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Lindblad dynamics of the quantum spherical model

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Abstract. The purely relaxational non-equilibrium dynamics of the quantum spherical model as described through a Lindblad equation is analysed. It is shown that the phenomenological requirements of reproducing the exact quantum equilibrium state as stationary solution and the associated classical Langevin equation in the classical limit $g \rightarrow 0$ fix the form of the Lindblad dissipators, up to an overall time-scale. In the semi-classical limit, the models' behaviour becomes effectively the one of the classical analogue, with a dynamical exponent $z = 2$ indicating diffusive transport, and an effective temperature T_{eff} , renormalised by the quantum coupling g . A different behaviour is found for a quantum quench, at zero temperature, deep into the ordered phase $g \ll g_c(d)$, for $d > 1$ dimensions. Only for $d = 2$ dimensions, a simple scaling behaviour holds true, with a dynamical exponent $z = 1$ indicating ballistic transport, while for dimensions $d \neq 2$, logarithmic corrections to scaling arise. The spin-spin

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correlator, the growing length scale and the time-dependent susceptibility show the existence of several logarithmically different length scales.

Keywords: dissipative systems, quantum criticality, quantum dissipative systems, solvable lattice models

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1. Introduction

The statistical mechanics of non-equilibrium open system continues to pose many challenges, related to the absence of a unified framework for their formulation. Here, we shall be concerned with non-equilibrium relaxations of open *quantum* systems. In the vicinity of equilibrium, linear-response theories such as the Kubo formula or the Landauer-Büttiker formalism may be used [49, 51, 56]. But such approaches cannot describe the system's behaviour far from equilibrium, for instance after a quench from one physical phase into another. Studies on the physical ageing of glassy and non-glassy systems after such quenches have led to a precise understanding of the associated phenomena and in particular have made it clear that the competition between several distinct, but equivalent, equilibrium states may prevent the system to relax to an equilibrium state at all, even if the microscopic dynamics does satisfy detailed balance [23, 43, 44, 57, 58, 80].

Often-used phenomenological approaches to classical dissipative systems include master equations for the probability distributions or Langevin equations for the observables. Various types of critical dynamics have been identified [11, 46, 58, 80]. Here, we shall concentrate on purely relaxational dynamics, often referred to as *model-A dynamics*. A major distinction of *quantum systems* with respect to classical ones is the presence of a conjugate momentum p_n for each classical observable s_n , both to be considered as operators, such that canonical commutation relations $[s_n, p_m] = i\hbar\delta_{n,m}$ hold true. From the point of view of a phenomenological classical description, this raises the

requirement to re-formulate the dynamics in such a way that these prescribed conservation laws should be obeyed. Therefore, simplistic approaches such as phenomenological Kramers equations, for the observables s_n and the momenta p_m , supplemented by phenomenological damping terms, are inadequate, since they lead to the violation of the canonical commutation relations, on time-scales of the order of the inverse damping rate, such that an effectively classical dynamics remains [17].

The open-system dynamics of a quantum system is most ideally studied using the concept of dynamical semi-groups and completely positive trace-preserving (CPTP) dynamics. In the Heisenberg picture, this may be implemented using the tools of quantum Langevin equations [37]. Conversely, in the Schrödinger picture, that is most readily accomplished using Lindblad master equations [10, 32, 55].

Formally, the Lindblad equation preserves the trace, the hermiticity and the positivity of the reduced density matrix ρ . On the other hand, it is not considered straightforward to write down explicit expressions for the Lindblad dissipators for generic many-body systems, although well-established formalisms exist for few-body systems, see e.g. [3, 4, 10, 74, 83, 85]. Finally, if such expressions have been obtained, actually solving a Lindblad equation is still far from obvious. Some results exist for one- or two-body problems, see [10, 74, 85]. For *fermionic* many-body *chains*, exact solutions have been found by establishing relationships with $1D$ quantum integrability, see [50, 66, 67] and [68] for a recent review. Indeed, integrable models are relevant for the understanding of a large range of experiments, see [6] for a recent review. But by their very mathematical nature, such techniques are limited to one-dimensional systems.

In order to provide insight beyond purely numerical studies, a versatile and non-trivial exactly solvable model is sought. In equilibrium statistical mechanics, the so-called *spherical model* of a ferromagnet (see section 2 for the precise definition) [7, 53] has since a long time served for such purposes. In the classical formulation with ferromagnetic nearest-neighbour interactions, it undergoes a continuous phase transition at a critical temperature $T_c > 0$ for spatial dimensions $d > 2$ (d can be treated as a continuous parameter). For $2 < d < 4$ dimensions, the critical exponents are distinct from those of mean-field theory. The standard formulation in terms of classical spins has the drawback that the third fundamental theorem of thermodynamics is not obeyed, since the specific heat $c_h = 1$ for temperatures $T < T_c$ [7]. This can be cured however, by adjoining to each spin variable s_n a canonically conjugate momentum p_n and adding a kinetic energy term, with a quantum coupling g , to the Hamiltonian H , thus arriving at the *quantum spherical model* (QSM) [61]. Then the specific heat c_h vanishes indeed as $T \rightarrow 0$, as it should be [60, 62, 73, 81]. The model's properties near the critical temperature $T_c(g) > 0$ are the same as in the classical spherical model. However, at temperature $T = 0$, there is for $d > 1$ dimensions a quantum critical point, at some $g = g_c > 0$, which is in the same universality class as the classical model in $d + 1$ dimensions [9, 29, 41, 52, 60, 62, 73, 78, 81, 82]. The formulation of the spherical model contains the so-called ‘spherical constraint’. The exact solution of the model reduces to establishing the constraint equation for the associated Lagrange multiplier which at equilibrium must be found from the solution of a transcendent equation. Turning to the dynamics, the kinetics of the classical spherical model can be described in terms of a Langevin equation, such that the spherical constraint reduces to a Volterra integral equation for the now time-dependent Lagrange multiplier [38, 72]. Many aspects of the

non-equilibrium dynamics of the model have been analysed in great detail, including extensions to the spherical spin glass and to the growth of interfaces [11, 20, 22, 27, 34, 35, 36, 38, 45, 64, 72, 80].

Here, we shall explore aspects of the non-equilibrium *quantum* dynamics of the QSM. In order to construct the Lindblad dissipators, we shall require that these are chosen such that (i) the correct quantum equilibrium state emerges as the stationary state of the dynamics and (ii) in the classical limit $g \rightarrow 0$, the correct classical Langevin dynamics should be recovered⁷. As we shall see, this fixes the form of the Lindblad dissipators, up to the choice of an overall time scale. To do so, we recall in section 2 the definition of the quantum spherical model and the main properties of its equilibrium phase diagram. In section 3, the Lindblad dissipators will be constructed in two different ways. First, we shall follow the traditional route of system-plus-reservoir methods [10, 74]. Inspired by recent constructions of free bosonic quantum systems [39, 75], we shall give an explicit description for the phonons which make up the reservoir. We also discuss how this construction must be amended to take the spherical constraint into account. In section 4, we derive the associated equations of motion for the observables. Independently of any specific model for the reservoir, we shall show how a comparison with the classical limit $g \rightarrow 0$ (whenever available) determines the form of the Lindblad dissipators. This also clarifies further the interpretation of the phonon reservoir model. The formal closed-form solution for spin- and momentum-correlators will be derived. The most difficult part of any spherical-model calculation is the solution of the spherical constraint, which becomes in our case a highly non-trivial integro-differential equation. Since a full solution of this equation is very difficult, we shall focus on two special cases. First, in section 5, we analyse the semi-classical limit, which can be used to describe the leading quantum correction to the order-disorder phase transition at temperature $T = T_c(g)$. By construction, the Lindblad equation does preserve quantum coherence. Still, we find from the explicitly computed spin-spin correlator that to leading order in g , the dynamical critical behaviour, for temperatures $T > T_c(g)$, $T = T_c(g)$ or $T < T_c(g)$ is exactly the one of the classical (purely relaxational model-A) dynamics, where quantum effects only manifest themselves through the appearance of a new effective temperature $T \mapsto T_{\text{eff}}(g)$. For quenches to $T \leq T_c(g)$, dynamical scaling holds with a dynamical exponent $z = 2$, which indicates diffusive motion of the basic degrees of freedom. Having thus confirmed the consistency of the Lindblad formalism applied to the quantum spherical model, we analyse in section 6 what happens for a quantum quench deeply into the ordered phase, through an exact analysis of the leading long-time and large-distance behaviour of the spin-spin and momentum-momentum correlators. We find a very rich behaviour which subtly depends on the spatial dimension d . Indeed, for $d = 2$, simple dynamical scaling holds true, while for $d > 2$, several logarithmically different time-dependent length scales appear, which implies a multi-scaling phenomenology. This leading behaviour is independent of the damping γ and the limit $\gamma \rightarrow 0$ of closed quantum systems can be taken. For $d < 2$, logarithmic corrections to scaling appear as well, but are of a different nature since the model's behaviour now depends on γ . The dynamical exponent is always $z = 1$, up to eventual logarithmic corrections, indicative of ballistic motion.

⁷ This classical dynamics is in the universality class of the $O(n)$ -model in the $n \rightarrow \infty$ limit with purely relaxational model-A dynamics [44, 46, 58, 80].

The technical details of the calculations are covered in several appendices. Appendix A describes how to derive the equilibrium form of the quantum spherical constraint, appendix B gives the analysis of the effective Volterra equation in the semi-classical limit. Appendices C and D give the necessary mathematical details for reducing asymptotically the spherical constraint to a transcendental equation involving Humbert functions, for the case of deep quantum quenches. This equation is solved asymptotically in appendices E and F. The scaling of the two-point correlator is analysed in appendix G.

2. Quantum spherical model: equilibrium

The spin-anisotropic quantum spherical model (SAQSM) [82] is defined by a set of ‘spin operators’ $s_{\mathbf{n}} = s_{\mathbf{n}}^\dagger$, attached to the sites \mathbf{n} of a d -dimensional hyper-cubic lattice $\mathcal{L} \subset \mathbb{Z}^d$ with $\mathcal{N} = N^d$ sites. For each spin variable we define the corresponding conjugated momentum $p_{\mathbf{n}} = p_{\mathbf{n}}^\dagger$ [61], which satisfies the canonical commutation relations

$$[s_{\mathbf{n}}, p_{\mathbf{m}}] = i\delta_{\mathbf{n}, \mathbf{m}}. \quad (2.1)$$

Throughout, we shall use units such that $\hbar = 1$. For nearest-neighbour interactions, and with periodic boundary conditions, the Hamiltonian is

$$H = \sum_{\mathbf{n} \in \mathcal{L}} \left[\frac{g}{2} \left(p_{\mathbf{n}}^2 - \frac{1-\lambda}{2\mathcal{S}} \sum_{\langle \mathbf{n}, \mathbf{m} \rangle} p_{\mathbf{n}} p_{\mathbf{m}} \right) + \mathcal{S} \left(s_{\mathbf{n}}^2 - \frac{1+\lambda}{2\mathcal{S}} \sum_{\langle \mathbf{n}, \mathbf{m} \rangle} s_{\mathbf{n}} s_{\mathbf{m}} \right) \right] \quad (2.2)$$

where $\langle \mathbf{n}, \mathbf{m} \rangle$ are pairs of nearest-neighbour sites \mathbf{m} and \mathbf{n} . The parameter λ describes the spin-anisotropy in the interactions (this can be seen explicitly by going over to bosonic degrees of freedom [82]) and the usually studied QSM [41, 62, 81] is the special case $\lambda = 1$. The parameter g is the quantum coupling, such that for $\lambda = 1$ and $g \rightarrow 0$, the spin operators become real numbers $s_{\mathbf{n}} \in \mathbb{R}$ and one recovers the classical spherical model [7]. Finally, the *spherical parameter* \mathcal{S} is a Lagrange multiplier, to be chosen self-consistently in order to satisfy the so-called mean *spherical constraint* [53]

$$\sum_{\mathbf{n} \in \mathcal{L}} \langle s_{\mathbf{n}}^2 \rangle = \mathcal{N}. \quad (2.3)$$

The quantum Hamiltonian is invariant under the duality transformation \mathcal{D} given by

$$\lambda \leftrightarrow -\lambda, \quad s_{\mathbf{n}} \leftrightarrow \sqrt{\frac{g}{2\mathcal{S}}} p_{\mathbf{n}}. \quad (2.4)$$

We shall strive to find a Lindblad dissipator which will preserve this symmetry. The equilibrium phases at temperature $T = 0$, and the dimension-dependent transition lines $g_c(\lambda, d)$ are shown in figure 1. A re-entrant phase transition is seen for $1 < d \lesssim 2.065$ when λ is small enough, without a known counterpart in the fermionic analogues of the SAQSM [82]. This illustrates the non-trivial nature of the ground state of H .

The Hamiltonian (2.2) is readily diagonalised by first going over to Fourier space. We define the (non-hermitian) operators $q_{\mathbf{k}} = q_{-\mathbf{k}}^\dagger$ and $\pi_{\mathbf{k}} = \pi_{-\mathbf{k}}^\dagger$, along with the inverse transformations

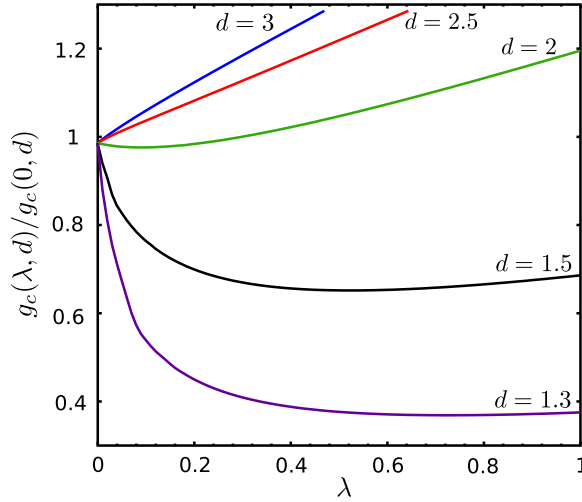


Figure 1. Equilibrium quantum phase diagram of the SAQSM at $T=0$, for dimensions $d = [1.3, 1.5, 2, 2.5, 3]$, from bottom to top. We show for each dimension the critical line $g_c(\lambda)$ below which the system is a quantum ferromagnet. Above these lines order is destroyed by quantum fluctuations. After [82].

$$q_{\mathbf{k}} := \mathcal{N}^{-1/2} \sum_{n \in \mathcal{L}} s_n e^{-i\mathbf{n} \cdot \mathbf{k}}, \quad \pi_{\mathbf{k}} := \mathcal{N}^{-1/2} \sum_{n \in \mathcal{L}} p_n e^{i\mathbf{n} \cdot \mathbf{k}} \quad (2.5a)$$

$$s_n = \mathcal{N}^{-1/2} \sum_{\mathbf{k} \in \mathcal{B}} q_{\mathbf{k}} e^{i\mathbf{n} \cdot \mathbf{k}}, \quad p_n = \mathcal{N}^{-1/2} \sum_{\mathbf{k} \in \mathcal{B}} \pi_{\mathbf{k}} e^{-i\mathbf{n} \cdot \mathbf{k}}. \quad (2.5b)$$

where the momentum \mathbf{k} lies in the first Brillouin zone $\mathcal{B} := \{\mathbf{k} = (k_1 \dots k_d) | k_i \in \{-\frac{\pi}{N} \dots \frac{\pi}{N}\}\}$. These operators obey the canonical commutation relations

$$[q_{\mathbf{k}}, \pi_{\mathbf{k}'}] = i \delta_{\mathbf{k}, \mathbf{k}'} \quad (2.6)$$

and the transformation (2.5a) casts the Hamiltonian (2.2) into the form

$$H = \sum_{\mathbf{k} \in \mathcal{B}} \left[\frac{g}{2\mathcal{S}} \Lambda_{-;\mathbf{k}}^2 \pi_{\mathbf{k}} \pi_{-\mathbf{k}} + \Lambda_{+;\mathbf{k}}^2 q_{\mathbf{k}} q_{-\mathbf{k}} \right] \quad (2.7)$$

where

$$\Lambda_{\pm;\mathbf{k}} := \sqrt{\mathcal{S} + \frac{1 \pm \lambda}{4} (\omega_{\mathbf{k}} - 2d)} \quad \text{with} \quad \omega_{\mathbf{k}} := 2 \sum_{j=1}^d (1 - \cos k_j). \quad (2.8)$$

In the same manner, the spherical constraint (2.3) is transformed as

$$\sum_{\mathbf{k} \in \mathcal{B}} \langle q_{\mathbf{k}} q_{-\mathbf{k}} \rangle = \mathcal{N}. \quad (2.9)$$

The Hamiltonian (2.7) is now diagonalised by introducing the *bosonic ladder operators*

$$q_{\mathbf{k}} = \alpha_{\mathbf{k}} \frac{b_{\mathbf{k}} + b_{-\mathbf{k}}^\dagger}{\sqrt{2}}, \quad \pi_{\mathbf{k}} = \frac{i}{\alpha_{\mathbf{k}}} \frac{b_{\mathbf{k}}^\dagger - b_{-\mathbf{k}}}{\sqrt{2}}, \quad (2.10a)$$

$$b_{\mathbf{k}} = \frac{\alpha_{\mathbf{k}}}{\sqrt{2}} \left(\frac{q_{\mathbf{k}}}{\alpha_{\mathbf{k}}^2} + i\pi_{-\mathbf{k}} \right) , \quad b_{\mathbf{k}}^\dagger = \frac{\alpha_{\mathbf{k}}}{\sqrt{2}} \left(\frac{q_{-\mathbf{k}}}{\alpha_{\mathbf{k}}^2} - i\pi_{\mathbf{k}} \right) \quad (2.10b)$$

where

$$\alpha_{\mathbf{k}} = \left(\frac{g}{2\mathcal{S}} \right)^{1/4} \sqrt{\frac{\Lambda_{-;\mathbf{k}}}{\Lambda_{+;\mathbf{k}}}}. \quad (2.11)$$

The operators $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$ obey the usual Weyl-Heisenberg algebra $[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'}$. The Hamiltonian in equation (2.7) then becomes

$$H = \sum_{\mathbf{k} \in \mathcal{B}} E_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \frac{1}{2} \right) , \quad E_{\mathbf{k}} = \sqrt{2 \frac{g}{\mathcal{S}}} \Lambda_{+;\mathbf{k}} \Lambda_{-;\mathbf{k}}. \quad (2.12)$$

2.1. The isotropic case

For technical simplicity, we shall focus on the Lindblad equation in the isotropic case $\lambda = 1$. Then, equation (2.8) reduces to $\Lambda_{-;\mathbf{k}} = \sqrt{\mathcal{S}}$ and

$$\Lambda_{+;\mathbf{k}} =: \Lambda_{\mathbf{k}} = \sqrt{\mathcal{S} - d + \omega_{\mathbf{k}}/2}. \quad (2.13)$$

The energy $E_{\mathbf{k}}$ in equation (2.12) and the parameter $\alpha_{\mathbf{k}}$ in equation (2.11) simplify to

$$E_{\mathbf{k}} = \sqrt{2g} \cdot \Lambda_{\mathbf{k}} = \sqrt{g} \sqrt{2(\mathcal{S} - d) + \omega_{\mathbf{k}}} , \quad \alpha_{\mathbf{k}} = \left(\frac{g}{2} \right)^{1/4} \frac{1}{\sqrt{\Lambda_{\mathbf{k}}}}. \quad (2.14)$$

In the long-wavelength limit we may expand the cosines and write

$$E_{\mathbf{k}} \simeq \sqrt{2g} \left[\sqrt{\mathcal{S} - d} + \frac{1}{2} \frac{k^2}{\sqrt{\mathcal{S} - d}} \right] \quad (2.15)$$

where $k^2 = k_1^2 + \dots + k_d^2$. The last term in (2.15) represents a non-relativistic massive dispersion relation, whereas the first term represents a chemical potential term. Clearly, thermodynamic stability is realised if $\mathcal{S} \geq d$. Furthermore, the zero-momentum energy gaps vanish for $\mathcal{S} = d$. In complete analogy with the equilibrium spherical model, classical [7] or quantum [41], this last condition defines the critical point.

3. Construction of the Lindblad master equation

Now, we discuss how to describe the dynamics of the QSM in contact with a heat bath. We shall explicitly admit the Markov property in the dynamics and the *weak-coupling limit* of the coupling between the system and the bath. It is well-established that under these hypotheses, the most general description of the quantum dynamics of a system interacting with a reservoir is a non-unitary time-evolution of the reduced density matrix ρ , via the *Lindblad equation* [10, 74]

$$\partial_t \rho = -i[H, \rho] + \mathcal{D}(\rho). \quad (3.1)$$

Herein, the *dissipator* $\mathcal{D}(\rho)$ describes the relaxation towards equilibrium. In the case of a single harmonic oscillator, interacting with a thermal bath, made of a phonon or photon gas, at the fixed temperature T [10, 55, 74]

$$\mathcal{D}(\rho) = \gamma(E) \left((\bar{n} + 1) \left[b\rho b^\dagger - \frac{1}{2} \{b^\dagger b, \rho\} \right] + \bar{n} \left[b^\dagger \rho b - \frac{1}{2} \{bb^\dagger, \rho\} \right] \right) \quad (3.2)$$

where E is the energy of the oscillator, b and b^\dagger are the bosonic ladder operators of the system, $\{A, B\} = AB + BA$ is the anti-commutator, $\gamma(E)$ is the *damping parameter* which also depends substantially on the bath and

$$\bar{n} = \bar{n}(E) = (e^{E/T} - 1)^{-1} \quad (3.3)$$

is the Bose–Einstein occupation number at bath temperature T . This quantum master equation (3.1) and (3.2) preserves essential properties of the density matrix ρ , namely *trace*, *complete positivity* and *hermiticity* [55]. In addition, the Schrödinger picture is used for the bosonic operators b, b^\dagger . Hence the commutator $[b, b^\dagger] = 1$ is time-independent and its conservation is an intrinsic property of the formalism [17].

For our many-body problem, further consistency requirements are necessary:

1. the quantum equilibrium state must be a stationary state of equations (3.1) and (3.2),
2. this should imply that in the $g \rightarrow 0$ limit, the classical equilibrium state must be a stationary state,
3. the classical Langevin dynamics must follow in the limit $g \rightarrow 0$, for all times.

It turns out that these requirements can all be met, in an essentially unique way. The final Lindblad equation of the SAQSM will come out to read

$$\begin{aligned} \partial_t \rho = & -i[H, \rho] + \gamma_0 \sum_{\mathbf{k} \in \mathcal{B}} \left[\left(\frac{1+\lambda}{2} \right)^2 \Lambda_{-;\mathbf{k}}^2 + \left(\frac{1-\lambda}{2} \right)^2 \Lambda_{+;\mathbf{k}}^2 \right] \frac{\Lambda_{+;\mathbf{k}}^2 \Lambda_{-;\mathbf{k}}^2}{\mathcal{S}^2} \\ & \times \left[(\bar{n}_{\mathbf{k}} + 1) \left(b_{\mathbf{k}} \rho b_{\mathbf{k}}^\dagger - \frac{1}{2} \{b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \rho\} \right) + \bar{n}_{\mathbf{k}} \left(b_{\mathbf{k}}^\dagger \rho b_{\mathbf{k}} - \frac{1}{2} \{b_{\mathbf{k}} b_{\mathbf{k}}^\dagger, \rho\} \right) \right]. \end{aligned} \quad (3.4)$$

Herein, the only free parameter is the constant γ_0 which sets the time-scale. Clearly, the dissipator does depend on the spherical parameter \mathcal{S} . The derivation of (3.4) is made first for a free bosonic system, without taking the spherical constraint into account. At the end, through the spherical constraint which must hold at all times, $\mathcal{S} = \mathcal{S}(t)$ becomes time-dependent. This will turn out to make the solution of the spherical constraint considerably more complicated than at equilibrium (and also with respect to the classical dynamics).

Two different ways of deriving (3.4) will be presented:

- (i) One may consider explicitly the system-reservoir coupling and go through the standard route, with the usual approximations [10]. The bath properties are taken into account through the explicit time-dependent phonon (or photon) correlators.

This gives a formal derivation of the Lindblad equation and will be carried out in the remainder of this section.

- (ii) For the purpose of model-building, an alternative and more phenomenological approach might be useful. As we shall show in section 4, one may start from a generic form of the dissipator, essentially a sum of terms of the form (3.2) for each mode, and with yet unspecified damping constants γ_k . We then derive quantum equations of motion for certain observables. Comparison of these equations of motion with the known classical $g \rightarrow 0$ limit (if available) then fixes the γ_k .

At the end, both procedures lead to the same Lindblad equation (3.4).

3.1. General structure of the system-bath coupling

Now, largely following [10], but with the few adaptations required for the QSM, we introduce the open-system dynamics.

For clarity, we begin treating just a single spin, say s_n , coupled to the bath. The coupling of several spins is readily obtained at the end. As usual, the bath will be modelled by an infinite number of bosonic ‘phonon’ degrees of freedom, with the bath hamiltonian

$$H_B = \sum_{\ell} \Omega_{\ell} \eta_{\ell}^{\dagger} \eta_{\ell} \quad (3.5)$$

with the bosonic operators η_{ℓ} and their corresponding frequencies Ω_{ℓ} . The system-bath interaction Hamiltonian is assumed to take the form

$$H_I = \sum_{\ell} f_{\ell} A_n \otimes (\eta_{\ell} + \eta_{\ell}^{\dagger}) \quad (3.6)$$

where $f_{\ell} \in \mathbb{R}$ are coupling constants and A_n is a hermitian system operator. There is a certain freedom in the choice of A_n . Here, rather than a simplistic coupling to only the spin operator $s_n = s_n^{\dagger}$ or only to the momentum operator $p_n = p_n^{\dagger}$, we prefer a coupling which preserves the invariance under the duality transformation \mathcal{D} , see equation (2.4). The most general linear operator compatible with duality is

$$A_n = \frac{1+\lambda}{2} \frac{s_n}{\sqrt{g}} + \frac{1-\lambda}{2} \frac{p_n}{\sqrt{2\mathcal{S}}} \quad (3.7)$$

In the weak-coupling limit, the action of the bath is described approximately by a Lindblad equation for the reduced density matrix ρ of the system

$$\partial_t \rho = -i[H, \rho] + \mathcal{D}_n(\rho) \quad (3.8)$$

where the first term describes the unitary evolution and $\mathcal{D}_n(\rho)$ is the Lindblad dissipator corresponding to the interaction (3.6). The expression for $\mathcal{D}_n(\rho)$ is most commonly derived using the method of eigenoperators [10].

