Wigner Entropy Production Rate

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The characterization of irreversibility in general quantum processes is an open problem of increasing technological relevance. Yet, the tools currently available to this aim are mostly limited to the assessment of dynamics induced by equilibrium environments, a situation that often does not match the reality of experiments at the microscopic and mesoscopic scale. We propose a theory of irreversible entropy production that is suited for quantum systems exposed to general, nonequilibrium reservoirs. We illustrate our framework by addressing a set of physically relevant situations that clarify both the features and the potential of our proposal.

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Introduction.—The entropy of an open system, unlike the energy, does not satisfy a continuity equation: in addition to entropic fluxes exchanged with the environment, some entropy may also be produced within the system. This contribution is called the entropy production and, according to the second law of thermodynamics, it is always non-negative, being zero only when the system and the environment are in thermal equilibrium. It therefore serves as a measure of the irreversibility of a physical process and may be used to characterize nonequilibrium systems in a broad range of situations and across all length scales. In symbols, if S is the entropy of the system, then its rate of change may always be written as

$$\frac{dS}{dt} = \Pi(t) - \Phi(t), \tag{1}$$

where $\Pi \geq 0$ is the entropy production rate and Φ is the entropy flux rate, from the system to the environment. The quantities Π and Φ are not direct observables and must therefore be related to experimentally accessible quantities via a theoretical framework. Unfortunately, a unified approach for this is still lacking.

In the past decades, several theories of entropy production have been developed in different contexts. The most prominent example is Onsager's theory of chemical kinetics [1–4], where the entropy production rate is related to particle and energy currents. Another widely used framework is that of Schnakenberg [5,6], which relates the entropy production rate to the transition rates of a system governed by a master equation. The generalization to other classical stochastic processes, such as dynamics described by a Fokker-Planck equation, have also been addressed [7–9].

The extension of these results to mesoscopic systems came into relevance with the discovery by Gallavotti, Cohen, and collaborators [10,11] that the total entropy production Σ of a process, when interpreted as a fluctuating quantity of the system's stochastic trajectory, satisfies a fluctuation theorem of the form $\langle e^{-\Sigma} \rangle = 1$, which is valid for processes arbitrarily far from equilibrium. Similar results were found by Jarzynski [12] and Crooks [13] for systems undergoing a work protocol, where the entropy production is proportional to the irreversible work. These developments and, in particular, their extensions to quantum systems, have shown that in meso- and microscopic systems, quantum fluctuations may play a prominent role in nonequilibrium processes.

Quantum systems also open up the possibility for exploring more general reservoirs, such as dephasing and squeezed baths [14]. The description of these systems extends beyond the usual paradigms of equilibrium environments. Despite the lack of equilibrium at the bath level, one should still be able to characterize processes by their irreversibility and entropy production. There is thus a strong need for the identification of suitable tools that are able to characterize nonequilibrium processes in a broad class of settings.

The goal of this Letter is to derive a theory of entropy production that is applicable to quantum systems subject to more general reservoirs. Differently from existing theories, instead of using the von Neumann entropy $S_{\rm vN} = -{\rm tr}(\rho \ln \rho)$, we shall characterize the irreversibility using the Rényi-2 entropy $S_2 = -\ln {\rm tr}\rho^2$, where ρ is the density matrix of the system. Both entropies have a similar behavior when used to characterize disorder. However, the Rényi-2 entropy is much more convenient to manipulate, since it is simply related to the purity ${\rm tr}\rho^2$ of the state. Recently, there have been several papers linking the general Rényi-α entropies $S_\alpha = (1-\alpha)^{-1} \ln {\rm Tr}(\rho^\alpha)$ to the thermodynamic properties of quantum systems, from the formulation of general fluctuation theorems to the derivation of a family of second laws of thermodynamics 17–15]]. Remarkably, all Rényi-α entropies tend asymptotically to the von Neumann one in the classical limit, corroborating their use in reformulating the theory of thermodynamic irreversibility. The subtleties implied by the differences between the von Neumann and Rényi entropies have been stressed in Ref. [18].

