Quantum Fluctuation Theorems beyond Two-Point Measurements

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We derive detailed and integral quantum fluctuation theorems for heat exchange in a quantum correlated bipartite thermal system using the framework of dynamic Bayesian networks. Contrary to the usual two-projective-measurement scheme that is known to destroy quantum features, these fluctuation relations fully capture quantum correlations and quantum coherence at arbitrary times. We further obtain individual integral fluctuation theorems for classical and quantum correlations, as well as for local and global quantum coherences.

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Fluctuation theorems are fundamental generalizations of the second law of thermodynamics for small systems. While the entropy production Σ is a non-negative deterministic quantity for macroscopic systems, it becomes random at the microscopic scale owing to the presence of non-negligible thermal [1,2] or quantum [3,4] fluctuations. Detailed fluctuation theorems quantify the probability of occurrence of negative entropy production events via the relation $P(\Sigma)/P(-\Sigma) = \exp(\Sigma)$ [5]. Integral fluctuation theorems take on the form $\langle \exp(-\Sigma) \rangle = 1$ after integration over Σ. The concavity of the exponential then implies that the entropy production is only positive on average, $\langle \Sigma \rangle \geq 0$. The generic validity of fluctuation theorems arbitrarily far from equilibrium makes them particularly useful in nonequilibrium physics. They have been extensively investigated for this reason, both theoretically and experimentally, for classical systems [6,7]. These studies have provided unique insight into the thermodynamics of microscopic systems, from colloidal particles to enzymes and molecular motors [1,2].

The situation is more involved in the quantum regime. Quantum fluctuation theorems are commonly studied within the two-point-measurement (TPM) scheme [3,4]. In this approach, the energy change, and in turn the entropy production, of a quantum system are determined for individual realizations by projectively measuring the energy at the beginning and at the end of a nonequilibrium protocol [8]. Equivalent formulations based on Ramsey-like interferometry [9,10] and generalized measurements [11,12] have also been proposed. These methods were used to perform experimental tests of quantum fluctuation theorems, both for mechanically driven [13–16] and thermally driven [17,18] systems, using NMR, trapped-ion, cold-atom, nitrogen-vacancy-center, and superconducting qubits setups. The TPM procedure successfully captures the discrete quantum energy spectrum of the system, as well as its nonequilibrium quantum dynamics between the two measurements [19]. However, due to its projective nature, it completely fails to account for quantum correlations and quantum coherence, two central features of quantum theory, that may be present in initial and final states of the system. In that sense, the TPM scheme may thus be viewed as not fully quantum.

In this Letter, we present detailed and integral quantum fluctuation theorems for heat exchange between quantum correlated bipartite thermal systems using a dynamic Bayesian network approach [20,21]. Global and local descriptions of a composite system usually differ because of quantum correlations. The dynamic Bayesian network offers a powerful framework to specify the local dynamics conditioned on the global states, hence preserving all the quantum properties of the system, including quantum correlations and quantum coherence, in contrast to the TPM strategy. Our findings reduce to the Jarzynski-Wójcik fluctuation theorem in the absence of correlations [22] and to the fluctuation relation of Jevtic and coworkers in the presence of classical correlations [23]. They additionally complement recent attempts to obtain fully quantum fluctuation theorems for mechanically driven systems [24–27] (see also Refs. [28,29]).

In the following, we first derive a detailed quantum fluctuation theorem for the ratio of the probability of a conditional local trajectory of the system and its reverse. We show that it accounts both for quantum correlations (in the form of a stochastic quantum mutual information [30]) and for quantum coherence (in the form of a stochastic relative entropy of coherence [31]). We further identify a contribution to the entropy production that stems from the randomness of the conditional local trajectory. Moreover, we obtain a detailed fluctuation relation for the joint probability of all quantum contributions and demonstrate that each of them, as well as their sum, individually satisfies an integral fluctuation theorem. Finally, we derive a modified quantum fluctuation relation for the heat variable alone, valid for any intermediate times.