To make this presentation self-contained, we rapidly recall the main steps before applying it to the SAQSM. Consider a Hamiltonian H with energy levels ϵ and let \mathcal{P}_{ϵ} denote the corresponding projection operator onto the subspace of eigenvectors that have energy ϵ . Moreover, assume that the system-bath coupling may be described by

an interaction Hamiltonian of the form $H_I = AB$, see also (3.6), where A and B are *hermitian* system and bath operators, respectively. Define the *eigenoperator* $A(\omega)$ corresponding to A via the relation

$$A(\omega) = \sum_{\epsilon, \epsilon'} \mathcal{P}_\epsilon A \mathcal{P}_{\epsilon'} \delta_{\epsilon' - \epsilon, \omega} \quad (3.9)$$

where the sum is over all distinct energies ϵ, ϵ' and $\delta_{a,b}$ is the Kronecker delta. It can be shown that

$$[H, A(\omega)] = -\omega A(\omega) \quad , \quad A^\dagger(\omega) = A(-\omega) \quad . \quad (3.10)$$

The quantities ω represent all allowed energy differences that may be produced by the action of the operator A .

It follows that the Lindblad dissipator corresponding to the interaction $H_I = AB$ reads, in the Born-Markov and rotating wave approximations [10]

$$\mathcal{D}(\rho) = \sum_{\omega} \Gamma(\omega) \left[A(\omega) \rho A^\dagger(\omega) - \frac{1}{2} \{ A^\dagger(\omega) A(\omega), \rho \} \right] \quad (3.11)$$

where

$$\Gamma(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \langle B(t) B(0) \rangle \quad (3.12)$$

is the Fourier transform of the bath correlation functions. This method therefore allows one to readily write down the dissipator corresponding to a given system-bath interaction. However, to do so we must compute the eigenoperator $A(\omega)$ from equation (3.9), which requires the full eigenstructure of the Hamiltonian. It is also worth noting that this method also produces a Lamb-shift correction to the Hamiltonian. However, this correction is usually small and, for simplicity, will be neglected.

3.2. Evaluation of bath correlation functions

Returning now to our problem, the interaction Hamiltonian (3.6) has $A = A_n$ and $B = \sum_{\ell} f_{\ell} (\eta_{\ell} + \eta_{\ell}^{\dagger})$. One must compute equation (3.12) for this choice of B . If the bath is in thermal equilibrium at a fixed temperature T , one has

$$\langle B(t) B(0) \rangle = \sum_{\ell} f_{\ell} f_{\ell} \left(e^{-i\Omega_{\ell} t} [\bar{n}(\Omega_{\ell}) + 1] + e^{i\Omega_{\ell} t} \bar{n}(\Omega_{\ell}) \right) \quad (3.13)$$

with the Bose–Einstein distribution \bar{n} defined in equation (3.3). Inserting this into equation (3.12) leads to

$$\Gamma(\omega) = 2\pi \sum_{\ell} f_{\ell} f_{\ell} \left(\delta_{\omega, \Omega_{\ell}} [\bar{n}(\Omega_{\ell}) + 1] + \delta_{\omega, -\Omega_{\ell}} \bar{n}(\Omega_{\ell}) \right). \quad (3.14)$$

If the bath frequencies Ω_{ℓ} vary continuously in the interval $[0, \infty)$, one may convert the sum to an integral, leading to

$$\Gamma(\omega) = \int_0^\infty d\Omega \, \gamma(\Omega) \left(\delta_{\omega, \Omega} [\bar{n}(\Omega) + 1] + \delta_{\omega, -\Omega} \bar{n}(\Omega) \right) = \begin{cases} \gamma(\omega) [\bar{n}(\omega) + 1] & , \text{ if } \omega > 0 \\ \gamma(|\omega|) \bar{n}(|\omega|) & , \text{ if } \omega < 0 \end{cases} \quad (3.15)$$

with the associated *spectral density*

$$\gamma(\Omega) = 2\pi \sum_{\ell} f_{\ell}^2 \delta(\Omega - \Omega_{\ell}). \quad (3.16)$$

In order to have a definite prediction for the spectral density $\gamma(\omega)$, additional physical information about the distribution of bath frequencies is needed. In general, one expects that

$$\gamma(\Omega) \sim \Omega^{\kappa} \quad (3.17)$$

for some exponent κ . The actual value of κ will depend sensibly on the microscopic details of the bath, which in our case we do not know. Instead, we shall be guided by the principle that the classical dynamics [38] should be recovered in an appropriate limit. As we shall show below, this turns out to imply the exponent $\kappa = 3$.

Interestingly, the value of this exponent also follows from another consideration which is common in the context of quantum optics. Suppose that our bath bosons have a linear dispersion linear relation (such as, for instance, photons or acoustic phonons). Then the index ℓ is replaced by the momentum \mathbf{k} and the dispersion relation is written as $\Omega_{\mathbf{k}} = c|\mathbf{k}|$ where c is the sound/light velocity. Transforming the sum in equation (3.16) into an integral gives

$$\gamma(\Omega) \sim f(\Omega)^2 \Omega^2. \quad (3.18)$$

We now see that we recover the exponent $\kappa = 3$ if we assume that $f_{\mathbf{k}}$ is proportional to $\sqrt{\Omega_{\mathbf{k}}}$. This turns out to be precisely the dipole approximation minimum coupling [10]. Thus, we conclude that we recover the classical Langevin dynamics if we assume a typical electric-field dipole coupling of the spins with the bath bosons. In summary, we emerge from this discussion with the result that

$$\gamma(\Omega) = \gamma_0 \Omega^3 \quad (3.19)$$

where the constant γ_0 describes the strength of the system-bath coupling.

3.3. Calculation of the eigenoperators

To finish the construction of the dissipator (3.11) one must find the eigenoperators $A(\omega)$ corresponding to $A = A_{\mathbf{n}}$. First, use equations (2.5b) and (2.10a) to write

$$A_{\mathbf{n}} = \frac{(g^{-3}/2\mathcal{S})^{\frac{1}{4}}}{\sqrt{2\mathcal{N}}} \sum_{\mathbf{k} \in \mathcal{B}} e^{i\mathbf{n} \cdot \mathbf{k}} \left(c_{\mathbf{k}} b_{\mathbf{k}} + c_{\mathbf{k}}^* b_{-\mathbf{k}}^{\dagger} \right) \quad \text{with} \quad c_{\mathbf{k}} = \frac{1+\lambda}{2} \sqrt{\frac{\Lambda_{-, \mathbf{k}}}{\Lambda_{+, \mathbf{k}}}} + i \frac{1-\lambda}{2} \sqrt{\frac{\Lambda_{+, \mathbf{k}}}{\Lambda_{-, \mathbf{k}}}}. \quad (3.20)$$

Next we note that, due to the diagonal structure of H in equation (2.12), it follows that $[H, b_{\mathbf{k}}] = -E_{\mathbf{k}} b_{\mathbf{k}}$. Hence, comparison with equation (3.10) shows that $b_{\mathbf{k}}$ is an eigenoperator of H with allowed transition frequency $\omega = E_{\mathbf{k}}$. The same is true for $c_{\mathbf{k}} b_{\mathbf{k}}$ as well. The full eigenoperator therefore reads

$$A_{\mathbf{n}}(\omega) = \frac{(g^{-3}/2\mathcal{S})^{\frac{1}{4}}}{\sqrt{2\mathcal{N}}} \sum_{\mathbf{k} \in \mathcal{B}} e^{i\mathbf{n} \cdot \mathbf{k}} \left(c_{\mathbf{k}} b_{\mathbf{k}} \delta_{E_{\mathbf{k}}, \omega} + c_{\mathbf{k}}^* b_{-\mathbf{k}}^{\dagger} \delta_{E_{\mathbf{k}}, -\omega} \right). \quad (3.21)$$

The dissipator (3.11) corresponding to $A_{\mathbf{n}}$ being coupled to the bath, will then be

$$\mathcal{D}_{\mathbf{n}}(\rho) = \sum_{\omega} \Gamma(\omega) \left[A_{\mathbf{n}}(\omega) \rho A_{\mathbf{n}}^{\dagger}(\omega) - \frac{1}{2} \{ A_{\mathbf{n}}^{\dagger}(\omega) A_{\mathbf{n}}(\omega), \rho \} \right]. \quad (3.22)$$

This expression may be simplified further. To do that, it suffices to look only at the first term

$$\begin{aligned} & \sum_{\omega} \Gamma(\omega) A_{\mathbf{n}}(\omega) \rho A_{\mathbf{n}}^{\dagger}(\omega) \\ &= \sum_{\omega, \mathbf{k}, \mathbf{k}'} \frac{e^{i\mathbf{n} \cdot (\mathbf{k} - \mathbf{k}')}}{2\mathcal{N}} \Gamma(\omega) \sqrt{\frac{g^{-3}}{2\mathcal{S}}} \left(c_{\mathbf{k}} b_{\mathbf{k}} \delta_{E_{\mathbf{k}}, \omega} + c_{\mathbf{k}}^* b_{-\mathbf{k}}^{\dagger} \delta_{E_{\mathbf{k}}, -\omega} \right) \rho \left(c_{\mathbf{k}'}^* b_{\mathbf{k}'}^{\dagger} \delta_{E_{\mathbf{k}'}, \omega} + c_{\mathbf{k}'} b_{-\mathbf{k}'}^{\dagger} \delta_{E_{\mathbf{k}'}, -\omega} \right). \end{aligned}$$

Since $E_{\mathbf{k}} > 0$, see equation (2.12), the only terms which will survive the constraints imposed by the δ 's are those with $E_{\mathbf{k}} = E_{\mathbf{k}'}$. Since the energies may be degenerate, this does not necessarily imply that $\mathbf{k} = \mathbf{k}'$. But if we carry out the sum over ω and use equation (3.15), we obtain

$$\begin{aligned} & \sum_{\omega} \Gamma(\omega) A_{\mathbf{n}}(\omega) \rho A_{\mathbf{n}}^{\dagger}(\omega) \\ &= \sqrt{\frac{g^{-3}}{8\mathcal{S}}} \sum_{\mathbf{k}, \mathbf{k}'} \delta_{E_{\mathbf{k}}, E_{\mathbf{k}'}} \frac{e^{i\mathbf{n} \cdot (\mathbf{k} - \mathbf{k}')}}{\mathcal{N}} \gamma(E_{\mathbf{k}}) \left[c_{\mathbf{k}} c_{\mathbf{k}'}^* (\bar{n}_{\mathbf{k}} + 1) b_{\mathbf{k}} \rho b_{\mathbf{k}'}^{\dagger} + c_{\mathbf{k}}^* c_{\mathbf{k}'} \bar{n}_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} \rho b_{\mathbf{k}'} \right]. \end{aligned}$$

The structure of the other terms in equation (3.22) will be similar. Finally, we define

$$\gamma_{\mathbf{k}, \mathbf{k}'}^{(n)} = \sqrt{\frac{g^{-3}}{8\mathcal{S}}} \gamma(E_{\mathbf{k}}) \frac{e^{i\mathbf{n} \cdot (\mathbf{k} - \mathbf{k}')}}{\mathcal{N}} c_{\mathbf{k}} c_{\mathbf{k}'}^*. \quad (3.23)$$

The final single-site dissipator (3.22) reads

$$\mathcal{D}_{\mathbf{n}}(\rho) = \sum_{\mathbf{k}, \mathbf{k}'} \delta_{E_{\mathbf{k}}, E_{\mathbf{k}'}} \left((\bar{n}_{\mathbf{k}} + 1) \gamma_{\mathbf{k}, \mathbf{k}'}^{(n)} \left[b_{\mathbf{k}} \rho b_{\mathbf{k}'}^{\dagger} - \frac{1}{2} \{ b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}, \rho \} \right] + \gamma_{\mathbf{k}', \mathbf{k}}^{(n)} \bar{n}_{\mathbf{k}} \left[b_{\mathbf{k}'}^{\dagger} \rho b_{\mathbf{k}} - \frac{1}{2} \{ b_{\mathbf{k}'}^{\dagger} b_{\mathbf{k}}, \rho \} \right] \right) \quad (3.24)$$

and describes the action of coupling a single degree of freedom to the heat bath. It couples to all normal modes $b_{\mathbf{k}}$. Furthermore, it is well-known that dissipators of this form will let evolve the system towards the correct thermal Gibbs state $\rho \sim e^{-H/T}$, although only a single spin was coupled to the bath [10].

The information which site \mathbf{n} is coupled to the bath is contained in the factor $\gamma_{\mathbf{k}, \mathbf{k}'}^{(n)}$.

3.4. Effect of coupling the entire system to the bath

We now extend this to the case where all spins are coupled to the bath. In this case, for each degree of freedom, at site \mathbf{n} , we shall have a dissipator $\mathcal{D}_{\mathbf{n}}(\rho)$ appearing in equation (3.8). But if we look at equation (3.24) we see that \mathbf{n} only appears in the quantities $\gamma_{\mathbf{k}, \mathbf{k}'}^{(n)}$. Thus if we sum all dissipators $\mathcal{D}_{\mathbf{n}}(\rho)$ we will get a result with a structure identical to equation (3.24), but with $\gamma_{\mathbf{k}, \mathbf{k}'}^{(n)}$ replaced by

$$\sum_{\mathbf{n}} \gamma_{\mathbf{k}, \mathbf{k}'}^{(\mathbf{n})} = \sqrt{\frac{1}{8g^3 \mathcal{S}}} \gamma(E_{\mathbf{k}}) |c_{\mathbf{k}}|^2 \delta_{\mathbf{k}, \mathbf{k}'} =: \delta_{\mathbf{k}, \mathbf{k}'} \gamma_{\mathbf{k}} \quad (3.25)$$

where, using also equation (3.19) along with (2.12) and (3.20), we find

$$\gamma_{\mathbf{k}} = \gamma_0 \left[\left(\frac{1+\lambda}{2} \right)^2 \Lambda_{-;\mathbf{k}}^2 + \left(\frac{1-\lambda}{2} \right)^2 \Lambda_{+;\mathbf{k}}^2 \right] \frac{\Lambda_{+;\mathbf{k}}^2 \Lambda_{-;\mathbf{k}}^2}{\mathcal{S}^2}. \quad (3.26)$$

Specific calculations will only be carried out for the isotropic case $\lambda = 1$ for which (3.26) simplifies to

$$\gamma_{\mathbf{k}} = \gamma_0 \Lambda_{\mathbf{k}}^2 \quad (3.27a)$$

see also equation (2.13). The final dissipator, after having summed over \mathbf{n} , reads

$$\mathcal{D}(\rho) = \sum_{\mathbf{k} \in \mathcal{B}} \gamma_{\mathbf{k}} \left((\bar{n}_{\mathbf{k}} + 1) \left[b_{\mathbf{k}} \rho b_{\mathbf{k}}^\dagger - \frac{1}{2} \{b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \rho\} \right] + \bar{n}_{\mathbf{k}} \left[b_{\mathbf{k}}^\dagger \rho b_{\mathbf{k}} - \frac{1}{2} \{b_{\mathbf{k}} b_{\mathbf{k}}^\dagger, \rho\} \right] \right) \quad (3.27b)$$

This is our final result (3.4) for the microscopic derivation of the Lindblad dissipator.

Recall that this dissipator satisfies detailed balance, as shown in [10]. Therefore, *modulo* an ergodicity assumption, the Lindblad equation (3.4) will thermalise the system, irrespective of the initial condition, to the unique steady-state with reduced density matrix $\rho \sim e^{-H/T}$.

This entire discussion did not take into account the spherical constraint (2.9). If one uses it in an ad hoc fashion, one would consider $\mathcal{S} = \mathcal{S}(t)$ as time-dependent. Then either the couplings to the bath or the bath properties themselves, described by $\gamma_{\mathbf{k}}$, $\bar{n}_{\mathbf{k}}$ and the operators $b_{\mathbf{k}}$ must be considered time-dependent. Pragmatically, one considers an effectively time-dependent dissipator \mathcal{D}_t which will always have as its target state the instantaneous Hamiltonian $H = H(t)$ of the system, such that formally $\mathcal{D}_t(e^{-\beta H(t)}) = 0$. Physically, that means that the time-dependent changes in H should be slow enough, which in turn should be the case if the changes in $\mathcal{S}(t)$ should be more slow than the typical bath correlation times. Since the eventual applications we are interested in concern the slow power-law relaxations after a quantum quench into the two-phase coexistence regime with formally infinite relaxation times, we expect that these kinds of physical requirements should be satisfied.

More systematically, one should not have imposed a spherical constraint, but rather have considered a second bath in order to implement it, at least on average. Since we expect that for sufficiently long times, the effective equations of motion should become the same as those we are going to study in the next section, we have not carried out this explicitly. At the present time, we consider it more urgent to arrive at some understanding of the qualitative consequences of the equations of motion on the long-time behaviour of the non-equilibrium correlators.

4. Dynamical equations for observables

In this section, we shall examine the dynamical equations governing the evolution of certain important observables under the influence of the heat bath. For any observable \mathcal{O} , the time-dependent average $\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle(t)$ is found from

$$\frac{d}{dt} \langle \mathcal{O} \rangle = \text{tr}(\mathcal{O} \partial_t \rho) + \text{tr}(\rho \partial_t \mathcal{O}). \quad (4.1)$$

In principle, all quantities $\alpha_{\mathbf{k}}$, $E_{\mathbf{k}}$, $\gamma_{\mathbf{k}}$ and $\bar{n}_{\mathbf{k}}$ should be considered as being time-dependent, if the spherical constraint is taken into account. These explicit time-dependencies come from the second term on the right-hand-side in (4.1). For the sake of simplicity of the presentation, we shall discard it for the moment but shall re-introduce it later.

Therefore, for any observable \mathcal{O} not depending explicitly on time, hence $\partial_t \mathcal{O} = 0$, inserting the Lindblad equation (3.8) into (4.1) gives

$$\frac{d}{dt} \langle \mathcal{O} \rangle = -i \langle [\mathcal{O}, H] \rangle + \langle \bar{\mathcal{D}}(\mathcal{O}) \rangle \quad (4.2)$$

with the adjoint dissipator

$$\bar{\mathcal{D}}(\mathcal{O}) = \sum_{\mathbf{k} \in \mathcal{B}} \gamma_{\mathbf{k}} \left((\bar{n}_{\mathbf{k}} + 1) \left[b_{\mathbf{k}}^\dagger \mathcal{O} b_{\mathbf{k}} - \frac{1}{2} \{b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \mathcal{O}\} \right] + \bar{n}_{\mathbf{k}} \left[b_{\mathbf{k}} \mathcal{O} b_{\mathbf{k}}^\dagger - \frac{1}{2} \{b_{\mathbf{k}} b_{\mathbf{k}}^\dagger, \mathcal{O}\} \right] \right). \quad (4.3)$$

Although the form of the $\gamma_{\mathbf{k}}$ was discussed in the previous section, we shall keep them here in a generic form. This will allow an alternative derivation of the Lindblad equation (3.4).

In order to understand how this adjoint dissipator arises, consider the second term as an example, namely $\mathcal{D}_2(\rho) = \gamma \bar{n} (b^\dagger \rho b - \frac{1}{2} \{b^\dagger b, \rho\})$, for a single mode. Then

$$\begin{aligned} \text{tr}(\mathcal{O} \mathcal{D}_2) &= \gamma \bar{n} \text{tr} \left(\mathcal{O} b^\dagger \rho b - \frac{1}{2} \mathcal{O} b^\dagger b \rho - \frac{1}{2} \mathcal{O} \rho b^\dagger b \right) \\ &= \gamma \bar{n} \text{tr} \left(\rho \left(b \mathcal{O} b^\dagger - \frac{1}{2} \mathcal{O} b^\dagger b - \frac{1}{2} b^\dagger b \mathcal{O} \right) \right) = \gamma \bar{n} \left\langle b \mathcal{O} b^\dagger - \frac{1}{2} \mathcal{O} b^\dagger b - \frac{1}{2} b^\dagger b \mathcal{O} \right\rangle \end{aligned}$$

which produces the second term in (4.3). The first term is obtained similarly.

For the single-particle observables $\mathcal{O} \in \{b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger, q_{\mathbf{k}}, \pi_{\mathbf{k}}\}$, we find from (4.1)–(4.3)

$$\frac{d}{dt} \langle b_{\mathbf{k}} \rangle = - \left(\frac{\gamma_{\mathbf{k}}}{2} + i E_{\mathbf{k}} \right) \langle b_{\mathbf{k}} \rangle + \langle \partial_t b_{\mathbf{k}} \rangle, \quad \frac{d}{dt} \langle b_{\mathbf{k}}^\dagger \rangle = - \left(\frac{\gamma_{\mathbf{k}}}{2} - i E_{\mathbf{k}} \right) \langle b_{\mathbf{k}}^\dagger \rangle + \langle \partial_t b_{\mathbf{k}}^\dagger \rangle \quad (4.4a)$$

$$\frac{d}{dt} \langle q_{\mathbf{k}} \rangle = - \frac{\gamma_{\mathbf{k}}}{2} \langle q_{\mathbf{k}} \rangle + \frac{g}{\mathcal{S}} \Lambda_{-, \mathbf{k}}^2 \langle \pi_{-\mathbf{k}} \rangle, \quad \frac{d}{dt} \langle \pi_{-\mathbf{k}} \rangle = - \frac{\gamma_{\mathbf{k}}}{2} \langle \pi_{-\mathbf{k}} \rangle - 2 \Lambda_{+, \mathbf{k}}^2 \langle q_{\mathbf{k}} \rangle \quad (4.4b)$$

where we also used the fact that $E_{\mathbf{k}}$, $\alpha_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ are all even in \mathbf{k} . In particular, the time-dependent magnetisation is expressed as

$$M = \sum_{n \in \mathcal{L}} \langle s_n \rangle = \sqrt{\mathcal{N}} \langle q_0 \rangle \quad (4.5)$$

where use was made of the orthogonality of the Fourier series.

Next, we turn to two-body correlators. We find, again using equations (4.1)–(4.3),

$$\frac{d}{dt} \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}'} \rangle = \langle \partial_t b_{\mathbf{k}}^\dagger b_{\mathbf{k}'} \rangle + \left[i(E_{\mathbf{k}} - E_{\mathbf{k}'}) - \frac{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}'}}{2} \right] \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}'} \rangle + \gamma_{\mathbf{k}} \bar{n}_{\mathbf{k}} \delta_{\mathbf{k}, \mathbf{k}'} \quad (4.6a)$$

$$\frac{d}{dt}\langle b_{\mathbf{k}}b_{\mathbf{k}'}\rangle = \langle \partial_t b_{\mathbf{k}}b_{\mathbf{k}'}\rangle + \left[-i(E_{\mathbf{k}} + E_{\mathbf{k}'} - \frac{\gamma_{\mathbf{k}} + \gamma_{\mathbf{k}'}}{2}) \right] \langle b_{\mathbf{k}}b_{\mathbf{k}'}\rangle \quad (4.6b)$$

From these equations we may also compute dynamical equations for the two-point correlators

$$Q_{\mathbf{k}}(t) := \langle q_{\mathbf{k}}q_{-\mathbf{k}}\rangle, \quad \Pi_{\mathbf{k}}(t) := \langle \pi_{\mathbf{k}}\pi_{-\mathbf{k}}\rangle, \quad \Xi_{\mathbf{k}}(t) := \frac{1}{2}\langle q_{\mathbf{k}}\pi_{\mathbf{k}} + \pi_{-\mathbf{k}}q_{-\mathbf{k}}\rangle. \quad (4.7)$$

The spherical constraint (2.9) then becomes in the $\mathcal{N} \rightarrow \infty$ limit

$$\sum_{\mathbf{k} \in \mathcal{B}} Q_{\mathbf{k}}(t) = \mathcal{N} \quad \Leftrightarrow \quad \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} Q_{\mathbf{k}}(t) = 1 \quad (4.8)$$

and we find the eqs of motion for the two-point correlators

$$\frac{dQ_{\mathbf{k}}}{dt} = -\gamma_{\mathbf{k}} \left[Q_{\mathbf{k}}(t) - \frac{1}{4} \sqrt{\frac{2g}{\mathcal{S}}} \frac{\Lambda_{-;\mathbf{k}}}{\Lambda_{+;\mathbf{k}}} (2\bar{n}_{\mathbf{k}} + 1) \right] + 2\frac{g}{\mathcal{S}} \Lambda_{-;\mathbf{k}}^2 \Xi_{\mathbf{k}}(t) \quad (4.9a)$$

$$\frac{d\Xi_{\mathbf{k}}}{dt} = -\gamma_{\mathbf{k}} \cdot \Xi_{\mathbf{k}}(t) + \frac{g}{\mathcal{S}} \Lambda_{-;\mathbf{k}}^2 \cdot \Pi_{\mathbf{k}}(t) - 2\Lambda_{+;\mathbf{k}}^2 \cdot Q_{\mathbf{k}}(t) \quad (4.9b)$$

$$\frac{d\Pi_{\mathbf{k}}}{dt} = -\gamma_{\mathbf{k}} \left[\Pi_{\mathbf{k}}(t) - \sqrt{\frac{\mathcal{S}}{2g}} \frac{\Lambda_{+;\mathbf{k}}}{\Lambda_{-;\mathbf{k}}} (2\bar{n}_{\mathbf{k}} + 1) \right] - 4\Lambda_{+;\mathbf{k}}^2 \cdot \Xi_{\mathbf{k}}(t) \quad (4.9c)$$

At this point we would like to stress again that the canonical commutation relation $[q_{\mathbf{k}}, \pi_{\mathbf{k}'}] = i\delta_{\mathbf{k},\mathbf{k}'}$ is preserved due to the fact that $q_{\mathbf{k}}$ and $\pi_{\mathbf{k}'}$ are Schrödinger operators. In particular, this is connected to the trace-preserving property of the dynamics as

$$\partial_t \langle [q_{\mathbf{k}}, \pi_{\mathbf{k}'}] \rangle = \text{tr} \left([q_{\mathbf{k}}, \pi_{\mathbf{k}'}] \partial_t \rho \right) = i\delta_{\mathbf{k},\mathbf{k}'} \partial_t \text{tr} \rho = 0. \quad (4.10)$$

Along with this, the commutation relation between the bosonic ladder operators is preserved since they present the same underlying algebra as

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = -\frac{i}{2} \left(\frac{\alpha_{\mathbf{k}'}}{\alpha_{\mathbf{k}}} [q_{\mathbf{k}}, \pi_{\mathbf{k}'}] - \frac{\alpha_{\mathbf{k}}}{\alpha_{\mathbf{k}'}} [\pi_{-\mathbf{k}'}, q_{-\mathbf{k}}] \right) = \delta_{\mathbf{k},\mathbf{k}'} . \quad (4.11)$$

Having completed these formal calculations, we must now take the spherical constraint (2.9) and (4.8) into account. From (4.9a), this becomes an integro-differential equation for the time-dependent spherical parameter $\mathcal{S} = \mathcal{S}(t)$. It follows that the parameter $\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}}(t)$, defined in (2.11), becomes time-dependent as well. It describes the transformation (2.10a) between the bosonic operators $b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger$ and the spins $q_{\mathbf{k}}$ and momenta $\pi_{\mathbf{k}}$. Therefore, one must decide whether either the pair $(b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger)$ or else the pair $(q_{\mathbf{k}}, \pi_{\mathbf{k}})$ is taken to be time-independent, and hence is described by the Schrödinger picture.

