MEASURES OF IRREVERSIBILITY USING QUANTUM PHASE SPACE

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SUMMARY

- Motivation and objectives.
- Quantum phase space for bosonic systems.
- Entropy production in quantum phase space.
- Irreversibility from exact dilations.
- Measures of non-Markovianity.
MOTIVATION AND OBJECTIVES

- Irreversibility is an emergent property, which is traditionally quantified by the *entropy production*.

- But entropy production is not an observable and must therefore be related to observables by means of a theoretical framework.

- A fully quantum mechanical theory of entropy production for open quantum systems is still lacking.

- Our goals are to understand entropy production

  - from master equations describing non-equilibrium reservoirs.

  - from the perspective of the environment and the S+E correlations.
We shall consider the relaxation of a bosonic mode in contact with a bath.

\[
\frac{d\rho}{dt} = -i[H, \rho] + D(\rho)
\]

\[
H = \omega (a^\dagger a + 1/2)
\]

\[
D(\rho) = \gamma (\bar{n} + 1) \left[ a\rho a^\dagger - \frac{1}{2} \{a^\dagger a, \rho\} \right] + \gamma \bar{n} \left[ a^\dagger \rho a - \frac{1}{2} \{a a^\dagger, \rho\} \right]
\]

\[
\bar{n} = \frac{1}{e^{\beta \omega} - 1}
\]
DESCRIPTION IN QUANTUM PHASE SPACE

Instead of working with density matrices, we work with the Wigner function:

\[ W_S(\alpha, \alpha^*) = \frac{1}{\pi \sqrt{|\Theta|}} \exp \left\{ -\frac{1}{2} (\alpha - \mu)^\dagger \Theta^{-1} (\alpha - \mu) \right\} \]

\[ \alpha = (\alpha, \alpha^*) \quad \Theta = \begin{pmatrix} \langle \delta a^\dagger \delta a \rangle + 1/2 & \langle \delta a \delta a \rangle \\ \langle \delta a^\dagger \delta a^\dagger \rangle & \langle \delta a^\dagger \delta a \rangle + 1/2 \end{pmatrix} \]

\[ \mu = (\langle a \rangle, \langle a^\dagger \rangle) \]

Then \( W \) will satisfy a Quantum-Fokker-Planck equation

\[ \partial_t W = \mathcal{U}(W) + \partial_\alpha J(W) + \partial_{\alpha^*} J^*(W) \]

\[ J(W) = \frac{\gamma}{2} \left[ \alpha W + (\bar{n} + 1/2) \partial_{\alpha^*} W \right] \quad J(W) \text{ is a probability current} \]

\[ J(W_{eq}) = 0 \]
WIGNER-RÉNYI-2 ENTROPY

For Gaussian states, the Wigner entropy coincides (up to a constant) with the Rényi-2 entropy:

\[ S(W) = - \int d^2 \alpha \ W \ln W \]

\[ = - \ln \text{tr}(\rho^2) \]

\[ = \frac{1}{2} \ln |\Theta| \]

It also satisfies the strong subadditivity inequality:

\[ S_{AB} + S_{BC} \geq S_{ABC} + S_B \]

It can “do” everything the von Neumann entropy can.

But it is much easier to work with.

Adesso, Girolami, Serafini
PRL 109, 190502 (2012)
WIGNER ENTROPY PRODUCTION

- Entropy does not satisfy a continuity equation.
- In addition to an entropy flow between the system and the environment, entropy may also be spontaneously produced.

\[ \frac{dS}{dt} = \Pi - \Phi \]

- The entropy production rate \( \Pi \) quantifies the instantaneous rate of irreversibility of a process.

\[ \Pi \geq 0 \quad \text{and} \quad \Pi = 0 \quad \text{iff} \quad W = W_{eq} \]
Our goal is then to relate $\Pi$ and $\Phi$ to the currents in phase space.

**Maxim:**

- $\Pi$ should be an even function of $J(W)$ and
- $\Phi$ should be an odd function.

As a result, we find:

$$\Pi = \frac{4}{\gamma(\bar{n} + 1/2)} \int d^2\alpha \frac{|J(W)|^2}{W} = -\frac{dS(W||W_{eq})}{dt}$$

$$\Phi = \frac{\gamma}{\bar{n} + 1/2} \left[ \langle a^\dagger a \rangle - \bar{n} \right] = \frac{\Phi_E}{\omega(\bar{n} + 1/2)}$$

At high temperatures $\omega(\bar{n} + 1/2) \approx T$ so we get $\Phi \simeq \frac{\Phi_E}{T}$.
DISCUSSION

- For Gaussian bosonic systems, the Wigner entropy appears to be the natural entropy measure.

