$$p(a|x,y) = \sum_b p(a,b|x,y) \tag{5}$$

This is simply the probability that Alice measures $a$, irrespective of what Bob gets. It would then be very crazy if this were to depend on $y$, because it would then imply that the mere conscious choice of Bob as to what he wishes to measure would affect Alice's experiment (No! Quantum Healing is not a real thing! Don't even think about it random internet person!)

Thus, physically we expect that

$$\boxed{p(a|x,y) = p(a|x)} \tag{6}$$

This is called the no-signalling condition. And it can be shown that any local measurements satisfy it.

From the prob. distribution we can of course compute some expectation values. For instance

$$\langle A_x \rangle = \sum_{a,b} a\, p(a,b|x,y) = \sum_a a\, p(a|x) \tag{7}$$

this is simply the average spin component in the direction $\vec{x}$. Or we can consider joint expectation values

$$\langle A_x B_y \rangle = \sum_{a,b} a b\, p(a,b|x,y) \tag{8}$$

With these expectation values we can now consider the most famous of all Bell inequalities, known as the CSCH inequality after Clauser, Horne, Shimony and Holt (1969).

Suppose that $x$ and $y$ can take on two values and label them as $x \in \{0,1\}$ and $y \in \{0,1\}$, just for simplicity. Then consider the quantity

$$S = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \tag{9}$$

We will now show that the locality condition (4) implies that

$$\boxed{S \leqslant 2} \tag{10}$$

then we will consider a quantum problem which can have $S > 2$, thus violating the inequality.