In this Letter, we shall focus on bosonic systems characterized by Gaussian states. In this case, the expression for S_2 is written simply as $S_2 = \frac{1}{2}\ln|\Theta|$, where Θ is the covariance matrix [19]. Moreover, as shown in Ref. [20], S_2 coincides up to a constant with the Wigner entropy

$$S = -\int d^2\alpha W(\alpha^*, \alpha) \ln W(\alpha^*, \alpha), \tag{2}$$

where $W(\alpha^*, \alpha)$ is the Wigner function and the integral is over the complex plane (as the state is Gaussian, $W > 0$ and hence S is real). This link between S and S_2 allows for a fundamental simplification of the problem of characterizing entropy production, as one can map the open system dynamics into a Fokker-Planck equation for W and hence employ tools of classical stochastic processes to obtain simple expressions for Π and Φ. This idea was already used in Refs. [21,22] via a quantum-to-classical correspondence to treat the case of simple heat baths. Here, instead, we present a full quantum mechanical treatment and show how to extend the framework to treat squeezed and dephasing reservoirs. The generalization to other types of baths is straightforward.

We shall assume that the system is modeled by a Lindblad master equation of the form

$$\partial_t \rho = -i[H, \rho] + \mathcal{D}(\rho), \tag{3}$$

where ρ is the density matrix of the system, H is its Hamiltonian, and $\mathcal{D}(\rho)$ describes the process arising from its coupling to the external reservoir. Let ρ^* denote the target state of $\mathcal{D}(\rho)$ (for thermal baths $\rho^* = \rho_{\rm eq} = e^{-\beta H}/Z$). In Refs. [23–26], it was shown that the von Neumann entropy production rate can be defined as

$$\Pi_{\rm vN} = -\partial_t K_{\rm vN}(\rho|\rho^*), \tag{4}$$

where $K_{\rm vN}(\rho|\rho^*) = {\rm tr}[\rho \ln(\rho/\rho^*)]$ is the von Neumann relative entropy. Equation (4) satisfies several properties expected from an entropy production. First, $\Pi_{\rm vN} \geq 0$, with the equality holding only for $\rho = \rho^*$. Second, for thermal baths, the corresponding total entropy production, when interpreted as a stochastic quantity, satisfies an integral fluctuation theorem [26]. Finally, Eq. (4) may be factored in the form of Eq. (1), with $S = S_{\rm vN}$ and

$$\Phi_{\rm vN}(t) = -\frac{1}{T}{\rm tr}[H\mathcal{D}(\rho)] \coloneqq \frac{\Phi_E}{T}, \tag{5}$$

where Φ_E is the energy flux from the system to the environment. This is a well known result of classical thermodynamics, relating heat and entropy flux.

Despite their clear physical interpretation, Eqs. (4) and (5) suffer from the problem that they diverge in the limit $T \to 0$. This is related to the divergence of the relative entropy when the reference state tends to a pure state [27,28]. This divergence is clearly an inconsistency of the theory. The limit $T \to 0$ is frequently used in quantum optics and the dynamics is known to be well behaved and to correctly reproduce experimental results in several situations. In fact, even dS/dt remains finite in this limit, and only Π and Φ diverge. In the past, several attempts have been made to overcome this problem [27–32] but a consistent theory is still lacking. To obtain a framework that does not suffer from this deficiency is another motivation for this Letter. As we will show, using the Rényi-2–Wigner entropy avoids this problem entirely.