- Formulating the theory of irreversibility in terms of it leads to expressions relating the entropy production with the microscopic currents in phase space.

  - Analogous to classical formulations.

- The theory recovers classical results for high temperatures.

- Can be readily extended to non-equilibrium baths (e.g. squeezed baths).
ZERO TEMPERATURE

✦ The Wigner formulation also remains valid at $T = 0$.

✦ The standard “von Neumann” method gives diverging results:

$$\Pi = -\frac{dS(\rho||\rho_{\text{eq}})}{dt} \quad S(\rho||\rho_{\text{eq}}) = \text{tr}\left\{\rho(\ln \rho - \ln \rho_{\text{eq}})\right\}$$

✦ The relative entropy diverges when the system tends to a pure state.

✦ Is this divergence physical (perhaps connected to the 3rd law?) or is it simply a mathematical limitation?
IRREVERSIBILITY FROM THE PERSPECTIVE OF THE ENVIRONMENT
Irreversibility is an emergent property.

Stems from the interaction of $S$ with a macroscopically large $E$.

How exactly does this take place?

What is the role of $S$-$E$ correlations?

Microscopic derivations of master equations involve approximations: we lose track of the contributions to $\Pi$. 
Instead, we shall look at exact dilations of a master equation.

We focus on the zero-temperature case:

\[
\frac{d\rho_S}{dt} = 2\kappa \left[ a\rho_S a\dagger - \frac{1}{2}\{a\dagger a, \rho_S\} \right]
\]

What is the most general dilation reproducing this equation at all times?

Assume only that bath is bosonic and starts in the vacuum.

Global vacuum must be a fixed point; Gaussianity must be preserved exactly; E-E interactions lead to nothing new.
EXACT SOLUTION

- The dynamics is entirely determined by means of 2 auxiliary functions:

\[
\frac{dg}{dt} = -i \sum_k \gamma_k e^{(\omega - \Omega_k) t} f_k(t), \quad g(0) = 1
\]
\[
\frac{df_k}{dt} = -i \gamma_k e^{-(\omega - \Omega_k) t} g(t), \quad f_k(0) = 0
\]

- Markovian behavior recovered in the Wigner-Weisskopf limit:

\[
g(t) = e^{-\kappa t} \quad f_k(t) = \frac{i \lambda_k}{\kappa + i(\omega - \Omega_k)} \left[ e^{-(\kappa + i(\omega - \Omega_k)) t} - 1 \right]
\]
Figure 1: Example of the behavior of the function $g(t)$, Eq. (32) for different numbers of bath oscillators, respectively $K = 10$, 30, 80 and 120. The functions were computed assuming $\omega = 1$, $\gamma_k = 1/K$ and $\Omega_k = 1/2 + (k - 1)/(K - 1)$ (i.e., a linear interpolation from 1/2 to 3/2). The red-dashed curve correspond to the Markovian solution (37). For this particular choice of frequencies, the function $g$ is real.
COVARIANCE MATRIX

- The initial conditions of S are specified by
  \[ \mu = \langle a \rangle_0, \quad N = \langle \delta a^\dagger \delta a \rangle, \quad M = \langle \delta a \delta a \rangle \]

- With \( g(t) \) and \( f(t) \) we then reconstruct the full S-E covariance matrix:

\[
\Theta_{SE}(t) = \begin{pmatrix}
\Theta_S & \Theta_{S,1} & \Theta_{S,2} & \ldots \\
\Theta_{S,1}^\dagger & \Theta_{1,1} & \Theta_{1,2} & \ldots \\
\Theta_{S,2}^\dagger & \Theta_{1,2}^\dagger & \Theta_{2,2} & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}, \quad \Theta_S(t) = \begin{pmatrix}
\langle \delta a^\dagger \delta a \rangle_t + 1/2 & \langle \delta a \delta a \rangle_t \\
\langle \delta a^\dagger \delta a^\dagger \rangle_t & \langle \delta a \delta a \rangle_t + 1/2 \\
\end{pmatrix} = \begin{pmatrix}
N|g|^2 + 1/2 & Mg^2 \\
M^*g^* & N|g|^2 + 1/2 \\
\end{pmatrix},
\]
We can also compute both S-E and E-E correlations.