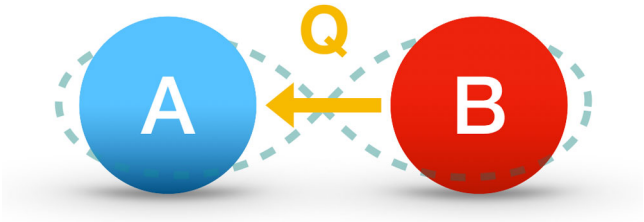

FIG. 1. Quantum correlated bipartite quantum system AB in local thermal states at different temperatures. The initial joint state is of the form $\rho_{AB}(0) = \rho_A^0 \otimes \rho_B^0 + \chi_{AB}$ with Gibbs states, $\rho_i^0 = \exp(-\beta_i H_i)/Z_i$ at inverse temperatures β_i $(i = A, B)$, and initial quantum correlations χ_{AB}. During thermal interaction, the two arbitrary subsystems exchange the amount of stochastic heat Q.

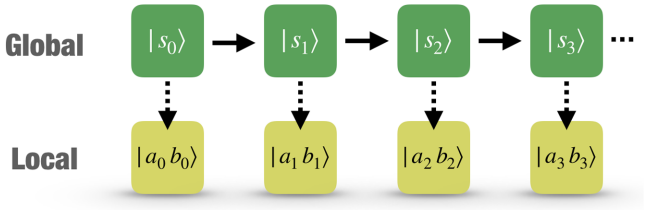

FIG. 2. Dynamic Bayesian network. The global quantum trajectory is specified by the state $|s(t)\rangle$ which evolves deterministically. At each instant t_n, the conditional probability of finding the reduced systems, A and B, in their local energy eigenstates $|a_n, b_n\rangle$, given the state $|s_n\rangle$, is specified by Eq. (2). The set of points $(s, a_0, b_0, a_1, b_1, \ldots)$ defines a conditional local trajectory Γ, with path probability $\mathcal{P}[\Gamma]$, Eq. (3), that accounts for the full quantum properties of the system.

Dynamic Bayesian networks.—We consider two quantum systems, A and B, with respective Hamiltonians H_A and H_B, initially prepared in the joint state,

$$\rho_{AB}(0) = \rho_A^0 \otimes \rho_B^0 + \chi_{AB}, \tag{1}$$

where $\rho_i^0 = \exp(-\beta_i H_i)/Z_i$ $(i = A, B)$ are local thermal Gibbs states at inverse temperatures β_i and $Z_i = \mathrm{tr}[\exp(-\beta_i H_i)]$ is the corresponding partition function (see Fig. 1). The operator χ_{AB} induces correlations between the two subsystems. It is assumed to satisfy $\mathrm{tr}_i[\chi_{AB}] = 0$, so that the reduced states, $\rho_i(0) = \mathrm{tr}_j[\rho_{AB}(0)]$, are locally thermal even though A and B are globally correlated [23]. This condition guarantees that the local systems have a well-defined temperature. Thermal contact between the two systems is established at $t = 0$ by letting them interact via an energy conserving unitary transformation $U(t)$ verifying $[U(t), H_A + H_B] = 0$. The global basis $|s_n\rangle$ at time t_n is defined as $\rho_{AB}(t_n) = U(t_n)\rho_{AB}(0)U^\dagger(t_n) = \sum_s P_s |s_n\rangle\langle s_n|$, where P_s is the initial population. On the other hand, the corresponding local bases, $|a_n\rangle$ and $|b_n\rangle$, follow from the decomposition of the reduced states, $\rho_A(t_n) = \sum_{a_n} P_{a_n}|a_n\rangle\langle a_n|$ and $\rho_B(t_n) = \sum_{b_n} P_{b_n}|b_n\rangle\langle b_n|$. We note that while the evolution of the global state is deterministic, with each eigenstate $|s\rangle$ simply evolving in time according to $U(t)|s\rangle$ and the initial populations P_s kept fixed, that of the local (reduced) states is stochastic.

Our aim is to assess the statistics of the heat exchanged between A and B at any given time, accounting for all the quantum properties of the process, including quantum correlations and quantum coherence. This endeavor faces a number of mathematical and physical difficulties. Mathematically, the global state is not diagonal in the energy representation because of the nonvanishing correlations. As a result, the global and local bases are not mutually orthogonal, $\langle a_n b_n|s_n\rangle \neq \delta_{a_n b_n, s_n}$, making their relationship nontrivial, except when $\chi_{AB} = 0$. The physical consequence is that the local bases, in which the exchanged heat variable is evaluated, do not contain the complete information about the composite system.