We choose $(q_{\mathbf{k}}, \pi_{\mathbf{k}})$ as time-independent operators. The Lindblad formalism then implies the time-independent commutator (2.6). Furthermore, the equations of motion equations (4.4b) and (4.9) remain valid. They will form the basis for our analysis of the dynamics of the QSM.

In consequence, in equations (4.4a) and (4.6) the contributions coming from the second term in (4.1) must be worked out. For example, the first equation of motion in (4.4a) now becomes, where the dot indicates the time derivative

$$\frac{d}{dt} \langle b_{\mathbf{k}} \rangle = - \left(\frac{\gamma_{\mathbf{k}}}{2} + iE_{\mathbf{k}} \right) \langle b_{\mathbf{k}} \rangle - \frac{\dot{\alpha}_{\mathbf{k}}}{\alpha_{\mathbf{k}}} \langle b_{-\mathbf{k}}^{\dagger} \rangle. \quad (4.12)$$

The other equations can be generalised similarly, but we shall not require them in this work.

Before we analyse the dynamics produced by equations (4.9), we shall first consider their steady-state properties.

4.1. Stationary solution and equilibrium properties

The correlators in equation (4.9) will relax to their stationary values, namely $\Xi_{\mathbf{k}}(\infty) = 0$ and

$$Q_{\mathbf{k}}(\infty) = \frac{1}{4} \sqrt{\frac{2g}{\mathcal{S}}} \frac{\Lambda_{-;\mathbf{k}}}{\Lambda_{+;\mathbf{k}}} (2\bar{n}_{\mathbf{k}} + 1), \quad \Pi_{\mathbf{k}}(\infty) = \sqrt{\frac{\mathcal{S}}{2g}} \frac{\Lambda_{+;\mathbf{k}}}{\Lambda_{-;\mathbf{k}}} (2\bar{n}_{\mathbf{k}} + 1). \quad (4.13)$$

These are precisely the equilibrium values expected from the SAQSM [82]. To see that more clearly, we substitute these results into the spherical constraint (2.9) and find

$$1 = \sqrt{\frac{g}{8\mathcal{S}}} \frac{1}{\mathcal{N}} \sum_{\mathbf{k} \in \mathcal{B}} \frac{\Lambda_{-;\mathbf{k}}}{\Lambda_{+;\mathbf{k}}} (2\bar{n}_{\mathbf{k}} + 1) \xrightarrow{\mathcal{N} \nearrow \infty} \sqrt{\frac{g}{8\mathcal{S}}} \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} \frac{\Lambda_{-;\mathbf{k}}}{\Lambda_{+;\mathbf{k}}} \coth(E_{\mathbf{k}}/2T). \quad (4.14)$$

This is indeed the spin-anisotropic spherical constraint at equilibrium. The derivation, through a canonical transformation, is given in appendix A. In view of the re-entrant phase diagram for a non-isotropic interaction with $\lambda \neq 1$, this is a non-trivial check of the formalism.

We have therefore confirmed that *the equilibrium state of the SAQSM is a stationary solution of the Lindblad equation*. Details on the form of the $\gamma_{\mathbf{k}}$ are not required to verify this.

In the isotropic case $\lambda = 1$, it is useful to let $\mathfrak{z} := 2(\mathcal{S} - d)$. Then (4.14) reduces to the familiar form [62]

$$\frac{\sqrt{g}}{2} \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{\sqrt{\mathfrak{z} + \omega_{\mathbf{k}}}} \coth \left(\frac{\sqrt{g}}{2T} \sqrt{\mathfrak{z} + \omega_{\mathbf{k}}} \right) = 1. \quad (4.15)$$

In the following sections, we shall mainly concentrate on either the semi-classical limit $g \rightarrow 0$ or else on the zero-temperature equilibrium limit $T = 0$. In these limit cases, the spherical constraint reduces to

$$\begin{cases} 1 - \frac{g}{12T} \simeq T \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} \left[\frac{1}{\mathfrak{z} + \omega_{\mathbf{k}}} + O(g^2) \right] & , \text{ for } g \rightarrow 0 \\ 1 = \sqrt{\frac{g}{4}} \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{\sqrt{\mathfrak{z} + \omega_{\mathbf{k}}}} & , \text{ for } T = 0 \end{cases} \quad (4.16)$$

The upper case in (4.16) reduces to the familiar classical form of the equilibrium spherical constraint [7, 53], where the temperature $T \mapsto T_{\text{eff}}(g) = T/(1 - \frac{g}{12T})$ is replaced

by an effective temperature. This also shows that in the $g \rightarrow 0$ limit, one recovers the classical equilibrium state. The lower case in (4.16) is the spherical constraint for the quantum phase transition at $T = 0$ [41, 62, 81]. For illustration, in figure 2 we show the Lagrange multiplier $\mathfrak{z} = \mathfrak{z}(d, T, g)$. In the classical limit $g = 0$ (left panel), the critical value $\mathfrak{z} = 0$ is reached for $d \leq 2$ only for a vanishing temperature $T = 0$ and there is no phase transition. On the other hand, for $d > 2$, the line $\mathfrak{z} = 0$ is already reached for a finite value $T = T_c(d) > 0$ which defines the critical temperature. A qualitatively analogous behaviour is seen for the quantum phase transition at $T = 0$ (right panel of figure 2). While for $d = 1$, the critical line $\mathfrak{z} = 0$ is only reached for $g = 0$, for any dimension $d > 1$ one finds a finite critical value $g_c(d) > 0$. A more detailed comparison reveals that the classical transition in $d + 1$ dimensions, at $g = 0$ and $T_c > 0$ and the quantum transition in d dimensions at $T = 0$ and $g_c(d) > 0$, are in the same equilibrium universality class [41, 62, 81, 82].

4.2. Formal solution of the non-equilibrium problem

To complete the microscopic derivation of the Lindblad dissipator, we now give a phenomenological discussion of how to choose the dissipator in a physically motivated fashion in order to include the correct classical many-body dynamics. To do so, we begin by writing down the formal solution of equations (4.9a)–(4.9c). This system can be re-written in a matrix form

$$\frac{d}{dt} \begin{pmatrix} Q_{\mathbf{k}}(t) \\ \Xi_{\mathbf{k}}(t) \\ \Pi_{\mathbf{k}}(t) \end{pmatrix} = -m_{\mathbf{k}}^{\lambda}(t) \begin{pmatrix} Q_{\mathbf{k}}(t) \\ \Xi_{\mathbf{k}}(t) \\ \Pi_{\mathbf{k}}(t) \end{pmatrix} + \mathbf{u}_{\mathbf{k}}^{\lambda}(t) \quad (4.17)$$

with the matrices

$$m_{\mathbf{k}}^{\lambda}(t) = \begin{pmatrix} \gamma_{\mathbf{k}}(t) & -2\frac{g}{\mathcal{S}}\Lambda_{-;\mathbf{k}}^2 & 0 \\ 2\Lambda_{+;\mathbf{k}}^2 & \gamma_{\mathbf{k}}(t) & -\frac{g}{\mathcal{S}}\Lambda_{-;\mathbf{k}}^2 \\ 0 & 4\Lambda_{+;\mathbf{k}}^2 & \gamma_{\mathbf{k}}(t) \end{pmatrix}, \quad \mathbf{u}_{\mathbf{k}}^{\lambda} = \gamma_{\mathbf{k}}(t)(2\bar{n}_{\mathbf{k}} + 1) \begin{pmatrix} -\sqrt{\frac{g}{8\mathcal{S}}} \frac{\Lambda_{-;\mathbf{k}}}{\Lambda_{+;\mathbf{k}}} \\ 0 \\ \sqrt{\frac{\mathcal{S}}{2g}} \frac{\Lambda_{+;\mathbf{k}}}{\Lambda_{-;\mathbf{k}}} \end{pmatrix}. \quad (4.18)$$

Here we suppressed the explicit time-dependence of the spherical parameter $\mathcal{S} = \mathcal{S}(t)$ of $\Lambda_{\pm;\mathbf{k}} = \Lambda_{\pm;\mathbf{k}}(t)$ for readability of the equation. Some more comments are in order:

- For a phenomenological discussion the damping rates $\gamma_{\mathbf{k}}$ were left unspecified. Since spin-anisotropy is a quantum-mechanical effect, this discussion must be carried out in the isotropic case $\lambda = 1$. Only at the end, we shall compare with the dissipator (3.27) derived from microscopic considerations in section 3.
- The time-dependence of the spherical parameter $\mathcal{S}(t)$ is to be found self-consistently from the formal solution and the spherical constraint $\sum_{\mathbf{k} \in \mathcal{B}} Q_{\mathbf{k}}(t) = \mathcal{N}$.
- Already the isotropic case $\lambda = 1$ turns out to be considerably more difficult than the classical non-equilibrium dynamics, so that we leave the non-isotropic case $\lambda \neq 1$ for future work.

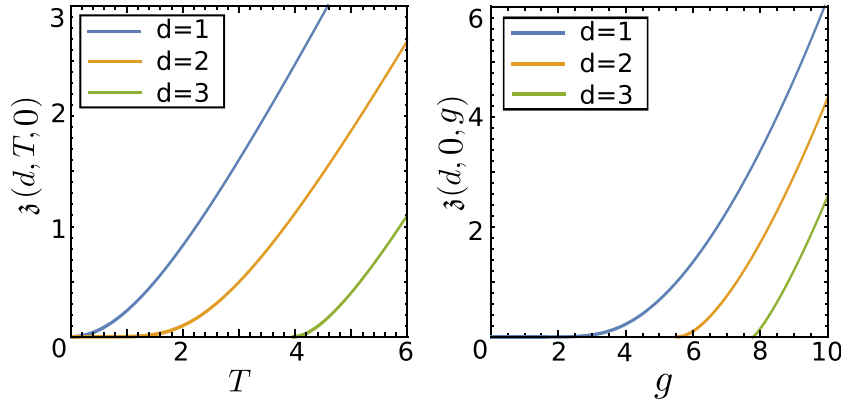


Figure 2. Spherical parameter $\mathfrak{z} = \mathfrak{z}(d, T, g)$ as a function of d , the temperature T and the coupling g . Left panel: classical limit $g = 0$. Right panel: quantum transition at $T = 0$.

Concentrating from now on only on the isotropic case $\lambda = 1$, we can simplify the matrices (4.18) by using the relations (2.13) and (3.27a) and find

$$m_{\mathbf{k}}(t) = \begin{pmatrix} \gamma_{\mathbf{k}}(t) & -2g & 0 \\ 2\Lambda_{\mathbf{k}}^2 & \gamma_{\mathbf{k}}(t) & -g \\ 0 & 4\Lambda_{\mathbf{k}}^2 & \gamma_{\mathbf{k}}(t) \end{pmatrix}, \quad \mathbf{u}_{\mathbf{k}}^{\lambda} = \gamma_{\mathbf{k}}(t)(2\bar{n}_{\mathbf{k}} + 1) \begin{pmatrix} \frac{1}{4} \frac{\sqrt{2g}}{\Lambda_{\mathbf{k}}} \\ 0 \\ \frac{\Lambda_{\mathbf{k}}}{\sqrt{2g}} \end{pmatrix}. \quad (4.19)$$

4.2.1. Choice of the damping parameters For $\lambda = 1$, a well-defined classical limit $g \rightarrow 0$ exists, and can be brought to coincide with the well-known purely relaxational model-A dynamics of the $O(n)$ model in the $n \rightarrow \infty$ limit [11, 38, 44, 46, 72, 80], as we shall now see. Then, the equation of motion for the spin correlator $Q_{\mathbf{k}}$ decouples and leads to (recall $\mathfrak{z} = 2(\mathcal{S} - d)$)

$$\frac{d}{dt} Q_{\mathbf{k}}(t) = -\gamma_{\mathbf{k}}(t) Q_{\mathbf{k}}(t) + \gamma_{\mathbf{k}}(t) \frac{T}{\mathfrak{z}(t) + \omega_{\mathbf{k}}}. \quad (4.20a)$$

We stress that this equation of motion is qualitatively different from the classical Kramers equation (see [83] for more details) since thermal fluctuations occur not just in the equation of motion of the momenta but already in the equation for the spins. The second term of the r.h.s. of equation (4.20a) comes from the assumed Lindblad dissipator and generates a coherent quantum dynamics. For an initial state at infinite temperature $Q_{\mathbf{k}}(0) = 1$. Then the formal solution of (4.20a) reads

$$Q_{\mathbf{k}}(t) = e^{-\int_0^t d\tau \gamma_{\mathbf{k}}(\tau)} \left(1 + T \int_0^t dt' \frac{\gamma_{\mathbf{k}}(t')}{\mathfrak{z}(t') + \omega_{\mathbf{k}}} e^{\int_0^{t'} d\tau \gamma_{\mathbf{k}}(\tau)} \right) \quad (4.20b)$$

We now compare this with the known dynamics of the classical model [38, equation (2.18)]. The spin-spin correlator obeys the following equation of motion, which can be derived from the Langevin equation of the classical spherical model

$$\frac{d}{dt} Q_{\mathbf{k}}(t) = -2(\mathfrak{z}(t) + \omega_{\mathbf{k}}) Q_{\mathbf{k}}(t) + 2T \quad (4.21a)$$

and has the solution

$$Q_{\mathbf{k}}(t) = e^{-2t\omega_{\mathbf{k}} - 2\int_0^t d\tau \mathfrak{z}(\tau)} \left(1 + 2T \int_0^t dt' e^{-2t'\omega_{\mathbf{k}} - 2\int_0^{t'} d\tau \mathfrak{z}(\tau)} \right) \quad (4.21b)$$

Our requirement that the $g \rightarrow 0$ limit should reduce to the classical Langevin equation means that equations (4.20) and (4.21) must be consistent. This is achieved if we choose

$$\gamma_{\mathbf{k}}(t) = 2\Lambda_{\mathbf{k}}^2(t) = 2(\mathfrak{z}(t) + \omega_{\mathbf{k}}) \quad (4.22)$$

and includes an implicit fixing of the time-scale in the classical dynamics [38]. Remarkably, the condition (4.22) is identical to the result equation (3.27a) obtained from the microscopic derivation of the Lindblad dissipator (3.27), up to a choice of the overall damping constant γ_0 . In particular, this sheds a different light on the heuristic argument we used above in order to fix the phenomenological exponent $\kappa = 3$.

Therefore, we have seen that *the requirements of reproducing (i) the correct quantum equilibrium state and (ii) the full classical dynamics in the limit $g \rightarrow 0$ are enough to fix the precise form of the Lindblad dissipator, up to an overall choice of the time scale.*

4.2.2. Closed formal solution With the final choice (4.22), we return to the dynamics for $g \neq 0$, but keep $\lambda = 1$. The formal solution of equation (4.17) is

$$\begin{pmatrix} Q_{\mathbf{k}}(t) \\ \Xi_{\mathbf{k}}(t) \\ \Pi_{\mathbf{k}}(t) \end{pmatrix} = e^{M_{\mathbf{k}}(t)} \begin{pmatrix} Q_{\mathbf{k}}(0) \\ \Xi_{\mathbf{k}}(0) \\ \Pi_{\mathbf{k}}(0) \end{pmatrix} + \gamma \int_0^t d\tau e^{M_{\mathbf{k}}(t) - M_{\mathbf{k}}(\tau)} (2\bar{n}_{\mathbf{k}}(\tau) + 1) \begin{pmatrix} \sqrt{\frac{g}{2}}\Lambda_{\mathbf{k}} \\ 0 \\ \sqrt{\frac{2}{g}}\Lambda_{\mathbf{k}}^3 \end{pmatrix} \quad (4.23)$$

where

$$M_{\mathbf{k}}(t) = \int_0^t d\tau m_{\mathbf{k}}(\tau) = -(Z(t) + t\omega_{\mathbf{k}}) \begin{pmatrix} \gamma & 0 & 0 \\ 1 & \gamma & 0 \\ 0 & 2 & \gamma \end{pmatrix} + gt \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.24)$$

and we defined the integrated spherical parameter

$$Z(t) := \int_0^t d\tau \mathfrak{z}(\tau). \quad (4.25)$$

At equilibrium, thermodynamic stability requires $\mathfrak{z} = \mathfrak{z}_{\text{eq}} \geq 0$, as we have seen in section 2.

Here, we are interested in the non-equilibrium dynamics after the systems undergoes a quench from an initial disordered state to a state characterised by certain values of (T, g) . Since then the initial values $\langle q_{\mathbf{k}} \rangle(0) = \langle \pi_{\mathbf{k}} \rangle(0) = 0$, the noise-averaged global magnetisation remains zero at all times, although fluctuations around this will be present. We therefore focus on two-body correlators. By analogy with classical dynamics we expect that if that quench goes towards a state in the disordered phase, with a single ground state of the Hamiltonian H , a rapid relaxation, with a finite relaxation time,

should occur towards that quantum equilibrium state. For sufficiently large times, $\mathfrak{z}(t) > 0$ is expected. On the other hand, for quenches either onto a critical point or else into the ordered phase (with at least two distinct, but equivalent ground states), the formal relaxation times become infinite. Then $\mathfrak{z}(t)$ may evolve differently. For the classical spherical model, quenched from a fully disordered high-temperature state to a temperature T , one finds for long times the leading behaviour [38, 72]

$$Z^{\text{cl}}(t) \sim \frac{F}{2} \ln t, \quad F = \begin{cases} -\frac{1}{2}(4-d) & ; \text{ if } T = T_c \text{ and } d < 4 \\ 0 & ; \text{ if } T = T_c \text{ and } d > 4 \\ -\frac{d}{2} & ; \text{ if } T < T_c \end{cases} \quad (4.26)$$

In contrast to the equilibrium situation, this is non-positive and by itself gives a clear indication that after a quench to $T \leq T_c$, the system never reaches equilibrium. In the next two sections, we shall work out what happens in the case of quantum dynamics. As we shall show, $Z(t) < 0$ may occur for non-equilibrium quantum quenches, but the time-dependence can be quite different from the classical result, in particular for quenches deep into the ordered phase.

In order to study what happens after a quench from the disordered phase, the system must be prepared by choosing initial two-point correlators. For a quantum equilibrium initial state, this requires $\Xi_{\mathbf{k}}(0) = 0$ and $Q_{\mathbf{k}}(0) =: \mathcal{A}_{\mathbf{k}}$ and $\Pi_{\mathbf{k}}(0) = \mathcal{C}_{\mathbf{k}}$, where $\mathcal{A}_{\mathbf{k}} = \mathcal{A}_{\mathbf{k}}(T_0, g_0)$ and $\mathcal{C}_{\mathbf{k}} = \mathcal{C}_{\mathbf{k}}(T_0, g_0)$ are chosen to specify the initial state further. The quench amounts to changing $T_0 \mapsto T$ and $g_0 \mapsto g$ to their final values which are kept fixed during the system's evolution. The two-point correlators are found from the system (4.23) by evaluating the matrix exponential which finally gives

$$\begin{aligned} Q_{\mathbf{k}}(t) = & e^{-\frac{\gamma}{g}\Delta_t} \left[\mathcal{A}_{\mathbf{k}} \cos^2 \sqrt{t\Delta_t} + \mathcal{C}_{\mathbf{k}} g^2 t \frac{\sin^2 \sqrt{t\Delta_t}}{\Delta_t} \right] \\ & + \frac{\gamma}{2} \int_0^t d\tau \sqrt{\Delta'_\tau} \left(\left[\frac{\cos \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt{\Delta_\tau/(\tau\Delta'_\tau)}} - \frac{\sin \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt{\Delta_t/(t\Delta'_\tau)}} \right]^2 \right. \\ & \left. + \left[\cos \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau} - \frac{\sin \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt{(\Delta_t/t)/(\Delta_\tau/\tau)}} \right]^2 \right) e^{\frac{\gamma}{g}(\Delta_\tau - \Delta_t)} \coth \frac{\sqrt{\Delta'_\tau}}{2T} \end{aligned} \quad (4.27a)$$

$$\begin{aligned} \Pi_{\mathbf{k}}(t) = & e^{-\frac{\gamma}{g}\Delta_t} \left[\mathcal{A}_{\mathbf{k}} g^2 t \frac{\sin^2 \sqrt{t\Delta_t}}{\Delta_t} + \mathcal{C}_{\mathbf{k}} \cos^2 \sqrt{t\Delta_t} \right] \\ & + \frac{\gamma}{2g^2} \int_0^t d\tau \sqrt{\Delta'_\tau} \left(\left[\frac{\sin \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt{t/\Delta_t}} + \frac{\cos \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt{\tau/\Delta_\tau}} \right]^2 \right. \\ & \left. + \left[\frac{\sin \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt{t/\Delta_t}} - \frac{\cos \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt{\tau/\Delta_\tau}} \right]^2 \right) e^{\frac{\gamma}{g}(\Delta_\tau - \Delta_t)} \coth \frac{\sqrt{\Delta'_\tau}}{2T} \end{aligned} \quad (4.27b)$$

$$\begin{aligned}
\Xi_{\mathbf{k}}(t) = & e^{-\frac{\gamma}{g}\Delta_t} \left[\frac{\mathcal{C}_{\mathbf{k}}g}{\sqrt{\Delta_t}} - \mathcal{A}_{\mathbf{k}}\sqrt{\Delta_t} \right] \sin 2\sqrt{t\Delta_t} + \frac{\gamma}{g} \int_0^t d\tau e^{\frac{\gamma}{g}(\Delta_\tau - \Delta_t)} \coth \frac{\sqrt{\Delta'_\tau}}{2T} \\
& \times \sqrt{\Delta_\tau} \left(\sqrt{\frac{t}{\tau}} \left[\frac{\cos \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_\tau/\tau)/(\Delta_t/t)}} + \frac{\sin \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_t/t)/(\Delta_\tau/\tau)}} \right] \left[\frac{\sin \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_\tau/\tau)/(\Delta_t/t)}} \right. \right. \\
& + \left. \frac{\cos \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_t/t)/(\Delta_\tau/\tau)}} \right] + \frac{\tau\Delta'_\tau}{\sqrt{\Delta_\tau}} \sqrt{\frac{t}{\tau}} \left[\frac{\sin \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_\tau/\tau)/(\Delta_t/t)}} - \frac{\cos \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_t/t)/(\Delta_\tau/\tau)}} \right] \\
& \times \left[\frac{\cos \sqrt{t\Delta_t} \sin \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_\tau/\tau)/(\Delta_t/t)}} - \frac{\sin \sqrt{t\Delta_t} \cos \sqrt{\tau\Delta_\tau}}{\sqrt[4]{(\Delta_t/t)/(\Delta_\tau/\tau)}} \right] \Bigg) \quad (4.27c)
\end{aligned}$$

with the definition (the \mathbf{k} -dependence is suppressed for readability)

$$\Delta_t := g(Z(t) + t\omega_{\mathbf{k}}) . \quad (4.28)$$

and the notation $\Delta'_t = \frac{d\Delta_t}{dt}$. This gives the full solution of the quantum problem and must be evaluated by using the spherical constraint (4.8), viz. $\int_B \frac{d\mathbf{k}}{(2\pi)^d} Q_{\mathbf{k}}(t) = 1$. This leads to a formidable integro-differential equation for spherical parameter $\mathfrak{z}(t)$.

4.2.3. Remark on the relaxation towards equilibrium In order to arrive at the first understanding of the correlator (4.27a), let us assume that there exists a *finite* relaxation time t_r such that the system is stationary for times $t \geq t_r$. For such times, we can write $\mathfrak{z} = \mathfrak{z}_\infty \simeq Z(t)/t$. Furthermore, the integration in (4.27a) can be split according to $[0, t] = [0, t_r] \cup [t_r, t]$. In the limit $t \rightarrow \infty$ we would have

$$Q_{\mathbf{k}}(\infty) = \frac{1}{2} \frac{\sqrt{g}}{\sqrt{\mathfrak{z}_\infty + \omega_{\mathbf{k}}}} \coth \left[\frac{\sqrt{g}}{2T} \sqrt{\mathfrak{z}_\infty + \omega_{\mathbf{k}}} \right] \quad (4.29)$$

and this is consistent with the equilibrium correlator equation (4.13). We can then conclude: *If the system relaxes towards a stationary state with a positive spherical parameter $\mathfrak{z}_\infty > 0$, this stationary state has to be the unique thermodynamic equilibrium.*

5. Semi-classical limit

Equations (4.27) contain two contributions of a different physical nature. The first one contains the contributions from the fluctuations in the initial state, while the second one describes the fluctuations generated by the coupling to the external bath. These latter contributions appear far too formidable to yield to a direct approach. We therefore restrict to the study of two limiting cases. In this section, we shall present a quasi-classical limit designed to reduce the complexity of the interaction with the bath considerably, so that it can be treated. In the next section, we consider a quench deep into the ordered phase, where the bath interactions are expected to produce only sub-leading terms in the long-time limit. The spin-spin correlator, equation (4.27a), contains complicated terms depending on Δ_t , which in turn depends on the quantum coupling g . This suggests that a semi-classical description should mean

that the quantum fluctuations generated by such terms should be small and could be achieved by letting $\Delta_t \rightarrow 0$. Simplifying, equation (4.27a) would then reduce to

$$Q_{\mathbf{k}}(t) \simeq \frac{e^{-\gamma t \omega_{\mathbf{k}}}}{G(t)} + \gamma \sqrt{\frac{g}{4}} \int_0^t d\tau \frac{G(\tau)}{G(t)} e^{-\gamma(t-\tau)\omega_{\mathbf{k}}} \sqrt{\mathfrak{z}(\tau) + \omega_{\mathbf{k}}} \coth \left[\frac{\sqrt{g}}{2T} \sqrt{\mathfrak{z}(\tau) + \omega_{\mathbf{k}}} \right] \quad (5.1)$$

with the definition

$$G(t) := e^{\gamma Z(t)}. \quad (5.2)$$

Inserted into the spherical constraint $\int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} Q_{\mathbf{k}}(t) = 1$, this gives a still complicated integro-differential equation for $G(t)$. Manageable expressions can be found by expanding the thermal occupation. We introduce as a small parameter

$$\varepsilon = \sqrt{\frac{g}{T}}. \quad (5.3)$$

which measures the relative importance of quantum and thermal fluctuations. For $\varepsilon \rightarrow 0$

$$\coth \left(\varepsilon \sqrt{\frac{\mathfrak{z}(\tau) + \omega_{\mathbf{k}}}{4T}} \right) = \frac{1}{\varepsilon} \sqrt{\frac{4T}{\mathfrak{z}(\tau) + \omega_{\mathbf{k}}}} + \frac{\varepsilon}{3} \sqrt{\frac{\mathfrak{z}(\tau) + \omega_{\mathbf{k}}}{4T}} + O(\varepsilon^3). \quad (5.4)$$

The first term in this expansion reproduces the classical model while the higher-order terms give successive quantum corrections.