Thermal bath.—We begin the construction of our formalism by considering a single bosonic mode with $H = \omega(a^\dagger a + 1/2)$ and dissipator

$$\mathcal{D}(\rho) = \gamma(\bar{n}+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right] + \gamma\bar{n}\left[a^\dagger \rho a - \frac{1}{2}\{a a^\dagger, \rho\}\right]. \tag{6}$$

Here, γ is the damping rate of the oscillator and $\bar{n} = (e^{\beta\omega} - 1)^{-1}$ is the mean number of excitations in the bath ($\beta = 1/T$ is its inverse temperature). The target state of this dissipator is the Gibbs thermal state $\rho^* = \rho_{\rm eq} = (1 - e^{-\beta\omega})e^{-\beta\omega a^\dagger a}$.

We define the Wigner function of the system as

$$W(\alpha^*, \alpha) = \frac{1}{\pi^2}\int d^2\lambda e^{-\lambda\alpha^* + \lambda^*\alpha}{\rm tr}\{\rho e^{\lambda a^\dagger - \lambda^* a}\}, \tag{7}$$

where λ and α are phase space variables. Using standard operator correspondences, Eq. (3) can be translated into the Fokker-Planck equation

$$\partial_t W = -i\omega[\partial_{\alpha^*}(\alpha^* W) - \partial_\alpha(\alpha W)] + \mathcal{D}(W), \tag{8}$$

where the dissipative part is written as a divergence in the complex plane:

$$\mathcal{D}(W) = \partial_\alpha J(W) + \partial_{\alpha^*} J^*(W) \tag{9}$$

with

$$J(W) = \frac{\gamma}{2}[\alpha W + (\bar{n} + 1/2)\partial_{\alpha^*} W]. \tag{10}$$

Equation (8) is a continuity equation in the complex plane. Hence, $J(W)$ can be interpreted as the irreversible component of the probability current. This picture is further corroborated by the fact that $J(W)$ will be zero only in the thermal state $W_{\rm eq} = [1/\pi(\bar{n}+1/2)]\exp[-|\alpha|^2/(\bar{n}+1/2)]$; i.e., $J(W_{\rm eq}) = 0$. This statement is stronger than $\mathcal{D}(W_{\rm eq}) = 0$ as it implies that the thermal equilibrium state is not only a fixed point of the dissipative dynamics, but also the state where all probability currents vanish identically.

Having defined the Wigner entropy as in Eq. (2), we now define the Wigner entropy production rate as

$$\Pi = -\partial_t K(W(t)||W_{\rm eq}), \tag{11}$$

where $K(W||W_{\rm eq}) = \int d^2\alpha W \ln W/W_{\rm eq}$ is the Wigner relative entropy. For a bipartite Gaussian state, this coincides (up to a constant) with the Rényi-2 mutual information [20]. Inserting the Fokker-Planck Eq. (8) in Eq. (11) and integrating by parts we get

$$\Pi = -\int d^2\alpha \mathcal{D}(W) \ln(W/W_{\rm eq}). \tag{12}$$

Next, we use Eq. (9) and integrate by parts again to obtain

$$\Pi = \int d^2\alpha \left\{ J\left(\frac{\partial_\alpha W}{W} - \frac{\partial_\alpha W_{\rm eq}}{W_{\rm eq}}\right) + \alpha \to \alpha^* \right\}. \tag{13}$$

Finally, one notes that, from Eq. (10),

$$\frac{\partial_\alpha W}{W} - \frac{\partial_\alpha W_{\rm eq}}{W_{\rm eq}} = \frac{2J^*}{\gamma(\bar{n}+1/2)}\frac{1}{W}. \tag{14}$$

Therefore, we conclude that the entropy production rate may be written as

$$\Pi = \frac{4}{\gamma(\bar{n}+1/2)}\int d^2\alpha \frac{|J(W)|^2}{W}. \tag{15}$$

This quantity is always non-negative (as we take $W > 0$) and null only at thermal equilibrium, which are precisely the properties expected from an entropy production rate.