In order to solve these issues, we employ the tools of dynamic Bayesian networks which are widely used in computer science and statistics [20,21]. They can be regarded as generalizations of hidden Markov models [32] which have been used to study classical fluctuation relations in the presence of hidden degrees of freedom [33,34] (see also Refs. [35,36]). These techniques allow the systematic analysis of probabilities of events conditioned on some other events. Concretely, at any given time t_n, the conditional probability of finding the local systems A and B in their respective energy eigenstates $|a_n\rangle$ and $|b_n\rangle$, given that the global system is in state $|s_n\rangle$, is,

$$P(a_n, b_n|s_n) = |\langle a_n b_n|s_n\rangle|^2. \tag{2}$$

For any sequence of times, $t_0, t_1, \ldots, t_N$, we may define a conditional trajectory $\Gamma = (s, a_0, b_0, a_1, b_1, \ldots, a_N, b_N)$ (see Fig. 2) and the corresponding path probability as

$$\mathcal{P}[\Gamma] = P_s P(a_0, b_0|s) P(a_1, b_1|s_1) \ldots P(a_N, b_N|s_N). \tag{3}$$

The corresponding probability for the local trajectory is obtained by summing over all quantum trajectories s,

$$\mathcal{P}(a_0, b_0, \ldots, a_N, b_N) = \sum_s \mathcal{P}[\Gamma]. \tag{4}$$

We may analogously introduce a reversed conditional local trajectory $\Gamma^* = (s^*, a_N, b_N, \ldots, a_0, b_0)$ with path probability $\mathcal{P}[\Gamma^*] = P_{s^*} \bar{P}(a_N, b_N|s^*) \ldots \bar{P}(a_0, b_0|s_N^*)$ where $\bar{P}(a_n, b_n|s_{N-n}^*) = |\langle a_n b_n|U^\dagger(t_{N-n})|s^*\rangle|^2$.

For concreteness and simplicity, we next focus on trajectories $\Gamma = (s, a_0, b_0, a_1, b_1)$ for a sequence of two times, $t_0 = 0$ and an arbitrary future time t_1. In this case, the multipoint distribution [Eq. (4)] reduces to a two-point distribution $\mathcal{P}(a_0, b_0, a_1, b_1)$. The generalization to multipoint distributions is straightforward. Marginalizing the conditional probability [Eq. (3)] over a_0, b_0 then yields,

$$\mathcal{P}(a_1,b_1)=\sum_{s,a_0,b_0}\mathcal{P}[\Gamma]=\langle a_1,b_1|\rho_{AB}(t_1)|a_1,b_1\rangle, \quad (5)$$

which is the result one would have expected on physical grounds. We furthermore have the two reduced probabilities $\mathcal{P}(a_1)=\sum_{b_1}\langle a_1,b_1|\rho_{AB}(t_1)|a_1,b_1\rangle$ and $\mathcal{P}(b_1)=\sum_{a_1}\langle a_1,b_1|\rho_{AB}(t_1)|a_1,b_1\rangle$ at time t_1. Similarly, by only marginalizing over the global trajectory s, we obtain the path probability for the local trajectory (a_0,b_0,a_1,b_1),

$$\mathcal{P}(a_0,b_0,a_1,b_1)=\sum_s P_s P(a_0,b_0|s)P(a_1,b_1|s_1). \quad (6)$$

Interestingly, these probabilities may also be cast in terms of the expectation value of a Choi matrix [37]. In the particular case where the initial state [Eq. (1)] is separable ($\chi_{AB}=0$), global and local bases are identical, $|s\rangle=|a_0b_0\rangle$, and Eq. (6) reduces to the TPM result [22],

$$\mathcal{P}(a_0,b_0,a_1,b_1)=P_{a_0}P_{b_0}|\langle a_1,b_1|U(t_1)|a_0,b_0\rangle|^2. \quad (7)$$

Expression Eq. (6) hence generally contains more information about the local quantum dynamics than Eq. (7).