5.1. Classical limit

Stopping at the first term in the expansion (5.4) and choosing $Q_{\mathbf{k}}(0) = 1$ for an infinite-temperature initial state gives the classical spin-spin correlator

$$Q_{\mathbf{k}}(t) = \frac{1}{G(t)} \left(e^{-\gamma t \omega_{\mathbf{k}}} + \gamma T \int_0^t d\tau G(\tau) e^{-\gamma(t-\tau)\omega_{\mathbf{k}}} \right). \quad (5.5)$$

From the spherical constraint equation (4.8), in the $\mathcal{N} \rightarrow \infty$ limit, one finds a Volterra integral equation for $G(t)$

$$G(t) = F(t) + \gamma T \int_0^t d\tau G(\tau) F(t - \tau) = F(t) + \gamma T (F \star G)(t) \quad (5.6)$$

where \star denotes a convolution and with the integral kernel

$$F(t) = \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} e^{-\gamma t \omega_{\mathbf{k}}} = (e^{-2\gamma t} I_0(2\gamma t))^d \quad (5.7)$$

and $I_0(x)$ is a modified Bessel function [1]. Up to a rescaling $\frac{1}{2}\gamma T \mapsto T$ of temperature, this reproduces the dynamical spherical constraint of the classical model, see [38, equation (2.23)]. Of course, this was to be expected from our derivation of the Lindblad dissipator (3.27).

For a deep quench to temperatures $T \ll T_c(d)$ (or $T = 0$), the solution of (5.6) trivially is $G(t) \simeq F(t)$, up to corrections to scaling. As we shall see in section 6, the solution of the analogous deep *quantum* quench is far from being trivial.

5.2. Leading quantum correction

New insight beyond the classical limit is found if one includes the first quantum correction from the expansion (5.4) in equation (5.1). We then get

$$Q_{\mathbf{k}}(t) \simeq \frac{e^{-\gamma t \omega_{\mathbf{k}}}}{G(t)} + \gamma \int_0^t d\tau \left[T + \frac{g}{12T} (\mathfrak{z}(\tau) + \omega_{\mathbf{k}}) \right] \frac{G(\tau)}{G(t)} e^{-\gamma(t-\tau)\omega_{\mathbf{k}}}. \quad (5.8)$$

The spherical constraint (4.8) becomes again a Volterra-integral equation for $G(t)$. This is seen as follows. From the definitions (5.7) and (5.2) we have

$$\frac{dF(t)}{dt} = -\gamma \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} \omega_{\mathbf{k}} e^{-\gamma t \omega_{\mathbf{k}}} \quad , \quad \frac{dG(t)}{dt} = \gamma \mathfrak{z}(t) G(t).$$

Integrating (5.8) gives

$$\begin{aligned} G(t) &= F(t) + \gamma T (G \star F)(t) + \frac{g}{12T} \int_0^t d\tau \left[\frac{dG(\tau)}{d\tau} F(t-\tau) + G(\tau) \frac{dF(t-\tau)}{d\tau} \right] \\ &= F(t) + \gamma T (G \star F)(t) + \frac{g}{12T} \int_0^t d\tau \frac{d}{d\tau} (G(\tau) F(t-\tau)) \end{aligned}$$

and using the initial values $G(0) = F(0) = 1$, this can be recast as

$$G(t) \left(1 - \frac{g}{12T} \right) = F(t) \left(1 - \frac{g}{12T} \right) + \gamma T (G \star F)(t).$$

The spherical constraint can now be written as

$$G(t) = F(t) + \gamma T^* \int_0^t d\tau G(\tau) F(t-\tau) \quad (5.9)$$

which is identical to the classical constraint equation (5.6), if one introduces an effective temperature

$$T^* = T^*(g) = \frac{T}{1 - \frac{g}{12T}} \simeq T \left(1 + \frac{g}{12T} \right). \quad (5.10)$$

Remarkably, $T^*(g) = T_{\text{eff}}(g)$ is exactly the effective temperature found in section 4 for the semi-classical *equilibrium* QSM, see equation (4.16). In figure 3 we show the phase diagram of the 3D isotropic QSM ($\lambda = 1$). There is an ordered ferromagnetic and a disordered paramagnetic phase. The quantum phase transition occurs on the horizontal axis $T = 0$ and the purely thermal phase transition is on the vertical axis $g = 0$. Clearly, the effective temperature $T^*(g)$ reproduces the exact critical line to first order in g . As expected, quantum fluctuations reduce the critical temperature $T_c(g) \leq T_c(0)$ with respect to the value of the classical model.

The identity $T_{\text{eff}}(g) = T^*(g)$ of the effective temperatures from the equilibrium and dynamical analysis corroborates the correctness of our proposed Lindblad formalism. On the other hand, we see that the effective long-time dynamics of the *semi-classical* spherical model becomes purely classical, although the underlying microscopic dynamics is described by a Lindblad equation and explicitly preserves quantum coherence. Quantum effects on the dynamics will only appear in second or higher order in g .

5.3. Equal-time spin–spin correlator

The analysis of the Volterra equation (5.9) is standard, with results identical to the ones for the classical $O(n)$ -model with model-A dynamics, in the $n \rightarrow \infty$ limit (see appendix B for details).

We have already seen that the formal expression for the single-time spin–spin correlator is

$$Q_{\mathbf{k}}(t) = \frac{e^{-\gamma t \omega_{\mathbf{k}}}}{G(t)} + \frac{g}{12T} \left[1 - \frac{e^{-\gamma t \omega_{\mathbf{k}}}}{G(t)} \right] + \frac{\gamma T}{G(t)} \int_0^t d\tau G(\tau) e^{-\gamma(t-\tau) \omega_{\mathbf{k}}}. \quad (5.11)$$

Its long-time behaviour depends both on the dimension d and on the effective temperature $T^* = T^*(g)$.

1. $T^* > T_c^*$. This corresponds to the paramagnetic phase at equilibrium and in particular to $d < 2$. The system relaxes within a finite time-scale τ_{eq} towards its (quantum) equilibrium state. For $d > 2$, the critical temperature $T_c^* > 0$ and

$$\gamma \tau_{\text{eq}} \stackrel{T^* \rightarrow T_c^*}{\simeq} \left[\frac{T_c^{*2}}{T^* - T_c^*} \frac{|\Gamma(1 - \frac{d}{2})|}{(4\pi)^{\frac{d}{2}}} \right]^{2/(d-2)}. \quad (5.12)$$

The limiting correlation function becomes rapidly constant in time and takes essentially an Ornstein-Zernicke form

$$Q_{\mathbf{k}}(t) \rightarrow Q_{\mathbf{k}}(\infty) = \frac{g}{12T} + \frac{T}{\omega_{\mathbf{k}} + \xi_{\text{eq}}^{-2}} \quad (5.13)$$

with the *equilibrium correlation length* $\xi_{\text{eq}}^2 = \gamma \tau_{\text{eq}}$. We also note a hard-core term, absent in the classical limit $g \rightarrow 0$ and which in direct space would give a contribution $\sim \frac{g}{12T} \delta(\mathbf{r})$.

2. $T^* < T_c^*$. For $d > 2$ dimensions, the critical point $T_c^* > 0$ and there is a ferromagnetic phase. In the scaling limit where $t \rightarrow \infty$ and $\mathbf{k} \rightarrow \mathbf{0}$ such that $\omega_{\mathbf{k}} t$ remains finite, we find the dynamical scaling form

$$Q_{\mathbf{k}}(t) \simeq \frac{g}{12T} + e^{-\gamma t \mathbf{k}^2} (4\pi \gamma t)^{d/2} \left(1 - \frac{g}{12T} \right) m^2. \quad (5.14)$$

Fourier-transforming to direct space gives the spin–spin correlator

$$C(t, \mathbf{r}) \simeq \frac{g}{12T} \delta(\mathbf{r}) + \left(1 - \frac{g}{12T} \right) m^2 e^{-\frac{r^2}{4\gamma t}} \quad (5.15)$$

and with the short-hand $m^2 = 1 - \frac{T^*}{T_c^*} \simeq 1 - \frac{T}{T_c^*}$, sufficiently close to the critical point. Indeed, up to the hard-core term, and a small g -dependent modification of the scaling amplitude, this has the same long-time behaviour as the classical spherical model [38, 72] to which one reverts when taking the limit $g \rightarrow 0$.

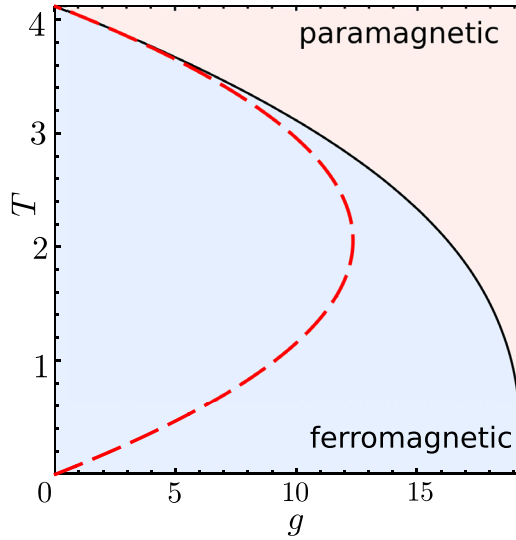


Figure 3. Phase diagram of the isotropic quantum spherical model in $d=3$ dimensions. The black curve is the exact critical line [62] which separates the paramagnetic and ferromagnetic phases. The red curve shows the critical line $T_c^*(g)$ according to equation (5.10), to first order in g .

The gaussian shape of the time-space correlator is a known property of the spherical model.

3. $T^* = T_c^*$. For quenches onto the critical line, we find the dynamical scaling form

$$Q_{\mathbf{k}}(t) = \begin{cases} \frac{g}{12T_c} + \frac{2\gamma T_c}{d-2} t {}_1F_1\left(1, \frac{d}{2}; -\gamma\omega_{\mathbf{k}}t\right) & ; \text{ if } 2 < d < 4 \\ \frac{g}{12T_c} + \frac{T_c}{\omega_{\mathbf{k}}} (1 - e^{-\gamma\omega_{\mathbf{k}}t}) & ; \text{ if } 4 < d. \end{cases} \quad (5.16)$$

Apart from the hard-core term, this agrees with what is known in the classical model.

In particular, we recover for $T^* \leq T_c^*$ the dynamical exponent $z=2$, characteristic for diffusive dynamics of the basic degrees of freedom.

6. Disorder-driven dynamics after a deep quench

We now turn to a different quench where quantum effects should be dominant for the long-time behaviour. The two-point correlators (4.9) contain contributions (i) from the fluctuations of the initial state and (ii) fluctuations which come from the exchange with the bath. In classical systems, the second term dominates for quenches onto the critical point, but only generates corrections to scaling for quenches into the two-phase coexistence region, where the first contribution dominates. Indeed, for classical systems the long-time scaling behaviour in the entire two-phase region is the same as for the deep quenches to zero temperature $T=0$. We anticipate that a similar result should also hold true for quantum systems, quenched deep into the ordered phase with $g \ll g_c(d)$.

At $T = 0$, this is possible for dimensions $d > 1$, where $g_c(d) > 0$. Therefore, instead of equations (4.9) or their formal solutions (4.27), we shall rather consider the correlators

$$Q_{\mathbf{k}}(t) = e^{-\frac{\gamma}{g}\Delta_t} \left[\mathcal{A}_{\mathbf{k}} \cos^2 \sqrt{t\Delta_t} + \mathcal{C}_{\mathbf{k}} g^2 t \frac{\sin^2 \sqrt{t\Delta_t}}{\Delta_t} \right] \quad (6.1a)$$

$$\Pi_{\mathbf{k}}(t) = e^{-\frac{\gamma}{g}\Delta_t} \left[\mathcal{A}_{\mathbf{k}} g^2 t \frac{\sin^2 \sqrt{t\Delta_t}}{\Delta_t} + \mathcal{C}_{\mathbf{k}} \cos^2 \sqrt{t\Delta_t} \right] \quad (6.1b)$$

$$\Xi_{\mathbf{k}}(t) = e^{-\frac{\gamma}{g}\Delta_t} \left[\frac{\mathcal{C}_{\mathbf{k}} g}{\sqrt{\Delta_t}} - \mathcal{A}_{\mathbf{k}} \sqrt{\Delta_t} \right] \sin 2\sqrt{t\Delta_t} \quad (6.1c)$$

along with $\Delta_t := g(Z(t) + t\omega_{\mathbf{k}})$. The terms neglected therein, with respect to (4.9), should for weak bath coupling γ and for a quench deep into the ordered phase only account for corrections to the leading scaling we seek. In our exploration of the coherent dynamics of the QSM, we conjecture that this is so and we shall inquire in particular whether a dynamical behaviour distinct from the one found in the quasi-classical case can be obtained.

As we shall see, and in contrast to the classical model, the solution of the dynamics is non-trivial.

6.1. Spherical constraint and asymptotic behaviour of the spherical parameter

Accepting the reduced form (6.1) for the two-point correlators, we concentrate on their dissipative dynamics, dominated by the *initial* disorder. The spherical constraint simplifies to

$$1 = \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} e^{-\gamma(Z(t)+t\omega_{\mathbf{k}})} \left(\mathcal{A}_{\mathbf{k}} \cos^2 \sqrt{t\Delta_t} + \mathcal{C}_{\mathbf{k}} g t \frac{\sin^2 \sqrt{t\Delta_t}}{Z(t) + t\omega_{\mathbf{k}}} \right) \quad (6.2)$$

and the initial conditions are characterised by the constants $\mathcal{C}_{\mathbf{k}}$ and $\mathcal{A}_{\mathbf{k}}$. This is still a difficult integro-differential equation without an obvious solution.

6.1.1. Initial conditions Consider a strongly disordered equilibrium initial state, situated far away from criticality. Then the equilibrium correlators are known [82]. Especially, $\Xi_{\mathbf{k}}(0) = 0$ and the spherical parameter $\mathfrak{z}_0 \gg 1$. We call this an *infinitely disordered state*. Such states are characterised by an equal occupation number of all modes \mathbf{k} , such that the equilibrium correlators equation (4.13) simplify to

$$Q_{\mathbf{k}}(0) \simeq \sqrt{\frac{g_0}{4\mathfrak{z}_0}} \coth \left(\sqrt{\frac{\mathfrak{z}_0 g_0}{4T_0^2}} \right) \stackrel{!}{=} 1, \quad \Pi_{\mathbf{k}}(0) \simeq \sqrt{\frac{\mathfrak{z}_0}{4g_0}} \coth \left(\sqrt{\frac{\mathfrak{z}_0 g_0}{4T_0^2}} \right) = \frac{\mathfrak{z}_0}{g_0} =: \mathcal{C} \quad (6.3)$$

where the first relation follows from the spherical constraint. Hence the single constant \mathcal{C} characterises the infinitely disordered initial state. Since $\mathfrak{z}(T_0, g_0)$ is defined self-consistently by the spherical constraint, no explicit expression $\mathcal{C}(g_0, T_0)$ is available. Solving (6.3) numerically, the parameter $\mathcal{C} = \mathcal{C}(g_0/T_0)$ is traced in figure 4. Two limit cases can be identified, which are both obtained for $\mathfrak{z}_0 \gg 1$.

1. *the strong classical-disorder limit* (SCDL), defined by the condition $g_0 \ll T_0^2$, along with $\mathfrak{z}_0 \gg 1$. A first-order Taylor series of \coth gives

$$\mathcal{C} \simeq \frac{T_0}{g_0} \gg 1. \quad (6.4)$$

The SCDL is obtained when \mathcal{C} is becoming large and positive.

2. *the strong quantum-disorder limit* (SQDL), defined by the condition $g_0 \gg T_0^2$, along with $\mathfrak{z}_0 \gg 1$. An asymptotic expansion now gives

$$\mathcal{C} \simeq \frac{1}{4} \quad (6.5)$$

which is the smallest admissible value for \mathcal{C} for a quantum *equilibrium* initial state.

Clearly, more general initial conditions interpolate between the limiting cases equations (6.4) and (6.5). It is conceptually significant that initial momentum correlators *must* be present.

At first sight, one might have appealed to an analogy with classical initial disordered states and expected that $\mathcal{C} = 0$ would be possible, but figure 4 shows that such a state does not correspond to a quantum disordered equilibrium state. Choosing $\mathcal{C} = 0$ means that one is considering an ‘artificial’ initial state, inconsistent with the laws of quantum mechanics.

We consequently parametrise our disordered initial state by

$$Q_{\mathbf{k}}(0) = 1, \quad \Pi_{\mathbf{k}}(0) = \mathcal{C} \quad (6.6)$$

and then quench the system to temperature $T = 0$ and a small coupling $g \ll g_c(T)$ far below the quantum critical point.

6.2. The spin–spin correlator

Our first task is to solve the spherical constraint. This requires in turn to cast the spin–spin correlator into a more manageable form. In the deep-quench scenario just defined, the spin–spin correlator becomes

$$Q_{\mathbf{k}}(t) = e^{-\gamma(Z(t)+t\omega_{\mathbf{k}})} \left[\cos^2 \left(\sqrt{gt(Z(t)+t\omega_{\mathbf{k}})} \right) + \frac{\mathcal{C}gt}{Z(t)+\omega_{\mathbf{k}}} \sin^2 \left(\sqrt{gt(Z(t)+t\omega_{\mathbf{k}})} \right) \right]. \quad (6.7)$$

Recall from the classical dynamics that $Z^{\text{cl}}(t) \simeq -\frac{d}{2} \ln t$ for $t \rightarrow \infty$ at $T = 0$. In order to prepare for the possibility that $Z(t) < 0$ also in the quantum case, it will turn out to be advantageous to rewrite the correlator in terms of a hyper-geometric function⁸

$$\begin{aligned} Q_{\mathbf{k}}(t) &= \frac{1}{2} \left[1 + \frac{\mathcal{C}gt}{Z+t\omega_{\mathbf{k}}} + \left(1 - \frac{\mathcal{C}gt}{Z+t\omega_{\mathbf{k}}} \right) {}_0F_1 \left(\frac{1}{2}; -gt(Z+t\omega_{\mathbf{k}}) \right) \right] e^{-\gamma(Z+t\omega_{\mathbf{k}})} \\ &= \left[1 + \frac{1}{2} \left(1 - \frac{\mathcal{C}gt}{Z+t\omega_{\mathbf{k}}} \right) \sum_{n=1}^{\infty} \frac{(-gt)^n}{\left(\frac{1}{2}\right)_n} \frac{(Z+t\omega_{\mathbf{k}})^n}{\Gamma(n+1)} \right] e^{-\gamma(Z+t\omega_{\mathbf{k}})} \end{aligned} \quad (6.8)$$

⁸ We suppress the explicit time-dependence of $Z = Z(t)$.

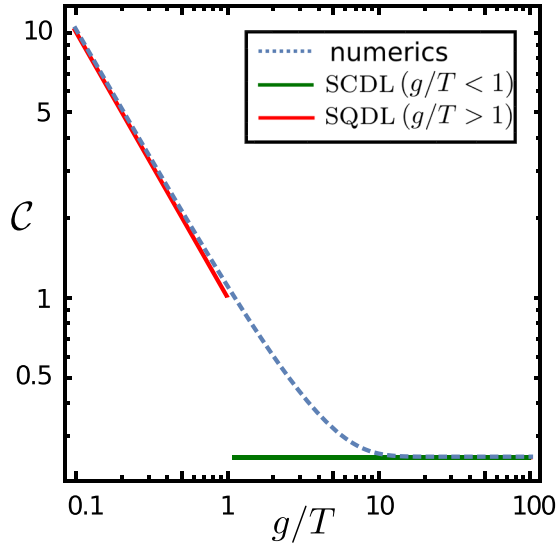


Figure 4. The initial parameter $\mathcal{C} = \mathcal{C}(g_0/T_0)$ and the two limits of SCDL and SQDL.

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ denotes the Pochhammer symbol. The evaluation of the spherical constraint becomes more simple if all dependence on \mathbf{k} is brought into the exponential. This will allow to derive factorised representations which in turn will permit to rewrite the expressions where the dimension d becomes a parameter which then can be generalised and considered as real $d \in \mathbb{R}$. This is easily achieved as

$$Q_{\mathbf{k}}(t) = \left[1 + \frac{1}{2} \sum_{n=1}^{\infty} (\partial_{\gamma}^n + \mathcal{C} g t \partial_{\gamma}^{n-1}) \frac{1}{\left(\frac{1}{2}\right)_n} \frac{(g t)^n}{n!} \right] e^{-\gamma(Z+t\omega_{\mathbf{k}})}. \quad (6.9)$$

6.3. The spherical constraint

We recall the spherical constraint (4.8), written as $1 = \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} Q_{\mathbf{k}}$, and define⁹

$$\mathfrak{f}(\gamma) := \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} e^{-\gamma(Z+t\omega_{\mathbf{k}})} = e^{-\gamma Z} (e^{-2\gamma t} I_0(2\gamma t))^d \stackrel{t \rightarrow \infty}{\simeq} e^{-\gamma Z} (4\pi\gamma t)^{-\frac{d}{2}}. \quad (6.10)$$

Thus, we can rewrite the constraint using equation (6.9) as¹⁰

$$1 = \mathfrak{f}(\gamma) + \sum_{n=1}^{\infty} \frac{(g t)^n}{2} \Gamma \left[\begin{matrix} \frac{1}{2} \\ n + \frac{1}{2} \end{matrix} \middle| n + 1 \right] (\partial_{\gamma}^n + \mathcal{C} g t \partial_{\gamma}^{n-1}) \mathfrak{f}(\gamma). \quad (6.11)$$

It is shown in appendix C that in the long-time limit, the derivative can be written as

$$\partial_{\gamma}^n \mathfrak{f}(\gamma) \simeq (-1)^n \mathfrak{f}(\gamma) \sum_{k=0}^n \Gamma \left[\begin{matrix} n+1 & \frac{d}{2} + k \\ \frac{d}{2} & n-k+1 & k+1 \end{matrix} \right] \gamma^{-k} Z^{n-k}. \quad (6.12)$$

⁹ In the short-hand $\mathfrak{f} = \mathfrak{f}(\gamma)$, the dependence on Z and t is suppressed.

¹⁰ We use throughout the notation $\Gamma \left[\begin{matrix} a_1 & \dots & a_n \\ b_1 & \dots & b_m \end{matrix} \right] = \frac{\Gamma(a_1) \dots \Gamma(a_n)}{\Gamma(b_1) \dots \Gamma(b_m)}$.