Going back to Eq. (12), the term proportional to $\mathcal{D}(W)\ln W$ is precisely dS/dt, with S defined in Eq. (2). Hence, comparing with Eq. (1) we find that the remainder must be the entropy flux rate

$$\Phi = \int d^2\alpha \mathcal{D}(W) \ln W_{\rm eq} = \frac{\gamma}{\bar{n}+1/2}\int d^2\alpha |\alpha|^2 W - \gamma,$$

where, in the last line, we integrated by parts and substituted the formulas for $\mathcal{D}(W)$ and $W_{\rm eq}$. Since $\int d^2\alpha|\alpha|^2 W = \langle a^\dagger a\rangle + 1/2$ we finally conclude that

$$\Phi = \frac{\gamma}{\bar{n}+1/2}(\langle a^\dagger a\rangle - \bar{n}). \tag{16}$$

Equations (15) and (16) are the main results for the Wigner entropy production and entropy flux rate. Equation (16) in particular is very useful, as it relates the entropy flux rate to a simple expectation value.

On the other hand, the energy flux rate may be computed from Eq. (3) and reads $\Phi_E = \gamma\omega(\langle a^\dagger a\rangle - \bar{n})$. We thus conclude that the entropy flux rate and the energy flux rate are related by

$$\Phi = \frac{\Phi_E}{\omega(\bar{n}+1/2)}. \tag{17}$$

When $T \gg \omega$ we may approximate $\omega(\bar{n}+1/2) \simeq T$, in which case we recover the traditional formula $\Phi \simeq \Phi_E/T$ [Eq. (5)]. Thus, Eq. (16) recovers the expected result at high temperatures. In addition, it tends to a finite value as $T \to 0$. Hence, as mentioned above, within the Wigner entropy formulation, both Π and Φ remain well behaved in the limit $T \to 0$.

We have opted to derive Eqs. (15) and (16) starting from the Wigner relative entropy, since this gives the most natural physical interpretation. In the Supplemental Material [33] we provide two alternative derivations of these formulas. The first is through a simple algebraic manipulation, which makes no mention at all of the relative entropy or the target state $W_{\rm eq}$. It may therefore be useful in situations where one does not know the target state of the dissipator *a priori*.

The second method is to map the Fokker-Planck Eq. (8) into a stochastic process in the complex plane. In this way, the total entropy production Σ of a process may be defined as a functional of the stochastic forward and backward trajectories. The entropy production rate is then obtained by averaging the stochastic entropy over an infinitesimal time interval, $\langle\Sigma\rangle = \Pi dt$, where $\langle\cdot\rangle$ stands for the average over all stochastic paths. The interesting aspect of this approach is that, as we show, Σ satisfies an integral fluctuation theorem, which is the fundamental property expected of the entropy production. This supports the interpretation of Eq. (15) as a valid entropy production rate.

Squeezed bath.—We now generalize the above results to the case of a bosonic mode subject to a nonequilibrium broadband squeezed bath. This type of reservoir appears frequently in quantum optics [14,34–38], whenever the squeeze radiation field is treated as an external bath. The bath is characterized by a temperature T (usually zero), a squeeze parameters $re^{i\theta}$, and a central frequency ω_s, related to the production of the squeezed field (usually by parametric down conversion).

The dissipator of the squeezed bath may be written in terms of the squeezed operators $b_z = S(z)aS^\dagger(z)$, where $S(z) = e^{(z^*a^2 - za^{\dagger 2})/2}$ and $z = re^{i(\theta - 2\omega_s t)}$. With these definitions, the formula for the squeezed Lindblad superoperator becomes identical to Eq. (6), with a replaced by b_z. Because of this correspondence, all results obtained above for the thermal bath remain valid for the squeezed bath, provided the calculations are all carried out in terms of the operators b_z instead of a. This allows us to readily write down the analogs of Eqs. (15) and (16) as

$$\Pi = \frac{4}{\gamma(\bar{n}+1/2)}\int d^2\beta \frac{|J_b(W)|^2}{W}, \qquad \Phi = \frac{\gamma(\langle b_z^\dagger b_z\rangle - \bar{n})}{\bar{n}+1/2}, \tag{18}$$

where $J_b(W)$ is defined exactly as in Eq. (10), but with β instead of α. As b_z and a are related by a unitary transformation, the Jacobian of the transformation from β to α is unity. Moreover, a straightforward calculation shows that

$$J_b(W) = J(W)\cosh r + [\gamma\alpha^* W - J^*(W)]e^{i(\theta - 2\omega_s t)}\sinh r. \tag{19}$$