Detailed quantum fluctuation theorem.—We next derive a detailed fluctuation theorem for the ratio of forward and reversed conditional trajectories using Eq. (3),

$$\frac{\mathcal{P}[\Gamma]}{\mathcal{P}[\Gamma^*]}=\frac{P_s}{P_{s^*}}\frac{P(a_0,b_0|s)P(a_1,b_1|s_1)}{\bar{P}(a_1,b_1|s^*)\bar{P}(a_0,b_0|s_1^*)}. \quad (8)$$

In order to obtain an explicit expression for the theorem, we begin by rewriting the first ratio in Eq. (8) as

$$\frac{P_s}{P_{s^*}}=\frac{P_{a_0}P_{b_0}}{P_{a_1}P_{b_1}}\exp\left(\ln\frac{P_s}{P_{a_0}P_{b_0}}-\ln\frac{P_{s^*}}{P_{a_1}P_{b_1}}\right), \quad (9)$$

where P_{a_1} and P_{b_1} are reference thermal occupations at time t_1 (at respective temperatures T_A and T_B) to which the actual probabilities $\mathcal{P}(a_1)$ and $\mathcal{P}(b_1)$ are compared to. This then leads to the quantum fluctuation relation,

$$\frac{\mathcal{P}[\Gamma]}{\mathcal{P}[\Gamma^*]}=\exp\left(Q_A\Delta\beta+I_0-I_1-\Sigma_A-\Sigma_B+\gamma\right). \quad (10)$$

We have here identified (i) the entropy production associated with heat exchange, $Q_A\Delta\beta=(E_{a_1}-E_{a_0})(\beta_A-\beta_B)$, where E_{a_n} are the eigenenergies of H_A, (ii) the stochastic quantum mutual information, $I_0=\ln[P_s/P_{a_0}P_{b_0}]$, that accounts for initial correlations between subsystems A and B, and (iii) the stochastic quantum mutual information, $I_1=\ln[P_{s^*}/\mathcal{P}(a_1)\mathcal{P}(b_1)]$, that characterizes quantum correlations at the final time. We have additionally introduced the stochastic quantum relative entropies, $\Sigma_A=\ln[\mathcal{P}(a_1)/P_{a_1}]$ and $\Sigma_B=\ln[\mathcal{P}(b_1)/P_{b_1}]$. Finally, we have discerned a contribution to the entropy production, $\gamma=\ln[P(a_0,b_0|s)P(a_1,b_1|s_1)/\bar{P}(a_1,b_1|s^*)\bar{P}(a_0,b_0|s_1^*)]$, that comes from the second ratio in Eq. (8). This term stems from the stochastic nature of the conditional dynamics, in analogy to the classical result of Ref. [38]. It vanishes on average, since the global dynamics is unitary and no extra energy is exchanged with an external bath.

Equation (10) is our first main result. It generalizes quantum fluctuation theorems for heat exchange beyond the standard TPM approach [22,23]. To make this point more precise, we express the stochastic quantum mutual information, $I_l=J_l+C_l$, $(l=0,\ 1)$, as a sum of the stochastic classical mutual information, $J_l=\ln(P_{a_lb_l}/P_{a_l}P_{b_l})$, and of the stochastic quantum relative entropy of coherence, $C_l=\ln(P_s/P_{a_lb_l})$, which is a proper measure of quantum coherence in a given basis [31]. The detailed fluctuation relation [Eq. (10)] therefore fully captures, at any time, the presence of quantum correlations between the two subsystems and of quantum coherence (both in the global and local systems), in the heat statistics. It provides, in particular, an extension of the fluctuation theorem of Jarzynski and Wójcik, $\mathcal{P}[\Gamma]/\mathcal{P}[\Gamma^*]=\exp(Q_A\Delta\beta)$ [22] and of Jevtic and coauthors, $\mathcal{P}[\Gamma]/\mathcal{P}[\Gamma^*]=\exp(Q_A\Delta\beta-\Delta J)$ [23,39].

By evaluating the average of the logarithm of Eq. (10), we furthermore obtain an expression for the mean heat exchanged between the subsystems A and B,

$$\langle Q_A\rangle\Delta\beta=\Delta\langle I\rangle+S(\rho_A||\rho_A^0)+S(\rho_B||\rho_B^0), \quad (11)$$

in agreement with the results of Ref. [40]. Here $S(\rho||\sigma)=\mathrm{tr}(\rho\ln\rho-\rho\ln\sigma)$ is the quantum relative entropy between two states ρ and σ [30]. Equation (11) indicates that the heat current may be reversed, thus flowing from cold to hot, when the initial correlations are such that $\Delta\langle I\rangle+S(\rho_A||\rho_A^0)+S(\rho_B||\rho_B^0)\leq 0$. This process is enabled by a trade-off between correlations and entropy [41]. The detailed fluctuation relation [Eq. (10)] extends this trade-off to the level of individual quantum realizations.