At this point, we have achieved a first goal: d merely enters as a parameter and from now, we can treat it as continuous by means of an analytic continuation. Consequently, the spherical constraint can be cast in the form

$$1 = \frac{\mathfrak{f}(\gamma)}{2} (1 + \mathfrak{s}_1 + \mathfrak{s}_2) \quad (6.13)$$

with the two double sums (see appendix D for the derivation)

$$\begin{aligned} \mathfrak{s}_1 &:= \sum_{n=0}^{\infty} \sum_{k=0}^n \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + \frac{1}{2} & \frac{d}{2} \end{matrix} \quad n - k + 1 \quad k + 1 \right] \left(-\frac{gt}{\gamma} \right)^n (\gamma Z)^{n-k} \\ &= \Phi_3 \left(\frac{d}{2}; \frac{1}{2}; -gtZ, -\frac{g}{\gamma}t \right) \end{aligned} \quad (6.14)$$

$$\begin{aligned} \mathfrak{s}_2 &:= -\gamma \mathcal{C} g t \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + \frac{1}{2} & \frac{d}{2} \end{matrix} \quad n - k \quad k + 1 \right] \frac{1}{n} \left(-\frac{gt}{\gamma} \right)^n (\gamma Z)^{n-1-k} \\ &= 2\mathcal{C} g^2 t^2 \int_0^1 dw \Phi_3 \left(\frac{d}{2}; \frac{3}{2}; -\frac{gt}{\gamma}w, -gtZw \right) \end{aligned} \quad (6.15)$$

that can be expressed in terms of the Humbert function Φ_3 [47, 48]. This function is a confluent of one of Appell's generalisations F_3 [2] of Gauss' hyper-geometric function to two independent variables [79]. The analysis of the spherical constraint requires the asymptotics of these functions when the absolute values of both arguments become simultaneously large. Since no information on these appears to be known in the mathematical literature, we shall derive it, as is outlined in appendix D. Indeed, very similar methods can be applied to different, but related confluent of the Appell function F_3 and will be presented elsewhere [84]. For our purposes, we simply state the main result: both sums can be expressed *exactly* as Laplace convolutions

$$\mathfrak{s}_1 = \Gamma \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} - \epsilon \end{matrix} \quad \epsilon \right] \sqrt{t} \int_0^t dv \frac{{}_1F_1 \left(\frac{d}{2}; \frac{1}{2} - \epsilon; -\frac{g}{\gamma}v \right)}{v^{\frac{1}{2}-\epsilon}} \frac{{}_0F_1 (\epsilon; -gZ(t-v))}{(t-v)^{1-\epsilon}} \quad (6.16)$$

$$\mathfrak{s}_2 = \mathcal{C} g^2 t^{3/2} \Gamma \left[\begin{matrix} \frac{1}{2} \\ \frac{3}{2} - \epsilon \end{matrix} \quad \epsilon \right] \int_0^1 dw \int_0^t dv \frac{{}_1F_1 \left(\frac{d}{2}; \frac{3}{2} - \epsilon; -\frac{g}{\gamma}wv \right)}{v^{\epsilon-\frac{1}{2}}} \frac{{}_0F_1 (\epsilon; -gZw(t-v))}{(t-v)^{1-\epsilon}} \quad (6.17)$$

(where $0 < \epsilon < \frac{1}{2}$ in \mathfrak{s}_1 and $0 < \epsilon < \frac{3}{2}$ in \mathfrak{s}_2). In appendix D, we first show how these integrals can be de-convoluted and then how their asymptotic limit for $t \rightarrow \infty$ can be found, using Tauberian theorems [33]. We then arrive at the following expression for the spherical constraint

$$\begin{aligned} 1 &\simeq \frac{\mathfrak{f}(\gamma)}{2} \left\{ 1 + \left[1 + \mathcal{C} \frac{gt}{Z} (e^{\gamma Z} - 1) \right] \left(\frac{\gamma}{gt} \right)^{\frac{d}{2}} \frac{{}_0F_1 \left(\frac{1-d}{2}; -gtZ \right)}{\Gamma \left(\frac{1-d}{2} \right) / \sqrt{\pi}} \right. \\ &\quad \left. + \gamma \mathcal{C} g t \left[\frac{{}_1F_1 \left(1; 2 - \frac{d}{2}; \gamma Z \right)}{\frac{d}{2} - 1} + \left(\frac{gt}{\gamma} \right)^{1-\frac{d}{2}} \frac{{}_1F_2 \left(1 - \frac{d}{2}; 2 - \frac{d}{2}, \frac{3-d}{2}; -gtZ \right) e^{\gamma Z}}{\left(1 - \frac{d}{2} \right) \Gamma \left(\frac{3-d}{2} \right) / \sqrt{\pi}} \right] \right\} \end{aligned} \quad (6.18)$$

and recall $f(\gamma)$ from (6.10). This representation, which depends on the initial condition through the parameter \mathcal{C} and contains the dimension d as a continuous parameter, will be the basis of our analysis of the physics contained in quantum spherical constraint.

We must solve this equation for $Z = Z(t)$, in the asymptotic limit $t \rightarrow \infty$, and for fixed parameters γ , g and \mathcal{C} and for a given dimension $d > 1$. The most simple case is given by the initial condition $\mathcal{C} = 0$ and serves as an illustration on how to solve the spherical constraint. We then have

$$\begin{aligned} 2e^{\gamma Z} (4\pi\gamma t)^{d/2} &= 1 + \left(\frac{\gamma}{gt}\right)^{d/2} \frac{{}_0F_1\left(\frac{1-d}{2}; -gtZ\right)}{\Gamma\left(\frac{1-d}{2}\right)/\sqrt{\pi}} \\ &= 1 + \gamma^{d/2} (\pi gt)^{1/2} \left(\frac{|Z|}{gt}\right)^{(d+1)/4} I_{-(d+1)/2}\left(2\sqrt{gt|Z|}\right) \end{aligned} \quad (6.19)$$

where we anticipated that the solution is negative $Z = -|Z(t)| < 0$ and I_ν is a modified Bessel function [1]. To illustrate this point, we display in figure 5 a typical example of the numerical solution $Z = Z(t)$ of (6.18) with $\mathcal{C} = 0$. Indeed, the solution is negative and we also observe that $Z(t) \rightarrow 0$ for $t \rightarrow \infty$. The asymptotic form of I_ν then leads to the following simplified form

$$2(4\pi g)^{d/2} = \left(\frac{\gamma^2 |Z|}{gt}\right)^{d/4} e^{2\sqrt{gt|Z|}}$$

which has the solution

$$|Z(t)| = \frac{d^2}{16gt} W^2\left(\frac{\pi}{d} 16^{\frac{1+d}{d}} gt^2\right) \simeq \frac{d^2}{16g} \frac{\ln^2 t}{t} \quad (6.20)$$

where $W = W_0$ denotes the principal branch of the Lambert-W function [21]¹¹. The agreement with the numerical solution is illustrated in figure 5. Clearly, this solution applies to all values of d and is distinct from the classical result (4.26). The logarithmic factor indicates corrections to a simple power-law scaling. We also notice that it is independent of the coupling γ between the system and the bath.

Any equilibrium initial state must have $\mathcal{C} \geq \frac{1}{4}$. Clearly, equation (6.18) with $\mathcal{C} \neq 0$ is still too complicated for an explicit solution. However, it turns out that a case distinction between the dimensions $1 < d < 2$, $d = 2$ and $d > 2$ leads to more manageable forms. The details of the calculations are given in appendix E for $d = 2$ and appendix F for $d \neq 2$. Here, we quote the results.

A. For $d > 2$, $Z = -|Z(t)| < 0$ turns out to be negative, in such a way that $t|Z|$ becomes large for large t . We have the equation

$$2e^{\gamma Z} (4\pi\gamma t)^{d/2} \simeq 1 + \frac{1}{2} \gamma^{d/2} \left(1 + \mathcal{C} \frac{gt}{|Z|}\right) \left(\frac{|Z|}{gt}\right)^{\frac{d}{4}} e^{2\sqrt{gt|Z|}}, \quad \text{for } d > 2. \quad (6.21a)$$

¹¹ Asymptotically, $W(x) \simeq \ln x - \ln \ln x + o(1)$ for $x \rightarrow \infty$.

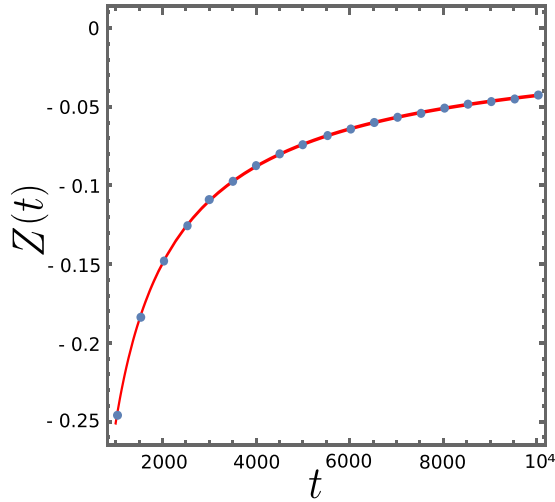


Figure 5. Time-dependence of the integrated Lagrange multiplier $Z(t)$, in $d = 2$ dimensions and for the parameters $g = 0.2$, $\gamma = 0.1$ and $\mathcal{C} = 0$. The full curve is the asymptotic form equation (6.20) and the dots come from solving numerically (6.18).

- B.** For $d = 2$, we find again $Z = -|Z(t)| < 0$ for large times, but now such that $t|Z| \rightarrow \varphi$ tends to a constant. This constant is given by the transcendent equation

$$\frac{4\pi}{\mathcal{C}g^2} = \varphi {}_2F_3\left(1, 1; \frac{3}{2}, 2, 2; g\varphi\right); \quad \text{for } d = 2. \quad (6.21b)$$

- C.** Finally, for $1 < d < 2$, the integrated Lagrange multiplier $Z = Z(t) > 0$ becomes positive for large enough times and it increases with increasing t beyond any bound. Its value is determined from

$$2(4\pi\gamma t)^{d/2} = \frac{\mathcal{C}gt}{Z} e^{-\gamma Z} + \frac{d\mathcal{C}\gamma^{d/2}}{2} \left(\frac{Z}{gt}\right)^{d/4-1} \\ \times \left[\frac{3(d+2)(4-d)}{64} \frac{\cos(2\sqrt{gtZ} + \frac{\pi d}{4})}{Z} - \frac{\sin(2\sqrt{gtZ} + \frac{\pi d}{4})}{\sqrt{gtZ}} \right]; \quad \text{for } 1 < d < 2 \quad (6.21c)$$

However, we also find an intermediate regime, with large but not enormous times, where $Z(t) < 0$ is still negative. In that regime the effective behaviour is analogous to the one found above for $d > 2$.

Summarising, for large times, the leading asymptotics of the solutions of equations (6.21) become

$$|Z(t)| \simeq \begin{cases} \frac{(d-2)^2}{4g} \frac{\ln^2 t}{t} & , \quad d > 2 \\ \varphi t^{-1} & , \quad d = 2 \\ \left(1 - \frac{d}{2}\right) \gamma^{-1} \ln t & , \quad \frac{4}{3} < d < 2 \end{cases} \quad (6.22)$$

where φ is given by (6.21b). Recall that $Z(t)$ is negative for $d \geq 2$ and positive for $1 < d < 2$. More precisely, for $\frac{4}{3} < d < 2$, the large-time behaviour is given by

$$Z(t) \simeq \left(1 - \frac{d}{2}\right) \gamma^{-1} \ln \gamma t + \mathcal{B}(d) \cos \left(2\sqrt{gtZ} + \frac{\pi d}{4}\right) \frac{t^{1-3d/4}}{\ln^{2-d/4} \gamma t} \quad (6.23)$$

and where $\mathcal{B}(d)$ is a known dimension-dependent amplitude. Hence the oscillatory term can no longer be treated as a mere correction for $d < \frac{4}{3}$ ¹².

The intermediate regime seen for dimensions $1 < d < 2$ for large, but not enormous times where $Z(t) < 0$, is effectively described by $|Z(t)| \approx \frac{(d-2)^2}{4g} \frac{\ln^2 t}{t}$.

In figure 6, we illustrate the solution for $d > 2$.

Several comments are in order:

1. Although the toy initial condition $\mathcal{C} = 0$ does indeed reproduce one instance of the long-time behaviour found from the physically more sensible equilibrium initial states with $\mathcal{C} \geq \frac{1}{4}$, it does not capture the full complexity of possible behaviours.
2. For equilibrium initial states, in $d = 2$ dimensions there is a qualitative change in the long-time behaviour of the solution $Z(t)$.
3. For $d < 2$ where the SAQSM undergoes a quantum phase transition at $T = 0$ but where the thermal critical temperature $T_c(d) = 0$ vanishes, the behaviour of $Z(t)$ is analogous to the one of the classical solution (4.26), although with the opposite sign. The Lagrange multiplier $\mathfrak{z}(t) \sim t^{-1}$ has a simple algebraic behaviour.

Very large times are required to see this regime. In addition, we find an intermediate regime of large, but not enormous times, where the system behaves effectively as for dimensions $d > 2$, up to an amplitude.

4. For $d > 2$ where the system also has a finite critical temperature $T_c(d) > 0$, strong logarithmic corrections modify the leading scaling behaviour, which is distinct from the classical one.
5. The case $d = 2$ is intermediate between the two, with a simple power-law scaling behaviour $|Z(t)| \sim t^{-1}$.
6. Surprisingly, the influence of the coupling of the coupling γ with the bath is also dimension-dependent. For $d \geq 2$ dimensions, γ disappears from the leading long-time behaviour of $Z(t)$, while it is present for $d < 2$. Therefore, for $d \geq 2$ dimensions, as well as in the intermediate regime for $d < 2$, the limit $\gamma \rightarrow 0$ can be formally taken.

The physical meaning of these properties will be understood by analysing the behaviour of the two-point correlators.

6.4. Correlation function and relevant length scales

For the deep-quench dynamics the spin–spin correlation function in Fourier space reads

¹² The occurrence of such a second ‘critical dimension’ which a qualitative change in the systems’ behaviour is a little reminiscent of the classical reaction-diffusion process reactions $2A \rightarrow \emptyset$ and $A \rightarrow 3A$, which has the critical dimensions $d_c = 2$ and $d'_c \simeq \frac{4}{3}$ [15, 16].

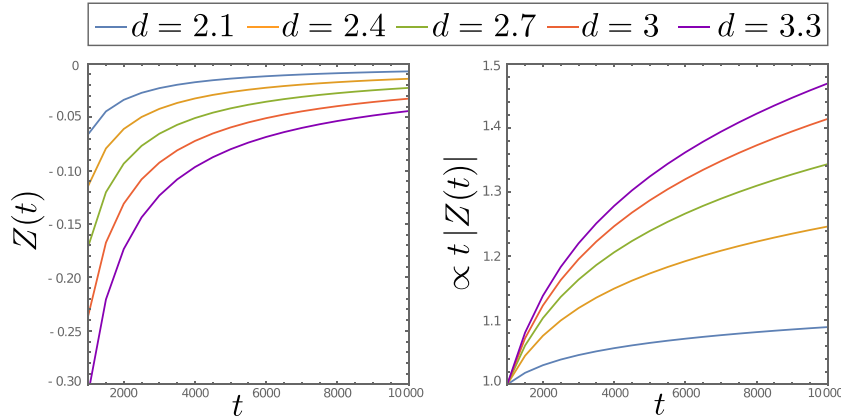


Figure 6. *Left panel:* integrated Lagrange multiplier $Z(t)$ as a function of time t obtained by solving equation (6.18) numerically, for $d = [2.1, 2.4, 2.7, 3, 3.3]$, from top to bottom, and for the parameters $\gamma = 1$, $g = 0.2$, $\mathcal{C} = 1$. *Right panel:* integrated Lagrange multiplier $t|Z(t)|$, normalised to unity at $t = 1000$, as a function of time and for $d = [2.1, 2.4, 2.7, 3, 3.3]$, from bottom to top, and the same parameters.

$$Q_{\mathbf{k}}(t) = \frac{1}{2} \left[1 + \frac{\mathcal{C}gt}{Z + t\omega_{\mathbf{k}}} + \left(1 - \frac{\mathcal{C}gt}{Z + t\omega_{\mathbf{k}}} \right) {}_0F_1 \left(\frac{1}{2}, -gt(Z + t\omega_{\mathbf{k}}) \right) \right] e^{-\gamma(Z + t\omega_{\mathbf{k}})}. \quad (6.24)$$

We are now interested in transforming this expression back to real space and studying the large-distance behaviour of the correlation. This is routinely revealed by a small- $|\mathbf{k}|$ expansion and in figure 7 we see on the 2D example that such an expansion is more than reasonable in the asymptotic limit $t \rightarrow \infty$. We consequently write $\omega_{\mathbf{k}} \approx |\mathbf{k}|^2 = k^2$ and observe that $Q_{\mathbf{k}}$ solely depends on $k = |\mathbf{k}|$. This leads to the following simplified expression for the d -dimensional inverse Fourier transform

$$f(R) \propto R^{1-\frac{d}{2}} \int_0^\infty dk k^{\frac{d}{2}} J_{\frac{d}{2}-1}(kR) \hat{f}(k) \quad (6.25)$$

- A.** We start the investigation with the case $d = 2$. In figure 7 we show a typical structure factor $Q_{\mathbf{k}}$ in 2D for different times and observe that the distribution is peaked around the zero momentum mode $k = 0$ and the peak sharpens for larger times. One can argue that the main contribution is given by the interval $[0, k^*]$ where k^* is the mode where the argument of the hyper-geometric function changes signs and ${}_0F_1$ reduces from an exponential contribution to a geometric function at this point. We can thus write

$$C(R) \propto \int_0^{k^*} dk \frac{1}{2} \left[1 + \frac{\mathcal{C}gt}{Z + tk^2} + \left(1 - \frac{\mathcal{C}gt}{Z + tk^2} \right) {}_0F_1 \left(\frac{1}{2}, -gt(Z + tk^2) \right) \right] e^{-\gamma(Z + tk^2)} k J_0(kR). \quad (6.26)$$

By introducing the *scaling variable* $\varrho := \sqrt{\varphi} \frac{R}{t}$ where φ is the solution to equation (6.21b), we find in a straightforward fashion using the variable transform $\mu = \frac{|Z| - tk^2}{|Z|}$ the scaling form

$$C(R) \propto \mathcal{C}g \int_0^1 d\mu \frac{{}_0F_1\left(\frac{1}{2}; g\varphi\mu\right) - 1}{\mu} J_0\left(\varrho\sqrt{1-\mu}\right) =: \mathcal{C}g\mathcal{W}(\varrho). \quad (6.27)$$

This shows explicitly the dynamical scaling behaviour of the spin–spin correlator with the dynamical exponent $z = 1$. In figure 8 we show the behaviour of the scaling function \mathcal{W} for different ranges of ϱ . For small ϱ the scaling function decays in a Gaussian fashion (left hand side) while it shows decaying oscillations for larger values. It is instructive to compare with the dynamical scaling seen in the classical spherical model, quenched to temperature $T \ll T_c(d)$. For a purely relaxational dynamics without any conservation law (model A), dynamical scaling is found [38, 72], whereas in the case of a conserved order-parameter (model B), the existence to two logarithmically distinct length scales was established long ago [20]. This logarithmic breaking of scale-invariance for conserved dynamics was later shown to be a peculiarity of the spherical model, see e.g. [58]. The quantum dynamics we are considering here actually has an infinite number of prescribed conservation laws, namely all canonical commutators between the spherical spins s_n and their conjugate moment p_n . Our finding that at least for $d = 2$ a standard dynamical scaling is found clearly suggests that the QSM should not be considered to be as special as its classical counterpart. Any breaking of dynamical scaling which we may find for different values cannot be as readily dismissed as a specific model property but could rather be a typical feature for more general models.

- B.** In the case $d > 2$ the treatment is similar to the case $d = 2$ since the argument of the hyper-geometric function presents once again a change of signs. However we have to respect that φ is no longer a constant but diverges logarithmically as it is shown in equation (6.22). This leads to a modified multi-scaling behaviour

$$C(R) \propto \mathcal{C}gR^{2-d} \underbrace{\int_0^1 d\mu \frac{{}_0F_1\left(\frac{1}{2}; \frac{(d-2)^2}{4}\mu \ln^2 t\right) - 1}{\mu} \left(\varrho\sqrt{1-\mu}\right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}\left(\varrho\sqrt{1-\mu}\right)}_{=: \mathcal{V}(\varrho, t)} \quad (6.28)$$

since $\varrho \simeq \frac{\frac{d}{2}-1}{\sqrt{g}} R \ln(t)/t$ which is illustrated in figure 9. The explicit logarithmic terms do break simple scale-invariance and point towards the existence of several length scales, which are distinguished by logarithmic factors. We observe a behaviour in terms of ρ which is qualitatively not too different from the case $d = 2$. However, the functional dependence on ρ changes strongly when the time is increased which is a manifest of breaking of simple scaling behaviour.

Phenomenologically, this looks analogous to the well-known behaviour of the classical spherical model with conserved order-parameter (model B) [20] but here we obtain this breaking of dynamical scaling by a mere change of the dimension d . Such a feature has never been seen before, to the best of our knowledge.

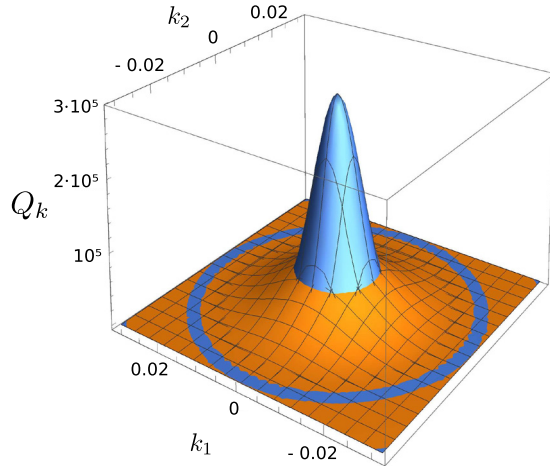


Figure 7. Structure factor $Q_{\mathbf{k}}$ in $d = 2$ dimensions shown in the first Brillouin zone for the parameter values $c = 1$; $g = 0.1$; $\gamma = 0.1$; $t = 500$ (orange); 1000 (blue). We observe that the function is sharply peaked around the centre of the Brillouin zone with the peak sharpening with time increasing.

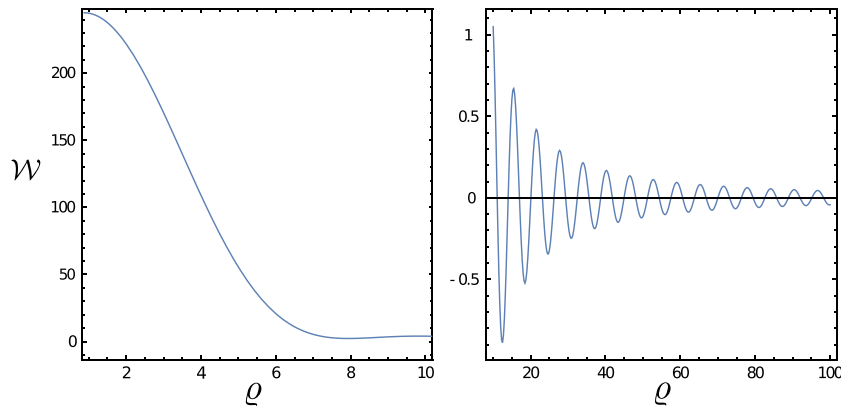


Figure 8. Illustration of the scaling function \mathcal{W} in $d = 2$ dimensions for different ranges of the scaling variable ϱ .

- C.** In $\frac{4}{3} < d < 2$ spatial dimensions the situation is different since the spherical parameter is $Z > 0$ positive and there is no intrinsic cut-off for the integral. We investigate first the structure factor $Q_{\mathbf{k}}$ by pointing out that the contribution $\sim \mathcal{C}gt$ is leading for large times.

$$Q_{\mathbf{k}} \simeq \frac{1}{2} \frac{\mathcal{C}gt}{Z + tk^2} \left[1 - {}_0F_1 \left(\frac{1}{2}; -gt(Z + tk^2) \right) \right] e^{-\gamma(Z + tk^2)}. \quad (6.29)$$

We show in appendix G that this expression can be rewritten as

$$Q_{\mathbf{k}} \simeq \mathcal{C}g^2t^2 {}_1F_2 \left(1; \frac{3}{2}, 2; -gt(Z + tk^2) \right) e^{-\gamma(Z + tk^2)}. \quad (6.30)$$

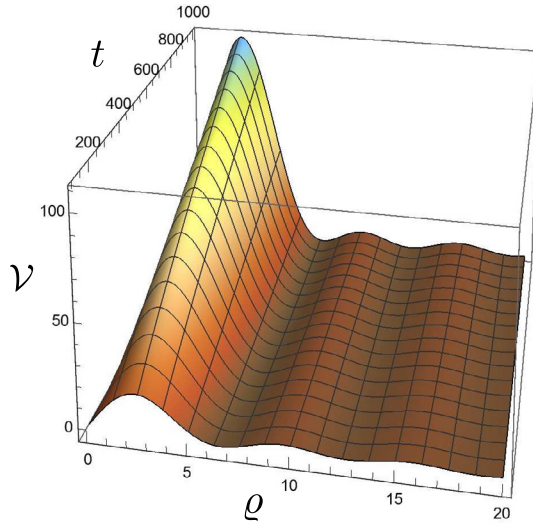


Figure 9. Functional dependence of the correlation function for $d = 3$ and $g = 0.1$. We do not find a single scaling function but rather find a dependence on the variable ρ and the time t .

Hence, its Fourier transform is readily cast in the form¹³

$$C(R) \propto \frac{Cg^2}{R^{\frac{d}{2}-1}} \int_0^\infty d\mu \mu^{\frac{d-2}{4}} J_{\frac{d}{2}+1}(\sqrt{\mu}R/t) {}_1F_2\left(1; \frac{3}{2}, 2; -g(tZ + \mu)\right) e^{-\gamma\mu/t}. \quad (6.31)$$

We observe that the integral is exponentially cut off and thus only small μ values contribute. Thus, we can omit the μ contribution in ${}_1F_2$ since $tZ \rightarrow \infty$ and the integral can be evaluated explicitly [69, equation (2.12.9.3)]

$$C(R) \propto \frac{Cg}{(2\gamma)^{d/2}} \frac{\sin^2(\sqrt{gtZ})}{Z} \exp\left[-\frac{R^2}{4\gamma t}\right] \quad (6.32)$$

revealing the dynamical exponent $z = 2$.

On the other hand, we can study a very large fixed time for which the exponential cutoff does not matter any more ($t \rightarrow \infty$ such that $R/t = \text{cste.}$) In this scenario, one can introduce a cutoff \mathcal{C} to regularise the integral and find

$$\begin{aligned} C(R) &\propto \frac{Cg^2}{R^{\frac{d}{2}-1}} \frac{\sin^2(\sqrt{gtZ})}{Z} \int_0^{\mathcal{C}} d\mu \mu^{\frac{d-2}{4}} J_{\frac{d}{2}+1}(\sqrt{\mu}R/t) \\ &= \frac{Cg^2}{R^{-2}} \frac{\sin^2(\sqrt{gtZ})}{Z} \left(\frac{\mathcal{C}}{2t}\right)^{\frac{d+2}{2}} {}_1F_2\left(\frac{d}{2} + 1; \frac{d}{2} + 2, \frac{d}{2} + 2; -\frac{R^2\mathcal{C}}{4t^2}\right). \end{aligned} \quad (6.33)$$

This implies a dynamical exponent $z = 1$ and we thus conclude that depending on the particular limit, the dynamical exponent varies between $z = 1$ and $z = 2$, an effect that will become more apparent in the following analysis of the relevant length scales.

¹³ One simply uses the change of variables $\mu = t^2 k^2$.

Having completed the analysis of the spin correlation function we mention that the momentum correlation function can be obtained from the spin correlator by simply exchanging $\mathcal{A}_{\mathbf{k}}$ and $\mathcal{C}_{\mathbf{k}}$ in equation (6.1). We thus expect a qualitatively analogous behaviour. The off-coherence $\Xi_{\mathbf{k}}(t)$ will be considered below in section 6.7.