\[ \Theta_{S,k} = \begin{pmatrix} \langle \delta a \delta b^\dagger_k \rangle_t & \langle \delta a \delta b_k \rangle_t \\ \langle \delta a^\dagger \delta b^\dagger_k \rangle_t & \langle \delta a^\dagger \delta b_k \rangle_t \end{pmatrix} \]

\[ = \begin{pmatrix} Ngf_k^* & Mg \Gamma_k \\ M^* g^* f_k^* & Ng^* f_k \end{pmatrix}, \]

\[ \Theta_{k,q} = \begin{pmatrix} \langle \delta b^\dagger_q \delta b_k \rangle_t + \delta_{k,q}/2 & \langle \delta b_k \delta b_q \rangle_t \\ \langle \delta b^\dagger_k \delta b^\dagger_q \rangle_t & \langle \delta b_k \delta b_q \rangle_t + \delta_{k,q}/2 \end{pmatrix} \]

\[ = \begin{pmatrix} Nf_k f_q^* + \delta_{k,q}/2 & Mf_k f_q \\ M^* f_k^* f_q^* & Nf_k^* f_q + \delta_{k,q}/2 \end{pmatrix}, \]

\[ \mathcal{I}_{SE} = S(\Theta_S) + S(\Theta_E) - S(\Theta_{SE}) \]
The global S-E Wigner function satisfy a unitary QFP:

\[ \partial_t W_{SE} = \partial_\alpha J_S - \partial_\alpha^* J_S^* + \sum_k (\partial_{\beta_k} J_k + \partial_{\beta_k^*} J_k^*) \]

The currents acting on S and E are:

\[ J_S(W_{SE}) = \frac{1}{g^*} (\sum_k \dot{f}_k^* \beta_k) W_{SE}, \]

\[ J_k(W_{SE}) = -\frac{\dot{f}_k}{\dot{g}} J_E(W_{SE}), \quad J_E(W_{SE}) = \frac{\dot{g}}{g} \alpha W_{SE}, \]
MARGINAL QFP

- Integrating over the bath we obtain the QFP for the system:

\[ \partial_t W_S = \partial_\alpha J_S(W_S) + \partial_{\alpha^*} J^*_S(W_S) \]

\[ J_S(W_S) = \int d^2 \beta \mathcal{J}_S(W_{SE}) = \Gamma(t) \left( \alpha + \frac{\partial_{\alpha^*}}{2} \right) W_S \]

\[ \Gamma(t) = -\frac{\dot{g}}{g} = \kappa \quad \text{(in the Markovian case)} \]

- Consequently, the entropy production continues to be:

\[ \Pi = -\frac{dS(W_S||W_{eq}^S)}{dt} = \frac{4}{\Gamma(t)} \int d^2 \alpha \frac{|J_S(W_S)|^2}{W_S} \]
ENTROPY PRODUCTION IN THE DILATION

- The global S-E unitary satisfies the following conservation law:

\[
\frac{dS(W_{SE}||W_{SE}^{eq}(W_E(0)))}{dt} = 0
\]

- i.e., the distance to the global vacuum remains constant.

- This allows us to separate \( \Pi \) as

\[
\Pi = \frac{dI_{SE}}{dt} + \frac{dS(W_E||W_E(0))}{dt}
\]

- 2nd term: irreversibility due to changes in \( E \) (local).

- 1st term: irreversibility due to S-E correlations (non-local)
Nothing in the dynamics this model indicates divergences occurring at $T = 0$. All properties are well behaved.

A particularly remarkable special case is when the system starts in a coherent state.

$S_E$ remain in a product state throughout.

$$|\psi(t)\rangle = |\mu g(t)\rangle \otimes \prod_k |\mu f_k(t)\rangle$$

$I_{SE} = 0$ at all times, but there is still an entropy production.
This model provides the perfect platform for experimenting with non-Markovian effects.

In this case Markovianity is fully determined from the sign of

\[ \Gamma(t) = -\frac{\dot{g}}{g} \]

An interesting question then is which contributions to the entropy production can be used as witnesses of non-Markovianity.
As a proof of principle, we consider the dynamics of the system $S$ which was initially entangled with an ancilla $A$ (Rivas, Huelga, Plenio, PRL, 105, 050403 (2010)).

Interpretation: thermal state of $S = \text{two-mode squeezed state of } S + A$

$$\Theta_{AS}(0) = \begin{pmatrix}
N + 1/2 & 0 & 0 & \sqrt{N(N+1)} \\
0 & N + 1/2 & \sqrt{N(N+1)} & 0 \\
0 & \sqrt{N(N+1)} & N + 1/2 & 0 \\
\sqrt{N(N+1)} & 0 & 0 & N + 1/2
\end{pmatrix}$$
Markovian
non-Markovian
CONCLUSIONS

- Bosonic Gaussian systems: complete theory of non-equilibrium entropy production in terms of the Wigner entropy.
- Relates entropy production with currents in phase space.
- Using exact dilations, we can separate $\Pi$ in two terms:
  - The displacement of the bath from equilibrium
  - The non-local system-environment correlations.


Thank you