With these transformations and $b_z = S(z) a S^\dagger(z)$, it is possible to express both Π and Φ solely in terms of quantities linked to a and $a^\dagger$.

To illustrate the new effects brought about by the squeezing of the bath, consider a cavity with frequency ω_c pumped by a radiation field with frequency ω_p and squeezed central frequency ω_s. The Hamiltonian describing the cavity mode is

$$H = \omega_c a^\dagger a + i(\mathcal{E} e^{-i\omega_p t} a^\dagger - \mathcal{E}^* e^{i\omega_p t} a), \qquad (20)$$

where $|\mathcal{E}| = \sqrt{2P\kappa/\hbar\omega_p}$ with P being the pump laser power and $\kappa = \gamma/2$ the cavity amplitude decay rate. The contact with the squeezed reservoir is modeled exactly by the Lindblad superoperator (6) with $a \to b_z$, $\gamma = 2\kappa$, and $\bar{n} = 0$. Because of the Gaussian nature of the problem, all calculations are straightforward [cf. the Supplemental Material [33]]. Here, we only emphasize the final result. First, the steady-state energy flux is given by

$$\Phi_E = \left\langle \frac{\partial H}{\partial t} \right\rangle = \frac{2\kappa\omega_p |\mathcal{E}|^2}{\kappa^2 + \Delta_{cp}^2}, \qquad (21)$$

where $\Delta_{ij} = \omega_i - \omega_j$. The heat current will thus be nonzero only in the presence of the pump. Second, at the steady state we have $dS/dt = 0$, so that $\Pi = \Phi$ and

$$\Pi = \frac{2\kappa\Delta_{sc}^2}{\kappa^2 + \Delta_{sc}^2} \sinh^2(2r) + \frac{4\kappa|\mathcal{E}|^2}{\kappa^2 + \Delta_{cp}^2} \cosh(2r) + 4\kappa \mathrm{Re}\left[\frac{\mathcal{E}^2 e^{-i(2\Delta_{ps} t + \theta)}}{(\kappa + i\Delta_{cp})^2}\right] \sinh(2r). \qquad (22)$$

If $\omega_p \neq \omega_s$, in a time-averaged picture the last term will oscillate in time with zero average and may thus be neglected. In the limit of zero squeezing ($r \to 0$) only the second term survives and we find that $\Phi = \Phi_E/2$. The important part of Eq. (22), however, is the first term, which would still be present even for no pumping ($\mathcal{E} = 0$). This term reflects the contribution coming from the nonequilibrium nature of the bath (encompassed by the degree of squeezing), and the one resulting from the mismatch between the central frequency ω_s of the broadband squeezed bath and the natural frequency ω_c of the cavity (which induces off-resonant exchanges of excitations between the cavity and the bath that are not present in the resonant case). We interpret this term as a signature of an (irreversible) nonequilibrium steady state that will occur even in the absence of a pump.