Having studied the real-space correlation function, we now investigate the *relevant length scale* given by

$$L^2(t) \sim -\frac{\partial_k^2 Q_k}{Q_k} \Big|_{k=0}. \quad (6.34)$$

This is readily evaluated to

$$L^2 \sim \frac{2t}{Z} \left(\frac{\mathcal{C}gt \left[1 + \gamma Z - (1 + \gamma Z)_0 F_1 \left(\frac{1}{2}, -gtZ \right) - 2gt Z_0 F_1 \left(\frac{3}{2}, -gtZ \right) \right]}{\mathcal{C}gt \left[1 - {}_0F_1 \left(\frac{1}{2}, -gtZ \right) \right] + Z \left[1 + {}_0F_1 \left(\frac{1}{2}, -gtZ \right) \right]} + \frac{\gamma Z^2 \left[1 + {}_0F_1 \left(\frac{1}{2}, -gtZ \right) + 2\frac{gt}{\gamma} {}_0F_1 \left(\frac{3}{2}, -gtZ \right) \right]}{\mathcal{C}gt \left[1 - {}_0F_1 \left(\frac{1}{2}, -gtZ \right) \right] + Z \left[1 + {}_0F_1 \left(\frac{1}{2}, -gtZ \right) \right]} \right). \quad (6.35)$$

For a vanishing quantum coupling $g \rightarrow 0$, the relevant length scale reduces to a purely diffusive behaviour introduced by the heat bath

$$L_\gamma^2 \sim 2\gamma t. \quad (6.36)$$

The length scale allows to read off the dynamical exponent z according to $L^2 \sim t^{2/z}$ and we deduce from equation (6.35) $z = 2$, as expected for the classical dynamics [38, 44]. A first impression on the different behaviour in the quantum case comes from the toy initial condition $\mathcal{C} = 0$. Simplifying equation (6.35), we find, for large enough times

$$L^2 \simeq 4gt^2 \frac{\tanh 2\sqrt{gt|Z|}}{2\sqrt{gt|Z|}} \sim \frac{t^2}{\ln t} \quad (6.37)$$

hence a logarithmic correction to a dynamical exponent $z = 1$, typical of quantum dynamics. In order to evaluate accurately the intrinsic length-scale, taking into account the quantum effects, we have to distinguish, once more the cases

- A.** $d = 2$: Here Z is negative and we can rewrite the hyper-geometric functions as hyperbolic functions. Moreover the correlation function obeys a clean scaling behaviour and we find

$$L^2 \simeq 2t \left(\gamma + \sqrt{\frac{gt}{|Z|}} \right) = 2\gamma t + 2\sqrt{\frac{g}{\varphi}} t^2 \quad (6.38)$$

indicating a crossover from *diffusive* to *ballistic transport*. The dynamical critical exponent crosses from $z = 2$ to $z = 1$ as we expect for a true quantum dynamics¹⁴ [12, 24, 28, 29].

¹⁴ An exception from the fast ballistic transport are many-body localised systems where information spreads much slower [5, 63]. Such slow transport has as well been observed in translation-invariant 1D quantum lattice models [59].

More commonly, quenches exactly *onto* the quantum critical point are studied. In these situations, one finds with increasing times a crossover from ballistic to diffusive transport, see e.g. [29]. Here, on the contrary we study a quantum quench from a totally disordered system deep into the quantum ordered region. Heuristically, the system should order locally and should form ‘bubbles’ which are locally in one of the equivalent quantum ground states, and whose size should increase with time. As long as these bubbles remain small enough, they should spread like single quantum particles for which one expects an effective diffusive behaviour. At later times, when the different ‘bubbles’ will interact with each other, many-body quantum properties should dominate and lead to ballistic transport.

- B.** $d > 2$: This case can be treated analogously to the case **A** since Z is still negative. Nevertheless, we do not have a clean scaling behaviour and logarithmic corrections are present in the long-time limit. The length scale reads

$$L^2 \simeq 2t \left(\gamma + \sqrt{\frac{gt}{|Z|}} \right) = 2\gamma t + 2g(d-2) \frac{t^2}{\ln t} \quad (6.39)$$

and up to logarithmic corrections, we observe the same diffusive to ballistic crossover as for $d = 2$ with $z = 1$.

- C.** $\frac{4}{3} < d < 2$: In this case, the spherical parameter is positive and the hyper-geometric functions reduce to trigonometric contributions. The length scale then reduces to

$$L^2 \simeq 2\gamma t - 2\sqrt{\frac{gt}{Z}} \frac{\mathcal{C}gt^2 \sin 2\sqrt{gtZ}}{\mathcal{C}gt \sin^2 \sqrt{gtZ} + Z \cos^2 \sqrt{gtZ}} \quad (6.40)$$

and can be recast up to a removable singularity as

$$L^2 \simeq 2\gamma t - 4\mathcal{C}gt^2 \sqrt{\frac{gt}{Z}} \frac{\tan \sqrt{gtZ}}{\mathcal{C}gt \tan^2 \sqrt{gtZ} + Z}. \quad (6.41)$$

This length scale shows an oscillatory behaviour which is shown in the left panel of figure 10 to which we shall come back later. For now we want to focus on the right panel where we show L^2/t^2 as a function of time. We see that the peaks are rather constant and $|L/t|$ remains bound for all times what indicates that the dynamical exponent should be $z \geq 1$. The specific value of z will depend strongly on the specific time window.

Furthermore, we observe a strongly kinked oscillatory behaviour that even renders L^2 negative. This can be better understood by referring to simple correlation functions as

$$C_1 = e^{-R/\xi} \cos(R/\Lambda), \quad C_2 = e^{-(R/\xi)^2} \cos(R/\Lambda). \quad (6.42)$$

For simplicity, we refer to $d = 1$ here, since dimensionality is not changing the key aspect and it is straightforward to generalise the calculation. The characteristic length scale L_i^2 with $i = 1, 2$ associated with the correlation function C_i is readily obtained from the scales second moment

$$L_1^2 \simeq 2\Lambda^2 \frac{(\Lambda/\xi)^2 - 3}{(1 + (\Lambda/\xi)^2)^2}, \quad L_2^2 \simeq \frac{\Lambda^2}{4} \frac{1}{(\Lambda/\xi)^2} \left(2 - \frac{1}{(\Lambda/\xi)^2} \right). \quad (6.43)$$

While the overall time-dependence of this effective length scale can still be used to extract the dynamical exponent from the scaling relation $L_i^2(t) \sim t^{2/z}$, the sign of the amplitude does depend on the ratio Λ/ξ . This change of sign, according to equation (6.43), is illustrated in figure 11.

Hence the change of signs in the effective squared length $L^2(t)$ can be attributed to oscillating correlators, and the competition between the two distinct length scales ξ and Λ . While $L(t)$ itself can no longer be interpreted as a length scale, it should still be possible to read off the value of the dynamical exponent. The oscillatory nature of equation (6.40) indicates consequently a competition between at least two different length scales in the system.

6.5. Dynamic susceptibility

By means of equation (6.22) we can calculate the dynamic susceptibility which is essentially proportional to Q_0

$$\chi \sim Q_0 = \frac{1}{2} \left[1 + \frac{Cgt}{Z} + \left(1 - \frac{Cgt}{Z} \right) {}_0F_1 \left(\frac{1}{2}, -gtZ \right) \right] e^{-\gamma Z}. \quad (6.44)$$

We find for the leading contribution for large times

$$\chi(t) \sim \begin{cases} Cg/\varphi \sinh^2(\sqrt{g\varphi}) t^2, & d = 2 \\ \frac{Cg^2}{(d-2)^2} t^d / \ln^2 t, & d > 2 \\ \frac{Cg\gamma}{2-d} \frac{t^{2-\frac{d}{2}}}{\ln t} \sin^2 \left(\sqrt{g/\gamma(1-d/2)t \ln t} \right), & \frac{4}{3} < d < 2 \end{cases}. \quad (6.45)$$

In general, for systems with simple scaling, one expects $\chi(t) \sim L(t)^d \sim t^{d/z}$, or said in words, the susceptibility is proportional to the volume explored up to time t [38]. In $d = 2$ dimensions, this expectation, is fully confirmed by our exact solution, since

$$\chi_{2D}(t) \sim t^2 \sim L^2 \quad (6.46)$$

and in particular, we see once more that indeed $z = 1$, in contrast to classical dynamics. For dimensions, $d > 2$, this scaling expectation for $\chi(t)$ is again confirmed, but only up to logarithmic corrections. In addition, the effective length scale $L_{\text{eff}}(t) \sim t (\ln t)^{-2/d}$ is different from the length scale extracted above from the second moment.

Finally, for $d < 2$, not only does the exponent of the leading time-dependence deviate from the expected value (to say nothing on the logarithmic correction), but furthermore, a strong time-dependent modulation of $\chi(t)$ is found.

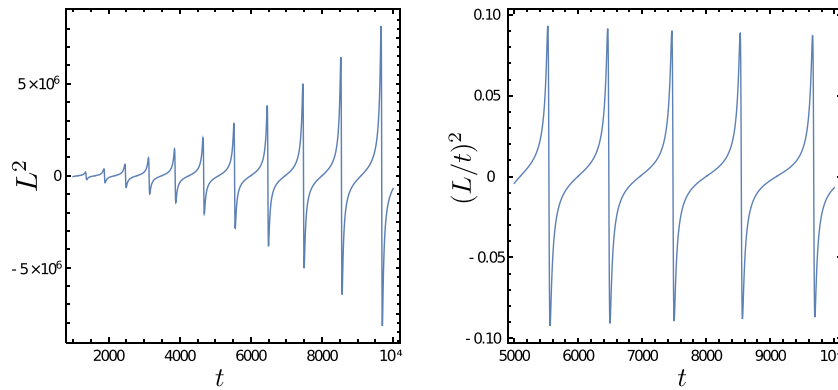


Figure 10. *left panel:* effective characteristic length L^2 for $d = 1.5$, $\gamma = 1$, $g = 0.1$ and $\mathcal{C} = 1$. *right panel:* $(L/t)^2$ for the same parameters as in the left panel.

We can understand this as a further justification of the strong competition between different length scales as we already discussed in the previous section.

6.6. Off-coherences

We now want to study the off-coherence term and which reads

$$\Xi_{\mathbf{k}}(t) = e^{-\frac{\gamma}{g}\Delta_t} \left[\frac{\mathcal{C}g}{\sqrt{\Delta_t}} - \sqrt{\Delta_t} \right] \sin 2\sqrt{t\Delta_t} \quad (6.47)$$

with $\Delta_t = g(Z(t) + t\omega_{\mathbf{k}})$.

- A.** For $d = 2$ we know that $Z < 0$ and thus Δ_t changes signs from negative to positive for after a time t^* for fixed $k \neq 0$. Consequently, all $\Xi_{\mathbf{k}} \rightarrow 0$ for $\mathbf{k} \neq 0$ due to the exponential damping. For the zero mode we find

$$\Xi_0 \simeq \mathcal{C} \sqrt{\frac{g}{\varphi}} \sinh(2\sqrt{g\varphi}) \sqrt{t} \xrightarrow{t \rightarrow \infty} \infty \quad (6.48)$$

and see a diverging off-coherence. This is a strong indicator that the system will not relax towards its thermal equilibrium but will rather stay in a non-equilibrium state for all times. It seems possible that this is a hint that the model should undergo physical ageing. Of course, a definite assertion would require a test of the three defining properties of physical ageing (slow dynamics, breaking of time-translation invariance, dynamical scaling) [44] and this requires at least an analysis of two-time correlators. We hope to return to an analysis of physical ageing in the qsm elsewhere.

- B.** For $d > 2$ the situation is, up to logarithmic corrections, similar to $d = 2$. We find immediately

$$\Xi_0 \simeq \frac{\mathcal{C}g}{d-2} \frac{t^{d-\frac{3}{2}}}{\ln t} \quad (6.49)$$

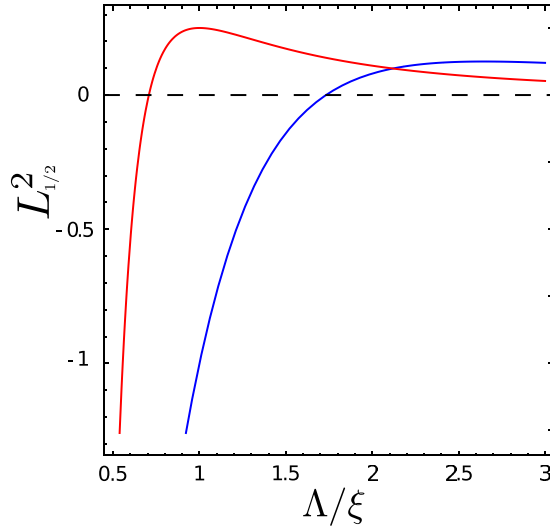


Figure 11. Effective squared length scale $L_i^2(t)$, $i = 1, 2$ as a function of Λ/ξ for a modulated exponential correlator (blue) and a modulated Gaussian correlator (red) introduced in equation (6.42).

while all non-zero mode off-coherences vanish in the asymptotic limit.

- C. In $\frac{4}{3} < d < 2$ the behaviour is qualitatively different. While it remains true, that all non-zero mode off-coherences vanish, we find for the zero mode

$$\Xi_0 \sim t^{-(1-\frac{d}{2})} \ln t \quad (6.50)$$

which decays to zero and thus at least indicates that a relaxation into thermal equilibrium might be possible. Moreover, we observe that in this scenario the solution $Z(t)$ depends on the bath quantity γ while for $d \geq 2$ the bath scales entirely out. All these observations point towards the fact that *the actual coupling to the reservoir becomes less important in higher-dimensional open quantum dynamics.*

7. Conclusions

We studied the QSM as a simple exactly solvable model in order to explore exact quantum dynamics and compare classical to quantum dynamical properties. We used certain consistency criteria, in order to construct the precise form of the Lindblad master equation, namely (i) the quantum equilibrium is a stationary state of the chosen dynamics and (ii) the classical Langevin dynamics is included in the limit $g \rightarrow 0$. This guarantees that the equilibrium state is a stationary solution and that the canonical commutator relations are obeyed. As in equilibrium, for the QSM the full N -body problem reduces to solving to a single integro-differential equation, for the time-dependent spherical parameter. The full solution of this equation is still an open and difficult problem.

We have focussed in this work on two special cases. First, we considered weakly quantum dynamics and calculated the leading quantum corrections to the classical dynamics. It turns out that the effective quantum dynamics is classical and quantum effects only renormalise the temperature and produce a hard-core effect in the spin-spin correlator. Therefore, the heuristic expectation that the thermal noise should wash out the quantum properties of the long-time dynamics is indeed confirmed and the dynamics is equivalent to the purely relaxational classical model-A dynamics. This confirmation serves as a useful consistency check of the formalism we set up to describe the open quantum dynamics of the spherical model.

Second, we studied the true quantum dynamics driven by the initial disorder for a quantum quench across the critical point and deep into the ordered phase. In this regime, not explored before to the best of our knowledge, the model's long-time behaviour is distinct from any heuristic expectation. We found that the conserved canonical quantum commutators lead to profound modifications of the dynamics, with respect to its classical limit. In order to carry out this analysis, we explored new mathematical methods that are related to asymptotic expansions of confluent hyper-geometric functions in two variables. It turns out that the long-time behaviour of the integrated spherical parameter $Z(t)$ is extremely complex to deduce and that it depends on the spatial dimension in a non-trivial fashion. We have found

$$|Z(t)| \simeq \begin{cases} \frac{(d-2)^2}{4g} \frac{\ln^2 t}{t} & , \quad d > 2 \\ \varphi t^{-1} & , \quad d = 2 \\ \left(1 - \frac{d}{2}\right) \gamma^{-1} \ln t & , \quad \frac{4}{3} < d < 2 \end{cases} . \quad (7.1)$$

This behaviour is qualitatively different from the classical case where simply $|Z^{\text{cl}}(t)| \sim \ln t$.

Due to this strong dependence on the dimensionality of the system, we observed prominent differences in the scaling behaviour. In $d=2$ dimensions we find a regular scaling with a unique characteristic length scale. Thus, the QSM is able to reliably predict general qualitative properties. In $d \neq 2$ dimensions, we find strong logarithmic corrections which destroy a simple scaling behaviour, through the presence of several time-dependent length scales which differ by power of $\ln t$. One might be tempted to view these corrections as a peculiarity of the SM, as found long ago for the classical spherical model with a conserved order parameter [20, 58] and interpreted in a multi-scaling scenario. However, since we find a clean scaling in $d=2$ dimensions, we believe that the logarithmic corrections through several logarithmically different length scales should not be too readily dismissed as a peculiarity of the QSM. These results are asymptotically independent of the damping rate γ such that the limit $\gamma \rightarrow 0$ towards closed quantum systems may be taken. The dynamical exponent turns out to be $z=1$, indicative of ballistic motion, as seen before in the quantum dynamics of models with fermionic degrees of freedom. While ballistic motion is common for quantum systems near their quantum critical point [13, 14, 29, 31, 87] and actually is expected to occur for very general reasons [24], here we find it for quenches deeply into the two-phase ordered region, and so far unstudied with field-theoretical methods. For dimensions $d < 2$ we find again logarithmic corrections to scaling, but of a different kind, and in addition strong time-dependent modulations of the spin-spin correlator $C(R)$ in terms of the distance R . Here, the damping constant γ does appear in the scaling amplitudes.

These features are confirmed and strengthened through an analysis of the leading long-time behaviour of the characteristic length scale $L(t)^2$ and of the time-dependent susceptibility $\chi(t)$. They show simple power-law scaling for $d = 2$, with associated logarithmic corrections whenever $d \neq 2$ and thereby confirm the existence of several logarithmically different length scales.

This work should be seen as a first, tentative, exploration of non-equilibrium quantum dynamics, far from a critical point, of an interacting many-body system such as the QSM. Several essential assumptions and hypotheses were admitted throughout in our exploration. First of these, are the intrinsic Born approximation and the Markov property which underlie the Lindblad approach. Second, the main new results come from our study of the deep quantum quenches into the ordered phase. Our results crucially depend on the conjecture that the integral term in equation (4.27a) is irrelevant, hence will give rise only to finite-time corrections to scaling. Testing this conjecture remains a difficult open problem. Another aspect which should be further analysed is the precise nature of the relaxation process. Is the quantum relaxation in the QSM in some way reminiscent to the *physical ageing* seen in the classical analogues? Although we have found some preliminary indications which might point into this direction, a full testing of this will require to analyse the behaviour of two-time correlators, via the quantum regression theorem [10, 74], or even to include an external field and look at two-time response functions. We hope to return to this elsewhere. It would also be important to compare our results with what can be found from different approaches, notably Keldysh field-theory [76, 77] or the generalised hydrodynamics of strongly interacting non-equilibrium quantum systems [8, 18, 19, 25, 26, 65].

An attractive feature of the QSM is that the rôle of the dimension d can be analysed explicitly. Our result suggest, to the extend that the QSM is a reliable guide for collective quantum dynamical behaviour, that $2D$ quenched quantum systems should show simple dynamical scaling, with an easily achieved data-collapse, whereas in $3D$ quenched quantum systems it should only be possible to find a data-collapse in small time-dependent windows with effective time-dependent exponents. To what extent such an expectation is borne out in more general quantum models remains an important challenge for the future.

Acknowledgments

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Appendix A. Equilibrium quantum spherical constraint

We present the exact derivation of the quantum spherical constraint in the equilibrium SAQSM, by diagonalising the hamiltonian via canonical transformations. Consider the following hamiltonian, with bosonic operators a_n such that $[a_n, a_m^\dagger] = \delta_{n,m}$

$$H = \sum_{n,m \in \mathcal{L}} \left[a_n^\dagger A_{nm} a_m - \frac{1}{2} (a_n B_{nm} a_m + \text{h.c.}) \right] + \sum_{n \in \mathcal{L}} C_n (a_n + a_n^\dagger) \quad (\text{A.1})$$

which for a specific choice of the matrices A, B reduces to the hamiltonian (2.2) of the SAQSM. In addition, the vector \mathbf{C} allows to consider the effects of an external field. We shall present an exact derivation of the equilibrium spherical constraint, which should also arise from the stationary state ($t \rightarrow \infty$ limit) of the dynamics. Many aspects of the treatment are analogous to the one of free fermion hamiltonians, see e.g. [42, 54]. For the sake of notational simplicity, we only treat the 1D case explicitly, the generalisation to any $d > 1$ being obvious.

Define the harmonic oscillator ladder operators [82]

$$s_n = \left(\frac{g}{8\mathcal{S}} \right)^{1/4} (a_n + a_n^\dagger) \quad , \quad p_n = -i \left(\frac{\mathcal{S}}{2g} \right)^{1/4} (a_n - a_n^\dagger) \quad (\text{A.2})$$

and the spherical constraint is then

$$\begin{aligned} \mathcal{N} \sqrt{\frac{g}{8\mathcal{S}}} &= \sum_{n \in \mathcal{L}} \left(\langle a_n a_n \rangle + \langle a_n^\dagger a_n^\dagger \rangle + 2 \langle a_n^\dagger a_n \rangle + 1 \right) \\ &= |\langle \mathbf{a} \rangle|^2 + |\langle \mathbf{a}^\dagger \rangle|^2 + 2 \langle \mathbf{a}^\dagger \cdot \mathbf{a} \rangle + \mathcal{N}, \end{aligned} \quad (\text{A.3})$$

where we have introduced the vector $\mathbf{a} = (a_1, a_2, \dots, a_{N-1}, a_N)$ and its element-wise adjoint. We now apply the canonical transformation, used for the diagonalisation in [82]

$$\mathbf{a} = \mathbf{r} + v^t \mathbf{b} - w^t \mathbf{b}^\dagger \quad (\text{A.4})$$

to the spherical constraint and find

$$\begin{aligned} \mathcal{N} \sqrt{\frac{g}{8\mathcal{S}}} &= 4 |\mathbf{r}|^2 + 4 \mathbf{r} \cdot (v - w)^t \langle \mathbf{b} + \mathbf{b}^\dagger \rangle \\ &\quad + \sum_{lmn} (v_{ml} v_{nl} - 2 v_{ml} w_{nl} + w_{ml} w_{nl}) \langle b_m b_n \rangle \\ &\quad + \sum_{lmn} (w_{ml} w_{nl} - 2 w_{ml} v_{nl} + v_{ml} v_{nl}) \langle b_m^\dagger b_n^\dagger \rangle \\ &\quad + \sum_{lmn} (2 v_{ml} v_{nl} - v_{ml} w_{nl} + w_{ml} v_{nl}) \langle b_m^\dagger b_n \rangle \\ &\quad + \sum_{lmn} (2 w_{ml} w_{nl} - v_{ml} w_{nl} + w_{ml} v_{nl}) \langle b_n b_m^\dagger \rangle + \mathcal{N}. \end{aligned} \quad (\text{A.5})$$

Following [82], we define the matrix

$$\underline{\Psi} := (v - w)^t \quad (\text{A.6})$$

with the eigenvectors Ψ_n of $(A - B)(A + B)$ as column entries, see (A.1). Analogously, we define

$$\underline{\Phi} := (A + B)\underline{\Psi} \quad (\text{A.7})$$

(for a full analysis of the diagonalisation of H via canonical transformations, see [82, appendix A]).

Since $[b_n, b_m] = [b_n^\dagger, b_m^\dagger] = 0$, we can exchange the indices m and n in line 2 and 3 of equation (A.5) to find the same prefactor for $\langle b_m b_n \rangle$ and $\langle b_m^\dagger b_n^\dagger \rangle$. In the fifth line we use the commutation relation to achieve a normal order and estimate the prefactor of $\langle b_m^\dagger b_n \rangle$ from this and the fourth line. We find

$$\begin{aligned} 2\mathcal{N}\sqrt{\frac{\mathcal{S}}{g}} &= \mathcal{N} + 4|\mathbf{r}|^2 + 4\mathbf{r} \cdot \underline{\Psi} \langle \mathbf{b} + \mathbf{b}^\dagger \rangle + \sum_n (|\Psi_n|^2 - \Psi_n \cdot \Phi_n) \\ &\quad + \sum_{mn} \Psi_m \cdot \Psi_n \left(\langle \widehat{b}_m b_n \rangle + \langle b_m^\dagger b_n^\dagger \rangle + 2\langle b_m^\dagger b_n \rangle \right). \end{aligned}$$

Using the property $\Phi_n \cdot \Psi_n = 1$ [82], we can rewrite the spherical constraint as

$$\frac{\mathcal{N}}{2}\sqrt{\frac{\mathcal{S}}{g}} = |\mathbf{r}|^2 + \mathbf{r} \cdot \underline{\Psi} \langle \mathbf{b} + \mathbf{b}^\dagger \rangle + \sum_{mn} \frac{\Psi_m \cdot \Psi_n}{4} (\langle b_m b_n \rangle + \langle b_m^\dagger b_n^\dagger \rangle + 2\langle b_m^\dagger b_n \rangle + \delta_{nm}). \quad (\text{A.8})$$

Finally, we use the orthogonality of the eigenvectors of Toeplitz matrices to find

$$\frac{\mathcal{N}}{2}\sqrt{\frac{\mathcal{S}}{g}} = |\mathbf{r}|^2 + \mathbf{r} \cdot \underline{\Psi} \langle \mathbf{b} + \mathbf{b}^\dagger \rangle + \sum_n \frac{|\Psi_n|^2}{4} (\langle b_n b_n \rangle + \langle b_n^\dagger b_n^\dagger \rangle + 2\langle b_n^\dagger b_n \rangle + 1). \quad (\text{A.9})$$

For systems without an external magnetic field $\mathbf{r} = \mathbf{0}$ which we shall admit from now on. The absolute value of the eigenvectors was found in [82] to be

$$|\Psi_n|^2 = \frac{\Lambda_k^-}{\Lambda_k^+} = \sqrt{\frac{\mathcal{S} - \frac{1-\lambda}{2} \cos k}{\mathcal{S} - \frac{1+\lambda}{2} \cos k}}. \quad (\text{A.10})$$

With this result we can write the final result, in zero external field

$$\sqrt{\frac{8}{g}}\sqrt{\mathcal{S}} = \int_B \frac{dk}{2\pi} \frac{\Lambda_{-,k}}{\Lambda_{+,k}} \left(\langle b_k b_k \rangle + \langle b_k^\dagger b_k^\dagger \rangle + 2\langle b_k^\dagger b_k \rangle + 1 \right) \quad (\text{A.11})$$

which is easily generalised to d dimensions.

In equilibrium, the off-diagonal averages $\langle b_k b_k \rangle = \langle b_k^\dagger b_k^\dagger \rangle \rightarrow 0$ decay to zero and the number operator $\langle b_k^\dagger b_k \rangle$ is given by the thermal occupation of the corresponding mode

$$\sqrt{\frac{8}{g}}\mathcal{S}^{1/2} = \int_B \frac{d\mathbf{k}}{(2\pi)^d} \frac{\Lambda_{-,k}}{\Lambda_{+,k}} (2n_k + 1) \quad (\text{A.12})$$

which is equivalent to equation (4.14) in the main text.

Appendix B. Analysis of the Volterra equation

Solving the linear Volterra equation (5.9), at an effective temperature T^* , is standard, e.g. [22, 38, 45, 72]. Define the Laplace transform

$$\bar{f}(p) = \int_0^\infty dt f(t) e^{-pt} \quad (\text{B.1})$$

such that the Laplace-transformed equation (5.9) reads simply

$$\bar{G}(p) = \frac{\bar{F}(p)}{1 - \gamma T^* \bar{F}(p)}. \quad (\text{B.2})$$

Tauberian theorems [33, ch. XIII] permit to extract the long-time behaviour of $G(t)$ from the behaviour of $\bar{G}(p)$ for $p \rightarrow 0$. We require $\bar{F}(p) = \bar{F}_{\text{uni}}(p) + \bar{F}_{\text{reg}}(p)$, for p small, conveniently decomposed into an universal and a regular part, which have been derived countless times before

$$\bar{F}_{\text{uni}}(p) \stackrel{p \rightarrow 0}{\approx} \frac{\Gamma(1 - \frac{d}{2})}{\gamma(4\pi)^{\frac{d}{2}}} \left(\frac{p}{\gamma}\right)^{\frac{d}{2}-1}, \quad \bar{F}_{\text{reg}}(p) = \frac{1}{\gamma} \left(A_1 - A_2 \frac{p}{\gamma} + A_3 \left(\frac{p}{\gamma}\right)^2 \mp \dots \right) \quad (\text{B.3})$$

where the last expansion can only be carried to the point where the coefficients

$$A_n = \int_{\mathcal{B}} \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{\omega_{\mathbf{k}}^n}. \quad (\text{B.4})$$

exist ($\mathcal{B} = [-\pi, \pi]^d$ is the Brillouin zone). For example, even A_1 does not exist for $d \leq 2$ and A_2 only exists for $d > 4$. We conclude that

$$\bar{F}(p) \stackrel{p \rightarrow 0}{\approx} \frac{1}{\gamma} \begin{cases} \Gamma(1 - \frac{d}{2}) (4\pi)^{-\frac{d}{2}} (p/\gamma)^{\frac{d}{2}-1} & , \text{ if } 0 < d < 2 \\ A_1 - |\Gamma(1 - \frac{d}{2})| (4\pi)^{-\frac{d}{2}} (p/\gamma)^{\frac{d}{2}-1} & , \text{ if } 2 < d < 4 \\ A_1 - A_2 p/\gamma - |\Gamma(1 - \frac{d}{2})| (4\pi)^{-\frac{d}{2}} (p/\gamma)^{\frac{d}{2}-1} & , \text{ if } 4 < d < 6 \end{cases} \quad (\text{B.5})$$

In the last, we included the regular term which dominates for $d < 6$. Inserting into (B.2) gives $\bar{G}(p)$ which in turn must be inserted into the generic expression (5.11) for the spin-spin correlator, which we repeat here for convenience

$$Q_{\mathbf{k}}(t) = \frac{e^{-\gamma t \omega_{\mathbf{k}}}}{G(t)} + \frac{g}{12T} \left[1 - \frac{e^{-\gamma t \omega_{\mathbf{k}}}}{G(t)} \right] + \gamma T \frac{1}{G(t)} \int_0^t d\tau G(\tau) e^{-\gamma(t-\tau)\omega_{\mathbf{k}}}. \quad (\text{B.6})$$

We shall now study the three cases from (B.5) separately.

B.1. $0 < d < 2$

In this case, $\bar{F}(p)$ is a monotonous and surjective function on the interval $(0, \infty)$, hence the equation $1 - \gamma T^* \bar{F}(p) = 0$ always has a solution at $p = p_0$. Hence $\bar{G}(p)$ has a simple pole at some $p_0 = t_{\text{eq}}^{-1}$, for all $T^* > 0$. The leading long-time behaviour of $G(t)$ is exponential, with the explicit relaxation time

$$G(t) \sim e^{t/t_{\text{eq}}}, \quad t_{\text{eq}} = \gamma^{-1} \left[T^* \Gamma\left(1 - \frac{d}{2}\right) (4\pi)^{-d/2} \right]^{-\frac{2}{d-2}}. \quad (\text{B.7})$$

Inserting this into (B.6) leads straightforwardly to (5.13).

B.2. $2 < d < 4$

Since for dimensions $d > 2$ the coefficient A_1 is finite, its value can be used to define a critical temperature

$$T_c^* = \frac{1}{A_1}. \quad (\text{B.8})$$

Then three distinct situations can arise: (i) The case $T^* > T_c^*$ is treated analogously to the case $d < 2$. Here, the relaxation time is modified, because the phase transition does occur at finite temperature, according to

$$t_{\text{eq}} = \gamma^{-1} \left[\frac{T^* - T_c^*}{T^* T_c^*} \left| \Gamma \left(1 - \frac{d}{2} \right) \right| (4\pi)^{-d/2} \right]^{-\frac{2}{d-2}} \quad (\text{B.9})$$

but the correlator retains the form (5.13).

(ii) For $T^* < T_c^*$ we have to analyse equation (B.2) carefully. Define the short-hand

$$m^2 = 1 - T^*/T_c^* \quad (\text{B.10})$$

and expand $\bar{G}(p)$ to lowest non-trivial order in p to find

$$\begin{aligned} \bar{G}(p) &= \frac{1}{\gamma} \frac{A_1 - (4\pi)^{-d/2} \left| \Gamma \left(1 - \frac{d}{2} \right) \right| (p/\gamma)^{d/2-1}}{m^2 + T^*(4\pi)^{-d/2} \left| \Gamma \left(1 - \frac{d}{2} \right) \right| (p/\gamma)^{d/2-1}} \\ &\stackrel{p \rightarrow 0}{\simeq} \frac{1}{\gamma} \left[\frac{A_1}{m^2} - \frac{(4\pi)^{-d/2} \left| \Gamma \left(1 - \frac{d}{2} \right) \right| \left(\frac{p}{\gamma} \right)^{\frac{d}{2}-1}}{m^4} \right] + \dots \end{aligned} \quad (\text{B.11})$$

A Tauberian theorem [33] then gives the long-time behaviour of $G(t)$ by a formal inverse Laplace transform ($\delta(t)$ is the Dirac distribution)

$$G(t) \simeq \frac{1}{m^2 \gamma T_c^*} \delta(t) + \frac{(4\pi \gamma t)^{-d/2}}{m^4}, \text{ for } t \rightarrow \infty \text{ and } 2 < d < 4. \quad (\text{B.12})$$

The singular term therein, of course, does not appear in the long-time limit, but is required to evaluate the correlator. Following [45], we insert into (B.6) and obtain

$$\begin{aligned} Q_{\mathbf{k}}(t) &= e^{-\gamma \omega_{\mathbf{k}} t} m^4 (4\pi \gamma t)^{d/2} \left(1 - \frac{g}{12T} \right) + \frac{g}{12T} \\ &\quad + e^{-\gamma \omega_{\mathbf{k}} t} m^4 (4\pi \gamma t)^{d/2} \frac{\gamma T}{m^2 \gamma T_c^*} + \gamma T t^{d/2} \mathcal{L}^{-1} \left(\Gamma(1 - d/2) p^{1-d/2} \frac{1}{p + \gamma \omega_{\mathbf{k}}} \right) (t) \\ &= e^{-\gamma \omega_{\mathbf{k}} t} m^4 (4\pi \gamma t)^{d/2} \left(1 - \frac{g}{12T} + \frac{1}{m^2} \frac{T}{T_c^*} \right) + \frac{g}{12T} \\ &\quad + \gamma T t \frac{1}{1 - d/2} {}_1F_1 \left(1, 2 - \frac{d}{2}; -\gamma \omega_{\mathbf{k}} t \right) \\ &= e^{-\gamma \omega_{\mathbf{k}} t} m^2 (4\pi \gamma t)^{d/2} \left(1 - \frac{g}{12T} \right) + \frac{g}{12T} \\ &\quad + \frac{\gamma T}{1 - d/2} t e^{-\gamma \omega_{\mathbf{k}} t} {}_1F_1 \left(1 - \frac{d}{2}, 2 - \frac{d}{2}; \gamma \omega_{\mathbf{k}} t \right). \end{aligned} \quad (\text{B.13})$$

Herein, in the first two lines the terms proportional to T come from the integral in (B.6). The first of those in the contribution from the singular term in (B.12) and the other is cast into an inverse Laplace transformation. In the next step, this inverse transformation is found using [71, equation (2.1.2.1)] and the coefficient of the other term is simplified using the definitions of m^2 and of T^* . Finally, we used the identity [1, equation (13.1.27)]. We are interested in the limit $\mathbf{k} \rightarrow \mathbf{0}$, $t \rightarrow \infty$ such that $\omega_{\mathbf{k}}t$ remains finite. Then the last term is sub-dominant and we arrive at (5.14).

(iii) For $T^* = T_c^* = 1/A_1$, the leading terms in small- p expansion are

$$\overline{G}(p) = \frac{1}{\gamma} \left(\frac{1}{T_c^*} \right)^2 \frac{(4\pi)^{d/2}}{|\Gamma(1 - \frac{d}{2})|} \left(\frac{p}{\gamma} \right)^{1-d/2} - \frac{1}{\gamma T_c^*} + o(p) \quad (\text{B.14})$$

hence

$$G(t) = G_d t^{d/2-2} - \frac{1}{\gamma T_c^*} \delta(t) \quad (\text{B.15})$$

where G_d is a known constant whose value will not be required. Inserting into (B.6) and taking into account the contribution of the singular term in the integral gives

$$\begin{aligned} Q_{\mathbf{k}}(t) &= \frac{e^{-\gamma\omega_{\mathbf{k}}t} t^{2-d/2}}{G_d} \underbrace{\left(1 - \frac{g}{12T_c} - \frac{T_c}{T_c^*} \right)}_{=0} + \frac{g}{12T_c} \\ &\quad + \gamma T_c t^{2-d/2} \Gamma(d/2 - 1) \mathcal{L}^{-1} \left(p^{1-d/2} \frac{1}{p + \gamma\omega_{\mathbf{k}}} \right) (t) \\ &= \frac{g}{12T_c} + \frac{\gamma T}{d/2 - 1} {}_1F_1 \left(1, \frac{d}{2}; -\gamma\omega_{\mathbf{k}}t \right) t. \end{aligned} \quad (\text{B.16})$$

Herein, the first term vanishes because of the definition of T^* and we re-used [71, equation (2.1.2.1)]. This gives the first equation (5.16).

B.3. $d > 4$

The discussion is analogous to the previous ones. At $T^* = T_c^*$, expansion gives for small p gives $\overline{G}(p) \simeq \frac{1}{T_c^{*2} A_2} \frac{1}{p} - \frac{1}{\gamma T_c^*}$, hence

$$G(t) \simeq -\frac{1}{\gamma T_c^*} \delta(t) + \frac{1}{(T_c^*)^2 A_2}. \quad (\text{B.17})$$

Inserting this into (B.6) leads to

$$Q_{\mathbf{k}}(t) = (T_c^*)^2 A_2 e^{-\gamma\omega_{\mathbf{k}}t} \underbrace{\left(1 - \frac{g}{12T_c} + \frac{T_c}{T_c^*} \right)}_{=0} + \frac{g}{12T_c} + \frac{T_c}{\omega_{\mathbf{k}}} (1 - e^{-\gamma\omega_{\mathbf{k}}t}) \quad (\text{B.18})$$

where we used again the definition of T^* and have thus found the second equation (5.16). Finally, below criticality, we must expand up to the first universal term. We obtain for p small (as it stands, this holds for $d < 6$, but extensions are obvious)

$$\overline{G}(p) \simeq \frac{1}{\gamma} \frac{A_1 - A_2 \frac{p}{\gamma} - |\mathcal{F}|_1 \left(\frac{p}{\gamma}\right)^{d/2-1}}{m^2 + T^* A_2 \frac{p}{\gamma} + \gamma T^* |\mathcal{F}|_1 \left(\frac{p}{\gamma}\right)^{d/2-1}} \simeq \frac{1}{\gamma} \left(\frac{A_1}{m^2} - \frac{A_2 p}{m^4 \gamma} \right) - \frac{|\mathcal{F}|_1}{m^4} \left(\frac{p}{\gamma}\right)^{d/2-1} \quad (\text{B.19})$$

which gives for large times

$$G(t) \simeq \frac{1}{m^2 \gamma T_c^*} \delta(t) - \frac{A_2}{m^4 \gamma^2} \delta'(t) + \frac{(4\pi \gamma t)^{-d/2}}{m^4} \quad (\text{B.20})$$

and from which one readily arrives again at equation (5.14).

We remark that the small- p expansions must be carried up to including (i) eventual constant terms and (ii) the leading universal contribution. The first contribution is required for the correct evaluation of the correlator (unless one prefers to derive sum rules instead, as carried out in [38]) and the second contribution gives the leading time-dependence.

We did not discuss the case $d = 4$ explicitly, although this can be done without much extra difficulty [30, 40, 45]. Below criticality, there is no dimension-dependent singularity and one may simply set $d = 4$ in the final result (5.14) and at criticality, additional logarithmic singularities will appear.

Appendix C. Proof of an identity

We prove the asymptotic identity equation (6.12).

Lemma. *The function $\mathfrak{f}(\gamma) = e^{-\gamma Z} (4\pi \gamma t)^{-d/2}$ obeys for all $d > 0$ and all Z, t the identity*

$$\partial_\gamma^n \mathfrak{f}(\gamma) = (-1)^n \mathfrak{f}(\gamma) \sum_{k=0}^n \Gamma \left[\begin{matrix} n+1 & \frac{d}{2} + k \\ \frac{d}{2} & n-k+1 & k+1 \end{matrix} \right] \gamma^{-k} Z^{n-k} \quad (\text{C.1})$$

Proof. This proceeds via mathematical induction, with the habitual two steps.

- *Basis $n = 1$:* it suffices to calculate the first derivative and compare with (C.1). We find straightforwardly, in both cases

$$\partial_\gamma \mathfrak{f}(\gamma) = -\mathfrak{f}(\gamma) \left[Z + \frac{d}{2\gamma} \right]. \quad (\text{C.2})$$

- *Step $n \rightarrow n + 1$:* We write

$$\partial_\gamma^{n+1} \mathfrak{f}(\gamma) = \partial_\gamma \partial_\gamma^n \mathfrak{f}(\gamma) \quad (\text{C.3})$$

and use the expression (C.1) to find

$$\partial_\gamma^{n+1} \mathfrak{f}(\gamma) = (-1)^{n+1} \mathfrak{f}(\gamma) \sum_{k=0}^n \Gamma \left[\begin{matrix} n+1 & \frac{d}{2} + k \\ \frac{d}{2} & k+1 & n-k+1 \end{matrix} \right] \gamma^{-k} Z^{n-k} \left\{ Z + \frac{d}{2\gamma} + \frac{k}{\gamma} \right\}. \quad (\text{C.4})$$

Shifting the index n to $n+1$ produces

$$\begin{aligned} \partial_\gamma^{n+1} \mathfrak{f}(\gamma) &= (-1)^{n+1} \mathfrak{f}(\gamma) \sum_{k=0}^{n+1} \left\{ \Gamma \left[\begin{matrix} n+2 & \frac{d}{2} + k \\ \frac{d}{2} & k+1 & n-k+2 \end{matrix} \right] \gamma^{-k} Z^{n+1-k} \right. \\ &\quad \times \left. \left[1 - \frac{k}{n+1} \right] \left[1 + \frac{d}{2Z\gamma} + \frac{k}{Z\gamma} \right] \right\}. \end{aligned}$$

Herein, the first line is already the sought expression for the $(n+1)^{\text{st}}$ derivative. It only remains to show that the residual terms

$$\sum_{k=0}^{n+1} \Gamma \left[\begin{matrix} n+2 & \frac{d}{2} + k \\ \frac{d}{2} & k+1 & n-k+2 \end{matrix} \right] \gamma^{-k} Z^{n+1-k} \left\{ \frac{d}{2Z\gamma} + \frac{k}{Z\gamma} - \frac{k}{n+1} \left[1 + \frac{d}{2Z\gamma} + \frac{k}{Z\gamma} \right] \right\} \quad (\text{C.5})$$

cancel. For simplicity we omit non-zero multiplicative factors and consider¹⁵

$$\begin{aligned} &\sum_{k=0}^{n+1} \Gamma \left[\begin{matrix} \frac{d}{2} + k \\ k+1 & n-k+2 \end{matrix} \right] (\gamma Z)^{-k} \left[\left(\frac{d}{2} + k \right) (n+1-k) + k\gamma Z \right] \\ &= \sum_{k=0}^n \Gamma \left[\begin{matrix} \frac{d}{2} + k+1 \\ k+1 & n-k+1 \end{matrix} \right] (\gamma Z)^{-k} - \sum_{k=1}^n \Gamma \left[\begin{matrix} \frac{d}{2} + k \\ k & n-k+2 \end{matrix} \right] (\gamma Z)^{-k-1} = 0 \end{aligned} \quad (\text{C.6})$$

which completes the proof. \square

Appendix D. Asymptotic analysis of some double series

In the main text, we introduced two double series

$$\mathfrak{s}_1 := \sum_{n=0}^{\infty} \sum_{k=0}^n \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + \frac{1}{2} & \frac{d}{2} & n-k+1 & k+1 \end{matrix} \right] \left(-\frac{gt}{\gamma} \right)^n (\gamma Z)^{n-k} \quad (\text{D.1})$$

$$\mathfrak{s}_2 := -\gamma \mathcal{C} g t \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + \frac{1}{2} & \frac{d}{2} & n-k & k+1 \end{matrix} \right] \frac{1}{n} \left(-\frac{gt}{\gamma} \right)^n (\gamma Z)^{n-1-k} \quad (\text{D.2})$$

and we require their asymptotic behaviour for $t \gg 1$ large, where Z is either being kept fixed or varies slowly with t .

1. We start our analysis with the treatment of \mathfrak{s}_1 . Begin with (D.1) and exchange the order of summation, followed by a shift in the second summation variable. This results in

¹⁵ In (C.5), bring the curly bracket to the common denominator, which does not depend on k and hence can be dropped.

$$\begin{aligned}\mathfrak{s}_1 &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + \frac{1}{2} & \frac{d}{2} & n - k + 1 & k + 1 \end{matrix} \right] \left(-\frac{gt}{\gamma} \right)^n (\gamma Z)^{n-k} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + k + \frac{1}{2} & \frac{d}{2} & n + 1 & k + 1 \end{matrix} \right] \left(-\frac{gt}{\gamma} \right)^{n+k} (\gamma Z)^n\end{aligned}\quad (\text{D.3a})$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{d}{2})} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(k + \frac{d}{2})}{\Gamma(k + n + \frac{1}{2})} \frac{(-gt/\gamma)^k}{k!} \frac{(-gtZ)^n}{n!}\quad (\text{D.3b})$$

Recalling the definition of the Humbert function [47, 48]

$$\Phi_3(\beta; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}\quad (\text{D.4})$$

we can identify $\mathfrak{s}_1 = \Phi_3\left(\frac{d}{2}; \frac{1}{2}; -gtZ, -\frac{gt}{\gamma}\right)$, as stated in (6.14) in the main text.

Sums such as (D.3a) would be easy to evaluate if they would factorise, but in fact they are coupled by the factor $\Gamma(n + k + \frac{1}{2})$ in the denominator. In order to achieve a factorisation, we use the following identity, which involves Euler's Beta function, with an arbitrary constant $0 < \epsilon < \frac{1}{2}$

$$\frac{1}{\Gamma(n + k + \frac{1}{2})} = \frac{B(n + \epsilon, \frac{1}{2} + k - \epsilon)}{\Gamma(\epsilon + n)\Gamma(\frac{1}{2} - \epsilon + k)} = \frac{t^{\frac{1}{2} - n - k}}{\Gamma(\epsilon + n)\Gamma(\frac{1}{2} - \epsilon + k)} \int_0^t dx x^{k - \epsilon - \frac{1}{2}} (t - x)^{n + \epsilon - 1}\quad (\text{D.5})$$

which is obtained from equations (6.2.1) and (6.2.2) in [1]. Now, insert this identity into (D.3a) such that the sums over n and k decouple. We then find

$$\mathfrak{s}_1 = \Gamma \left[\begin{matrix} \frac{1}{2} \\ \frac{1}{2} - \epsilon, \epsilon \end{matrix} \right] \sqrt{t} (\mathbf{u}_1 \star \mathbf{v}_1)(t)\quad (\text{D.6})$$

with $0 < \epsilon < \frac{1}{2}$ and the functions

$$\mathbf{u}_1(x) = x^{-\frac{1}{2} - \epsilon} {}_1F_1\left(\frac{d}{2}; \frac{1}{2} - \epsilon; -\frac{g}{\gamma}x\right), \quad \mathbf{v}_1(x) = x^{\epsilon - 1} {}_0F_1(\epsilon; -gZx).\quad (\text{D.7})$$

Inserting the functions from (D.7) then gives the exact representation of \mathfrak{s}_1 as a Laplace convolution, stated in (6.16) in the main text, where the Laplace transform is defined as

$$\bar{\mathfrak{h}}(p) := \mathcal{L}(\mathfrak{h})(p) = \int_0^{\infty} dx \mathfrak{h}(x) e^{-px}.\quad (\text{D.8})$$

The *Laplace convolution theorem* states $(\mathbf{u}_1 \star \mathbf{v}_1)(t) = \mathcal{L}^{-1}(\bar{\mathbf{u}}_1(p)\bar{\mathbf{v}}_1(p))(t)$.

In addition, combining the representation (6.16) and (D.6) with the Laplace convolution theorem gives access to the large- t asymptotics of \mathfrak{s}_1 , via a Tauberian theorem [33]: *find the small- p behaviour for $\bar{\mathbf{u}}_1(p)$ and $\bar{\mathbf{v}}_1(p)$ and then carry out the inverse Laplace transform*. Therefore, we use equation (3.38.1.1) from [70] and find

$$\bar{\mathbf{u}}_1(p) = \Gamma\left(\frac{1}{2} - \epsilon\right) p^{\epsilon - \frac{1}{2}} \left(1 + \frac{g}{\gamma p}\right)^{-d/2}, \quad \bar{\mathbf{v}}_1(p) = \Gamma(\epsilon) p^{-\epsilon} e^{-\frac{gZ}{p}}. \quad (\text{D.9})$$

The small- p expansion of the product $\bar{\mathbf{u}}_1(p)\bar{\mathbf{v}}_1(p)$ yields¹⁶

$$\mathfrak{s}_1 \stackrel{p \searrow 0}{\simeq} \sqrt{\pi t} \left(\frac{\gamma}{g}\right)^{\frac{d}{2}} \mathcal{L}^{-1}\left(p^{\frac{d-1}{2}} e^{-\frac{gZ}{p}}\right)(t) \quad (\text{D.10})$$

and the inverse Laplace transform can be extracted from equation (2.2.2.1) in [71]

$$\mathfrak{s}_1 \simeq \sqrt{\pi} \left(\frac{\gamma}{gt}\right)^{\frac{d}{2}} \frac{{}_0F_1\left(\frac{1-d}{2}; -gtZ\right)}{\Gamma\left(\frac{1-d}{2}\right)}. \quad (\text{D.11})$$

2. For \mathfrak{s}_2 our approach is analogous. Starting from (D.2), we shift variables and exchange the order of summation to arrive at

$$\begin{aligned} \mathfrak{s}_2 &= -\gamma \mathcal{C} g t \sum_{n=1}^{\infty} \sum_{k=1}^n \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} - 1 + k \\ n + \frac{1}{2} & \frac{d}{2} \end{matrix} \quad n - k + 1 \quad k \right] \frac{1}{n} \left(-\frac{gt}{\gamma}\right)^n (\gamma Z)^{n-k} \\ &= -\gamma \mathcal{C} g t \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} - 1 + k \\ n + \frac{1}{2} & \frac{d}{2} \end{matrix} \quad n - k + 1 \quad k \right] \frac{1}{n} \left(-\frac{gt}{\gamma}\right)^n (\gamma Z)^{n-k} \\ &= -\gamma \mathcal{C} g t \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ n + 1 & \frac{d}{2} \end{matrix} \quad n + k + \frac{3}{2} \quad k + 1 \right] \frac{1}{n + k + 1} \left(-\frac{gt}{\gamma}\right)^{n+k} (\gamma Z)^n \\ &= \gamma \mathcal{C} g^2 t^2 \int_0^{\infty} dv e^{-v} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \Gamma \left[\begin{matrix} \frac{1}{2} & \frac{d}{2} + k \\ \frac{d}{2} & n + k + \frac{3}{2} \end{matrix} \right] \frac{(-gte^{-v}/\gamma)^k}{k!} \frac{(-gtZe^{-v})^n}{n!}. \end{aligned} \quad (\text{D.12})$$

With the definition (D.4) of the Humbert function Φ_3 , we can also identify

$$\mathfrak{s}_2 = 2\mathcal{C} g^2 t^2 \int_0^1 dw \Phi_3\left(\frac{d}{2}; \frac{3}{2}; -\frac{gt}{\gamma} w, -gtZw\right), \text{ as stated in (6.15) in the main text.}$$

The two sums can be decoupled via the identity, with $0 < \epsilon < \frac{3}{2}$

$$\frac{1}{\Gamma\left(n + k + \frac{3}{2}\right)} = \frac{t^{\frac{1}{2} - n - k}}{\Gamma(\epsilon + n)\Gamma\left(\frac{3}{2} - \epsilon + k\right)} \int_0^t dx x^{k - \epsilon - \frac{3}{2}} (t - x)^{n + \epsilon - 1} \quad (\text{D.13})$$

¹⁶ Here we explicitly treat the quantum case $g \neq 0$. Admitting $g = 0$ leads to a different small- p expansion that results in the well-known classical zero-temperature quench dynamics [38].

such that we finally recast the double sum into an integrated convolution

$$\mathfrak{s}_2 = \mathcal{C}g^2 t^{\frac{3}{2}} \Gamma \left[\begin{matrix} \frac{1}{2} \\ \frac{3}{2} - \epsilon, \epsilon \end{matrix} \right] \int_0^1 dw (\mathfrak{u}_2 \star \mathfrak{v}_2)(t) \quad (\text{D.14})$$

with

$$\mathfrak{u}_2(x) = x^{\frac{1}{2}-\epsilon} {}_1F_1 \left(\frac{d}{2}, \frac{3}{2} - \epsilon, -\frac{g}{\gamma} wx \right), \quad \mathfrak{v}_2(x) = x^{\epsilon-1} {}_0F_1 (\epsilon, -gZwx) \quad (\text{D.15})$$

as stated in (6.17) in the main text. Finally, the asymptotics for $t \rightarrow \infty$ is found as before from a Tauberian theorem. The Laplace transforms of the above functions read [70, 3.38.1.1]

$$\bar{\mathfrak{u}}_2(p) = \Gamma \left(\frac{3}{2} - \epsilon \right) p^{\epsilon-\frac{3}{2}} \left(1 + \frac{gw}{\gamma p} \right)^{-\frac{d}{2}}, \quad \bar{\mathfrak{v}}_2(p) = \Gamma(\epsilon) p^{-\epsilon} e^{-gZ\frac{w}{p}}. \quad (\text{D.16})$$

Inserting leads to the expression

$$\mathfrak{s}_2 = \mathcal{C}g^2 \sqrt{\pi t^3} \mathcal{L}^{-1} \left(p^{\frac{d-3}{2}} \int_0^1 dw \left(p + \frac{g}{\gamma} w \right)^{-\frac{d}{2}} e^{-gZw/p} \right) (t). \quad (\text{D.17})$$

The w -integration can be expressed exactly as an incomplete Gamma function [1]

$$\mathfrak{s}_2 = \gamma \mathcal{C}g \sqrt{\pi t^3} (\gamma Z)^{\frac{d}{2}-1} e^{\gamma Z} \mathcal{L}^{-1} \left[\Gamma \left(1 - \frac{d}{2}, \gamma Z \right) \frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p}} \Gamma \left(1 - \frac{d}{2}, Z\frac{g}{p} + \gamma Z \right) \right] (t). \quad (\text{D.18})$$

Since we now want to study this expression in the $p \rightarrow 0$ limit, it is adequate to use an asymptotic expansion for the last term, which we extract from equation [1, 6.5.30]

$$\Gamma(a, x+y) \stackrel{x \rightarrow \infty}{\simeq} \Gamma(a, x) - e^{-x} x^{a-1} (1 - e^{-y}). \quad (\text{D.19})$$

In order to evaluate the inverse Laplace transform, we consult equations (2.2.2.1), (3.10.2.2) and (2.1.1.3) in [71] and find

$$\mathfrak{s}_2 \simeq \gamma \mathcal{C}gt \left\{ \frac{{}_1F_1 \left(1; 2 - \frac{d}{2}; \gamma Z \right)}{\frac{d}{2} - 1} + \sqrt{\pi} \left(\frac{\gamma}{gt} \right)^{\frac{d}{2}-1} \left[\frac{{}_1F_2 \left(1 - \frac{d}{2}; 2 - \frac{d}{2}, \frac{3-d}{2}; -gtZ \right)}{\left(1 - \frac{d}{2} \right) \Gamma \left(\frac{3-d}{2} \right)} e^{\gamma Z} + \frac{e^{\gamma Z} - 1}{} {}_0F_1 \left(\frac{1-d}{2}; -gtZ \right)}{\Gamma \left(1 - \frac{d}{2} \right)} \right] \right\}. \quad (\text{D.20})$$

Finally, combining equations (D.11) and (D.20) and inserting into the constraint (6.13), we arrive at the asymptotic form (6.18) of the spherical constraint.