It is remarkable that this information is not present in the energy flux rate (21), but only in the entropy production. This thus provides a clear exception to the usual assumption that nonequilibrium steady states always have an associated energy current. In addition, our formulation reveals a genuinely quantum effect, and one that in principle could be measured experimentally. Similar counterintuitive results have been reported for the efficiency of quantum Carnot cycles under squeezed reservoirs [39]. We can also analyze this effect from the viewpoint of the irreversible current $J_b(W)$ appearing in Eq. (18). Using the results detailed in the Supplemental Material [33], one may readily show that for $\mathcal{E} = 0$

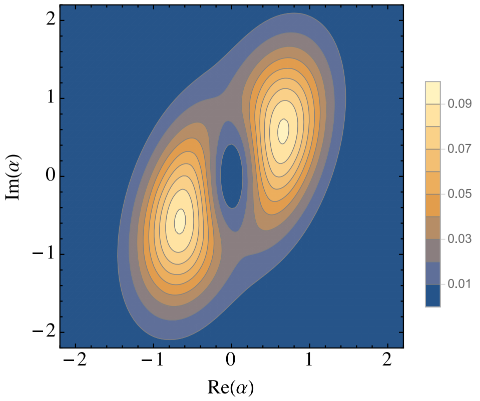

FIG. 1. $|J_b|^2/W$ as a function of α, computed using Eq. (23) with $\Delta_{cs}/\kappa = 0.9$, $r = 0.5$, and $\theta - 2\omega_s t = 0.0$.

$$\frac{|J_b(W)|^2}{W^2} = \frac{\kappa^2 \Delta_{sc}^2 \sinh^2(2r)}{\kappa^2 + \Delta_{cs}^2 \cosh^2(2r)} |\beta|^2, \qquad (23)$$

where $\beta = \alpha \cosh r + \alpha^* e^{i(\theta - 2\omega_s t)} \sinh r$. Thus, the magnitude of the current will be zero when either $\Delta_{sc} = 0$ or $r = 0$. Figure 1 shows $|J_b|^2/W$ against α.

Dephasing bath.—Finally, we turn to the problem of a dephasing bath, characterized by the Lindblad superoperator

$$\mathcal{D}_{\mathrm{deph}}(\rho) = \lambda\left[a^\dagger a \rho a^\dagger a - \frac{1}{2}\{(a^\dagger a)^2, \rho\}\right]. \qquad (24)$$

The action of the environment is to suppress quantum coherences without the exchange of energy with the system, so that $\Phi_E = 0$. The corresponding operator in Wigner space reads $\mathcal{D}_{\mathrm{deph}}(W) = \partial_\alpha I(W) + \partial_{\alpha^*} I^*(W)$, where $I(W) = \lambda\alpha[\alpha^* \partial_{\alpha^*} W - \alpha \partial_\alpha W]/2$. The target state of this dynamics is not unique, as any Gibbs thermal state will be a target state. Using Eq. (11), we find

$$\left.\frac{dS}{dt}\right|_{\mathrm{deph}} = \Pi_{\mathrm{deph}} = \frac{2}{\lambda} \int \frac{d^2\alpha}{W} \frac{|I(W)|^2}{|\alpha|^2}. \qquad (25)$$

Clearly, for such a dephasing bath the entropy flux Φ is null, which agrees intuitively with the idea that the energy flux will also be zero, and demonstrates the suitable nature of the approach that we have proposed.

Conclusions.—We have addressed the difficulty of studying irreversibility in a general quantum process incorporating an out-of-equilibrium environment. While relevant, experimentally and technologically, in a number of physical situations (especially in solid-state quantum information processing), the successful addressing of this problem has so far been hindered by the lack of formal tools suited to encompass the complexity of the effects arising from the environment. We have put forward a new, alternative picture for irreversible entropy production based on the use of the Rényi-2 entropy, which is able to address the open-system dynamics of a quantum system in contact with nonequilibrium reservoirs in a successful way. Three independent methods of obtaining the entropy production rate were provided, which serves to corroborate the generality of our approach. Whether it is possible to generalize this theory to arbitrary Rényi-α entropies remains an open question. The illustrations that we have discussed, including squeezed and dephasing baths, show both the potential of the proposed approach and the breath of physically relevant situations that it is able to address. We have opted to focus our approach on a single bosonic mode. The generalization to a multimode process is straightforward and will be the subject of a future publication.

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