Similar methods can be applied to find the asymptotics of several confluent Appell's hyper-geometric function F_3 [2, 79], when both arguments become large. This will be presented elsewhere [84].

Appendix E. Spherical constraint in two spatial dimensions

The constraint (6.13) requires a special analysis in two spatial dimensions, due to apparent divergences in equation (D.20) for $d \rightarrow 2$. We carry this out by writing $d = 2(1 + \varepsilon)$ and studying the limit $\varepsilon \rightarrow 0$. We want to show that equation (D.20) is indeed well-defined in the $d \rightarrow 2$ limit and to find this limit.

The critical sum is \mathfrak{s}_2 , which may be rewritten as

$$\mathfrak{s}_2 \simeq \mathcal{C}gt\gamma \left[e^{\gamma Z} \left(\frac{1}{\varepsilon} - \frac{{}_1F_2(-\varepsilon; 1, \frac{1}{2}; -gtZ)}{\varepsilon} \right) + \frac{e^{\gamma Z} - 1}{gtZ} \sqrt{\pi} \frac{{}_0F_1(-\frac{1}{2}; -gtZ)}{\Gamma(-\frac{1}{2})} \right]. \quad (\text{E.1})$$

The limit where ε goes to zero can be taken using the formula (derived below)

$$\lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} - \frac{{}_1F_2(-\varepsilon; 1, \frac{1}{2}; x)}{\varepsilon} \right) = 2x {}_2F_3\left(1, 1; \frac{3}{2}, 2, 2; x\right) \quad (\text{E.2})$$

and renders the sum \mathfrak{s}_2 into the form

$$\mathfrak{s}_2 \simeq \mathcal{C}gt\gamma \left[-e^{\gamma Z} 2gtZ {}_2F_3\left(1, 1; \frac{3}{2}, 2, 2; -gtZ\right) - 2 \frac{e^{\gamma Z} - 1}{gtZ} {}_0F_1\left(-\frac{1}{2}; -gtZ\right) \right]. \quad (\text{E.3})$$

Recalling (D.11), we can now study the constraint (6.13) in 2D. Solving the spherical constraint numerically, see figure E1, we remark that the observations $Z = -|Z| < 0$ and $Z \rightarrow 0$ still hold true in the long-time limit $t \rightarrow \infty$. However, an asymptotic expansion for $t|Z| \rightarrow \infty$ fails. Therefore, we must consider $t|Z| =: \varphi \rightarrow \text{cste.}$ and proceed to determine this constant. Asymptotically, the constraint (6.13) reads

$$8\pi\gamma t \simeq 1 - \sqrt{\frac{1}{4t}} \frac{\gamma}{g} {}_0F_1\left(-\frac{1}{2}; g\varphi\right) + \mathcal{C}g\gamma t \left[2g\varphi {}_2F_3\left(1, 1; \frac{3}{2}, 2, 2; g\varphi\right) - \frac{2\gamma}{gt} {}_0F_1\left(-\frac{1}{2}; g\varphi\right) \right] \quad (\text{E.4})$$

where we replaced $e^{\gamma Z} \mapsto 1$. We also observe that the first and the last term on the right-hand side are sub-dominant. For the constant φ we thus find the transcendental equation

$$\frac{4\pi}{\mathcal{C}g^2} = \varphi {}_2F_3\left(1, 1; \frac{3}{2}, 2, 2; g\varphi\right) \quad (\text{E.5})$$

which is equation (6.21b) in the main text. It always has an unique solution since the image of the right-hand side is \mathbb{R}^+ and the function is monotonous. The spherical parameter then reads $Z \simeq -\varphi/t$.

In the limit of an extreme SCDL with $\mathcal{C} \rightarrow \infty$, see figure 4, we have simply $\varphi\mathcal{C} = 4\pi g^{-2}$. The opposite limit of an extreme SQDL with $\mathcal{C} = \frac{1}{4}$ gives an upper bound for the admissible values of φ .

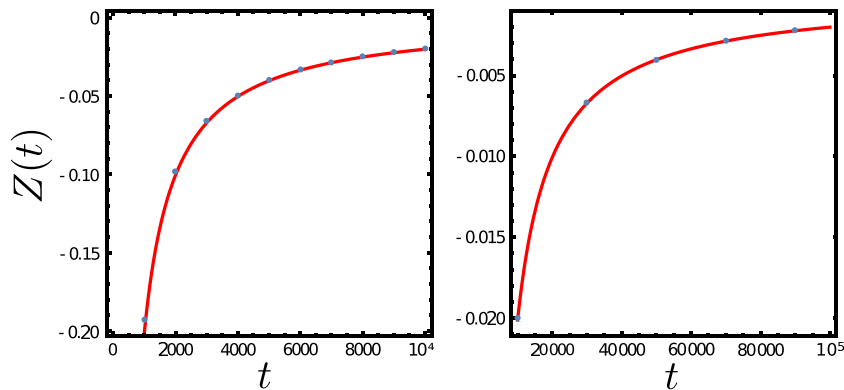


Figure E1. Solution $Z(t)$ of the spherical constraint for $d = 2$, $g = 0.1$, $\gamma = 1$ and $\mathcal{C} = 1/4$. The full curve is from equation (E.5) and the dots are numerical data. The left and right panel display different intervals for t .

Proof of equation (B.2). Insert the expansion $\Gamma(n - \varepsilon)/\Gamma(-\varepsilon) \simeq -\Gamma(n)\varepsilon + O(\varepsilon^2)$ into

$$\begin{aligned} \frac{1}{\varepsilon} \left[1 - {}_1F_2 \left(-\varepsilon; 1, \frac{1}{2}; x \right) \right] &= -\frac{1}{\varepsilon} \sum_{n=1}^{\infty} \frac{x^n}{n! (1)_n (\frac{1}{2})_n} \frac{\Gamma(n - \varepsilon)}{\Gamma(-\varepsilon)} \\ &\simeq \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \frac{n! (1)_n}{(2)_n (2)_n (\frac{3}{2})_n \frac{1}{2}} + O(\varepsilon) = 2x {}_2F_3 \left(1, 1; 2, 2, \frac{3}{2}; x \right) + O(\varepsilon) \end{aligned}$$

by also using $(a)_{n+1} = (a+1)_n \frac{\Gamma(a+1)}{\Gamma(a)}$. □

Appendix F. Analysis of the spherical constraint for $d \neq 2$

We present the asymptotic analysis of the spherical constraint (6.18) in generic dimensions $d \neq 2$.

F.1. $d > 2$

In order to define the goals of an asymptotic analysis, we first consider the qualitative behaviour of the numerical solution $Z = Z(t)$, illustrated in figure F1. Therein, both the left-hand side (LHS) and the right-hand-side (RHS) are displayed as a function of Z , for certain values of t , and for typical values of \mathcal{C} , g and γ . The solution $Z = Z(t)$ is given by the intersections of the black and one of the coloured lines, respectively. For large times and for dimensions $d > 2$, the numerical examples suggest the following properties, which we shall need for our further analysis:

1. The solution to the spherical constraint is *unique* and *negative*, which is clear from figure F1¹⁷.
2. In the asymptotic limit where $t \rightarrow \infty$, the solution tends to $Z \rightarrow 0^-$. This is apparent in figure F1 and further shown in the left panel of figure 6 in the main text.

¹⁷ We have checked numerically that $Z < 0$ for times up to $t \approx 10^{51}$.

- the decay of Z is slower than $O(t^{-1})$, such that $t|Z(t)|$ still increases with t , as further illustrated in the right panel of figure 6.

In figure 6 in the main text, the time-dependence of $Z(t)$ is further illustrated for the generic spatial dimensions $d \neq 2$. The qualitative shape of these curves does not depend much on the specific values of the other parameters. Therefore, these examples suggest that the sought long-time behaviour can be obtained by studying the asymptotics for $t|Z(t)| \rightarrow \infty$ in (6.18), at least when $d \neq 2$. A more detailed study further suggests that this growth is more slow than any power-law.

Therefore, we need the following expansions of the various hyper-geometric functions in (6.18) for $t|Z(t)| \gg 1$ and $|Z(t)| \ll 1$. This is achieved by the asymptotic identities, see [1] and especially [86, 07.22.06.0005.01]

$${}_1F_2\left(1 - \frac{d}{2}; 2 - \frac{d}{2}, \frac{3-d}{2}; gt|Z|\right) \simeq -\left(1 - \frac{d}{2}\right) \frac{\Gamma(\frac{3}{2} - \frac{d}{2})}{\Gamma(-\frac{1}{2})} \frac{e^{2\sqrt{gt|Z|}}}{(gt|Z|)^{1-d/4}} \quad (\text{F.1a})$$

$${}_0F_1\left(\frac{1-d}{2}; gt|Z|\right) \simeq -\frac{\Gamma(\frac{1}{2} - \frac{d}{2})}{\Gamma(-\frac{1}{2})} (gt|Z|)^{d/4} e^{2\sqrt{gt|Z|}} \quad (\text{F.1b})$$

$${}_1F_1\left(1; 2 - \frac{d}{2}; \gamma Z\right) \simeq 1 + \frac{\gamma Z}{2 - \frac{d}{2}} \quad (\text{F.1c})$$

which simplify the constraint (6.18) to the following form

$$e^{\gamma Z} (4\pi\gamma t)^{d/2} \simeq \frac{1}{2} + \left(1 + \mathcal{C}gt\gamma \left[1 + \frac{1}{\gamma|Z|}\right]\right) \left(\frac{\gamma^2|Z|}{gt}\right)^{\frac{d}{4}} \frac{e^{2\sqrt{gt|Z|}}}{4} + \frac{\gamma\mathcal{C}gt}{d-2} \left[1 + \frac{4}{d-4}\gamma|Z|\right]. \quad (\text{F.2})$$

Herein, the last term on the right-hand side is sub-dominant. We can therefore neglect it and arrive at the following final form of the constraint

$$2e^{\gamma Z} (4\pi\gamma t)^{d/2} \simeq \left[1 + \frac{\gamma^{\frac{d}{2}}}{2} \left(1 + \mathcal{C}\frac{gt}{|Z|}\right) \left(\frac{|Z|}{gt}\right)^{\frac{d}{4}} e^{2\sqrt{gt|Z|}}\right] \quad (\text{F.3})$$

which is equation (6.21a) in the main text. As before in the toy case where $\mathcal{C} = 0$ and analysed in the main text, the constraint can be solved explicitly in terms of W -functions, but some care is needed to select the correct real-valued branch [21], which is either W_0 or W_{-1} , such that positive values for $|Z(t)|$ are produced. We find

$$t|Z(t)| \simeq \frac{(d-4)^2}{16g} \begin{cases} W_{-1}^2\left(\frac{2g}{d-4} \left[\frac{(8\pi)^d}{\mathcal{C}^2}\right]^{\frac{1}{d-4}} t^{2\frac{d-2}{d-4}}\right) & , d < 4 \\ \left(\frac{2}{d-4}\right)^2 \ln^2\left(\frac{(8\pi t)^2}{\mathcal{C}}\right) & , d = 4 \\ W_0^2\left(\frac{2g}{d-4} \left[\frac{(8\pi)^d}{\mathcal{C}^2}\right]^{\frac{1}{d-4}} t^{2\frac{d-2}{d-4}}\right) & , d > 4 \end{cases} \quad (\text{F.4})$$

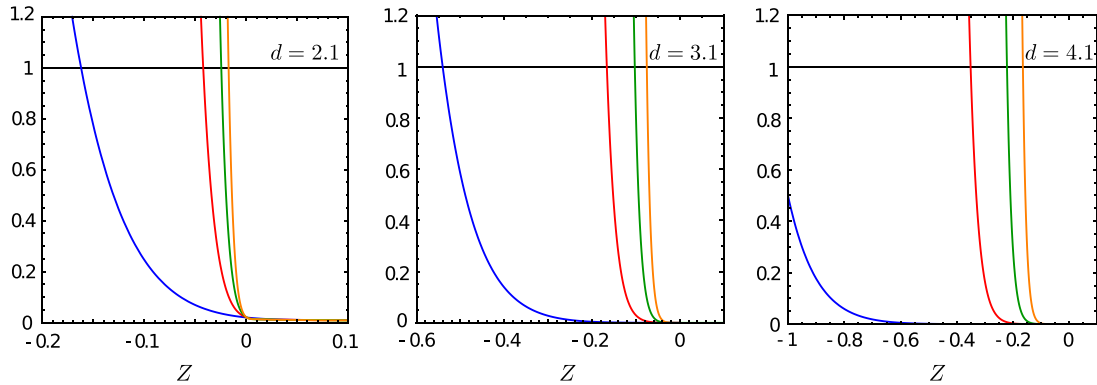


Figure F1. Solving the constraint (6.18) as a function of Z : the LHS is shown in black and the RHS is shown for different times $t = [1000, 4000, 7000, 10\,000]$ corresponding to the blue, red, green and orange lines, from left to right. The other parameters are $\mathcal{C} = 1$, $\gamma = 0.1$ and $g = 0.1$. Different frames correspond to different dimensions: left panel $d = 2.1$, middle panel $d = 3.1$, right panel $d = 4.1$.

The leading behaviour is found from the known asymptotics of the W -function¹⁸ to be

$$|Z(t)| \simeq \frac{(d-2)^2 \ln^2 t}{4g} \frac{1}{t} \quad (\text{F.5})$$

for all dimensions $d > 2$. This asymptotic result does neither depend explicitly on the initial condition \mathcal{C} nor on the coupling γ to the bath.

F.2. $1 < d < 2$

Again, we try to identify the correct mathematical setting by looking at some numerical solutions of the constraint (6.18). We illustrate in figure F2 some typical behaviour, for several values of d . Clearly, the left panel shows that for $d < 2$ the qualitative behaviour is different from what was seen for $d > 2$. We observe as generic features

1. For large enough times, the solution to the spherical constraint becomes *positive*.
2. In the asymptotic limit $t \rightarrow \infty$, the solution $Z(t)$ grows beyond all bounds, but its growth is very slow compared to t .
3. Strong oscillations are superposed onto this growth, the frequency of whom apparently increase with t , while the amplitude decreases.
4. There is a regime of large intermediate times, where the solution $Z = -|Z(t)| < 0$ is negative and qualitatively behaves as seen above for dimensions $d > 2$. This is illustrated in the middle panel of figure F2, which is very similar to figure F1. In the right panel, it is further shown that for truly enormous times the final true asymptotic regime with $Z > 0$ is reached.

Therefore, for intermediate times, we can take over the analysis for $d > 2$ and recover equation (F.5) as an effective description¹⁹. One can estimate the order of the time-scale

¹⁸ One uses $W_{-1}(x) \simeq \ln(-x) - \ln(-\ln(-x)) + o(1)$ for $x \rightarrow 0^-$ [21].

¹⁹ Equation (F.4) with $d < 4$ applies.

t_{\times} where this cross-over happens by setting $Z = 0$ in the constraint (6.18). For $d = 2 - \varepsilon$ dimensions, we find $t_{\times} \approx \frac{\gamma}{g} e^{8\pi/Cg}$ which for the chosen parameters can become very large indeed.

In order to find the true final asymptotics for really large values of t , we must re-analyse (6.18) in the limit where $t \rightarrow \infty$ and $Z \gg 1$. We then require the following asymptotic expansions, see [1] and [86, 07.22.06.0011.01]

$${}_1F_1\left(1; 2 - \frac{d}{2}; \gamma Z\right) = \left(1 - \frac{d}{2}\right) e^{\gamma Z} (\gamma Z)^{d/2-1} \left[\Gamma\left(1 - \frac{d}{2}\right) - \Gamma\left(1 - \frac{d}{2}, \gamma Z\right)\right] \\ \simeq \left(1 - \frac{d}{2}\right) e^{\gamma Z} (\gamma Z)^{d/2-1} \Gamma\left(1 - \frac{d}{2}\right) - \left(1 - \frac{d}{2}\right) (\gamma Z)^{-1} \quad (\text{F.6a})$$

$$\frac{{}_0F_1\left(\frac{1-d}{2}; -gtZ\right)}{\Gamma(1-d/2)} = (gtZ)^{(d+1)/4} J_{-(d+1)/4}\left(2\sqrt{gtZ}\right) \\ \simeq \frac{(gtZ)^{(d+1)/4}}{\pi^{1/2}} \left[\cos\left(2\sqrt{gtZ} + \frac{\pi d}{4}\right) \left(1 - \frac{d(d+2)[(d+1)^2 - 9]}{512} \frac{1}{gtZ}\right) \right. \\ \left. - \sin\left(2\sqrt{gtZ} + \frac{\pi d}{4}\right) \frac{d(d+2)}{16} \frac{1}{\sqrt{gtZ}} \right] \quad (\text{F.6b})$$

$$\frac{{}_1F_2\left(1 - \frac{d}{2}; 2 - \frac{d}{2}, \frac{3}{2} - \frac{d}{2}; -gtZ\right)}{\Gamma(\frac{3}{2} - \frac{d}{2})(1 - \frac{d}{2})\pi^{-1/2}} \simeq \Gamma\left(1 - \frac{d}{2}\right) (gtZ)^{d/2-1} \\ + \cos\left(2\sqrt{gtZ} + \frac{\pi d}{4}\right) (gtZ)^{d/4-1} \left[-1 + \frac{d(d+2)(d^2 - 14d + 56)}{512} \frac{1}{gtZ}\right] \\ + \sin\left(2\sqrt{gtZ} + \frac{\pi d}{4}\right) (gtZ)^{d/4-3/2} \frac{d(d-6)}{16} \quad (\text{F.6c})$$

and where J_{ν} is a Bessel function and $\Gamma(a, x)$ an incomplete Gamma function [1]. Inserting these expansions into (6.18), several leading terms cancel. The constraint takes the form

$$2(4\pi\gamma t)^{d/2} = \frac{\mathcal{C}gt}{Z} e^{-\gamma Z} + \frac{d\mathcal{C}\gamma^{d/2}}{2} \left(\frac{Z}{gt}\right)^{d/4-1} \\ \times \left[\frac{3(d+2)(4-d)}{64} \frac{\cos\left(2\sqrt{gtZ} + \frac{\pi d}{4}\right)}{Z} - \frac{\sin\left(2\sqrt{gtZ} + \frac{\pi d}{4}\right)}{\sqrt{gtZ}} \right] \quad (\text{F.7})$$

which is equation (6.21) in the main text. In order to solve this equation, consider first only the first term on the right-hand side. If one assumes that asymptotically $e^{\gamma Z} \sim t^{\alpha}$, matching the left-hand side with the right-hand side gives $\alpha = 1 - \frac{d}{2}$. Then, the second term on the right-hand side is of the order $t^{1-d/4+\alpha}$, up to logarithmic or oscillating factors. If $\alpha < d/4$, this second term merely generates a correction. This is so for $d > \frac{4}{3}$. Similarly, the third term is of the order $t^{1/2-d/4+\alpha}$, hence it only generates a finite-time correction for $d > 1$.

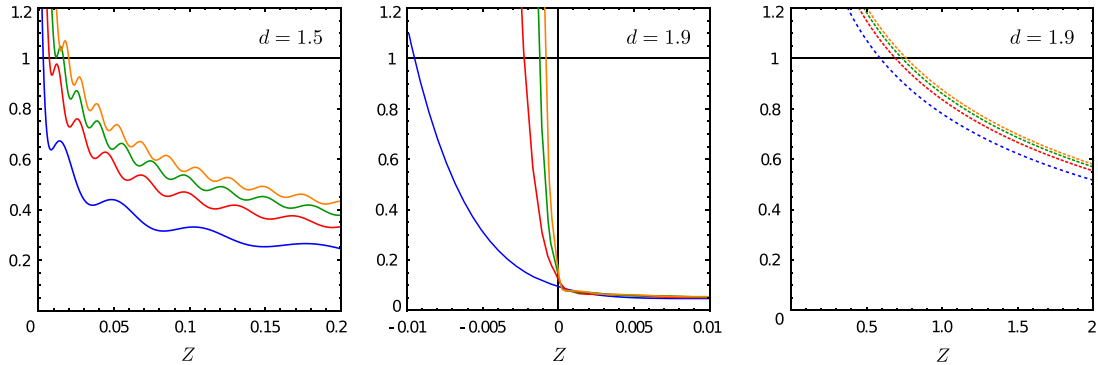


Figure F2. Solving the constraint (6.18) as a function of Z : the LHS is shown in black and the RHS is shown for different times $t = [10\,000, 40\,000, 70\,000, 100\,000]$ corresponding to the blue, red, green and orange lines, from left to right, in the left and middle panels. In the right panel, the RHS with $t = [1, 4, 7, 10] \cdot 10^{40}$ corresponds to the blue, red, green and orange dashed lines, from left to right. The other parameters are $\mathcal{C} = 1$, $\gamma = 0.1$ and $g = 0.1$. Different frames correspond to different dimensions: left panel $d = 1.5$, middle and right panels $d = 1.9$.

Hence, for $\frac{4}{3} < d < 2$, it is enough to concentrate on the first term on the right-hand-side in (F.7). Analogously to previous cases, the constraint is solved via the Lambert-W function

$$\gamma Z = W\left(\frac{\mathcal{C}g}{2^{d+1}\pi^{d/2}}(\gamma t)^{1-d/2}\right) \simeq \left(1 - \frac{d}{2}\right) \ln \gamma t + O(\ln \ln t). \quad (\text{F.8})$$

For a better approximation, one can re-inject this solution into the second and third terms on the right-hand-side of (F.7). Then one obtains an oscillatory correction, of the form quoted in the main text.

Appendix G. Structure factor for $4/3 < d < 2$

We derive the identity equation (6.30). We neglect all prefactors, focus on the functional dependence and treat the expression

$$Q_{\mathbf{k}} \propto \frac{1 - {}_0F_1(1/2; -x)}{x} \quad (\text{G.1})$$

where $x = gt(Z + tk2)$. Clearly, replacing the hyper-geometric function by its series representation gives

$$\frac{1 - {}_0F_1(1/2; -x)}{x} = \sum_{n=1}^{\infty} \frac{1}{(1/2)_n} \frac{(-x)^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{1}{(1/2)_{n+1}} \frac{(-x)^n}{(n+1)!}. \quad (\text{G.2})$$

We now rewrite

$$\frac{1}{(1/2)_{n+1}} \frac{1}{(n+1)!} = \Gamma \left[\begin{matrix} n+1 & 3/2 & 2 & 1/2 \\ 1 & n+3/2 & n+2 & 2 \end{matrix} \right] \frac{1}{n!} = 2 \frac{(1)_n}{(3/2)_n (2)_n} \frac{1}{n!} \quad (\text{G.3})$$

and can consequently recast equation (G.2) as

$$\frac{1 - {}_0F_1(1/2; -x)}{x} = 2 \sum_{n=0}^{\infty} \frac{(1)_n}{(3/2)_n (2)_n} \frac{(-x)^n}{n!} = 2 {}_1F_2(1; 3/2, 2; -x) \quad (\text{G.4})$$

which is the representation used in the main text.

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