Integral quantum fluctuation theorems.—An integral fluctuation relation that incorporates all the quantum contributions may be derived from Eq. (10) by integrating over all conditional trajectories Γ. We find

$$\langle\exp\left(Q_A\Delta\beta+I_0-I_1-\Sigma_A-\Sigma_B+\gamma\right)\rangle=1. \quad (12)$$

Interestingly, by using the rules of Bayesian networks, one may show that each contribution satisfies an individual quantum fluctuation theorem [37]. We have, for example,

$$\langle e^{-I_0}\rangle=\sum_\Gamma\mathcal{P}[\Gamma]\exp\left(-\ln\frac{P_s}{P_{a_0}P_{b_0}}\right) \quad (13)$$

$$=\sum_{s,a_0,b_0}P(a_0,b_0|s)P_{a_0}P_{b_0}=\sum_{a_0,b_0}P_{a_0}P_{b_0}=1. \quad (14)$$

In a similar fashion (see Ref. [37] for details), we obtain

$$\langle e^{-I_l}\rangle = \langle e^{-J_l}\rangle = \langle e^{-C_l}\rangle = \langle e^{-\Sigma_i}\rangle = \langle e^{-\gamma}\rangle = 1. \quad (15)$$

We therefore conclude that contributions from both classical and quantum correlations, J_l and I_l, as well as from quantum coherence, C_l, separately obey an integral fluctuation relation, generalizing the recent findings of Refs. [42,43] for the quantum mutual information. Equation (15) is our second main result.

Modified detailed quantum fluctuation theorem for heat.—The detailed fluctuation relation [Eq. (10)] is formulated in terms of the probabilities of forward and reversed conditional trajectories. However, it is often convenient, both from a theoretical and an experimental point of view, to express it as a function of the joint probability of the different variables that appear in the exponent [44–46]. To this end, it is important to separate variables according to their properties under time reversal [46]. We therefore introduce the odd (information) variable, $K = I_1 - I_0 + \Sigma_A + \Sigma_B$, and define the forward joint probability distribution of K, the odd variable Q and γ as $P_f(Q,K,\gamma) = \langle \delta(Q - Q[\Gamma])\delta(K - K[\Gamma, s^*])\delta(\gamma - \gamma[\Gamma, s^*])\rangle$. The corresponding reversed joint probability distribution is $P_r(-Q,-K,\bar{\gamma}) = \langle \delta(Q - Q[\Gamma^*])\delta(K - K[\Gamma^*, s])\delta(\gamma - \bar{\gamma}[\Gamma^*, s])\rangle$ with $\bar{\gamma}[\Gamma, s^*] = -\ln(|\langle a_0 b_0|s\rangle|^2|\langle a_1 b_1|U_t^\dagger|s\rangle|^2)/(|\langle a_1 b_1|s^*\rangle|^2|\langle a_0 b_0|U_t|s^*\rangle|^2)$. The relation Eq. (10) then implies the detailed quantum fluctuation theorem [37],

$$\frac{P_f(Q,K,\gamma)}{P_r(-Q,-K,\bar{\gamma})} = \exp(Q\Delta\beta - K + \gamma). \quad (16)$$

In like manner, a more general fluctuation relation of the form Eq. (16) can be derived for all the individual quantum contributions by considering the joint probability distribution $P_f(Q, J_0, C_0, J_1, C_1, \Sigma_A, \Sigma_B, \gamma)$. Integrating Eq. (16) over K and γ, we eventually arrive at the modified detailed quantum fluctuation relation for heat,

$$\frac{P_f(Q)}{P_r(-Q)} = \frac{\exp(Q\Delta\beta)}{\Psi(Q)}, \quad (17)$$

where the factor $\Psi(Q) = \int dK d\gamma P(K,\gamma|Q)e^{-K-\gamma}$ depends on the correlations between Q, K, and γ. In the absence of correlations between the two subsystems A and B, we recover the Jarzynski-Wójcik result, $\Psi_{\mathrm{JW}}(Q) = 1$ [22]. Quantum correlations thus modify the heat distributions (a consequence of the difference between Eqs. (6) and (7) [37]) and, in turn, the exponential dependence on the heat variable on the right-hand side of Eq. (17) through the function $\Psi(Q)$. This is our third main result.

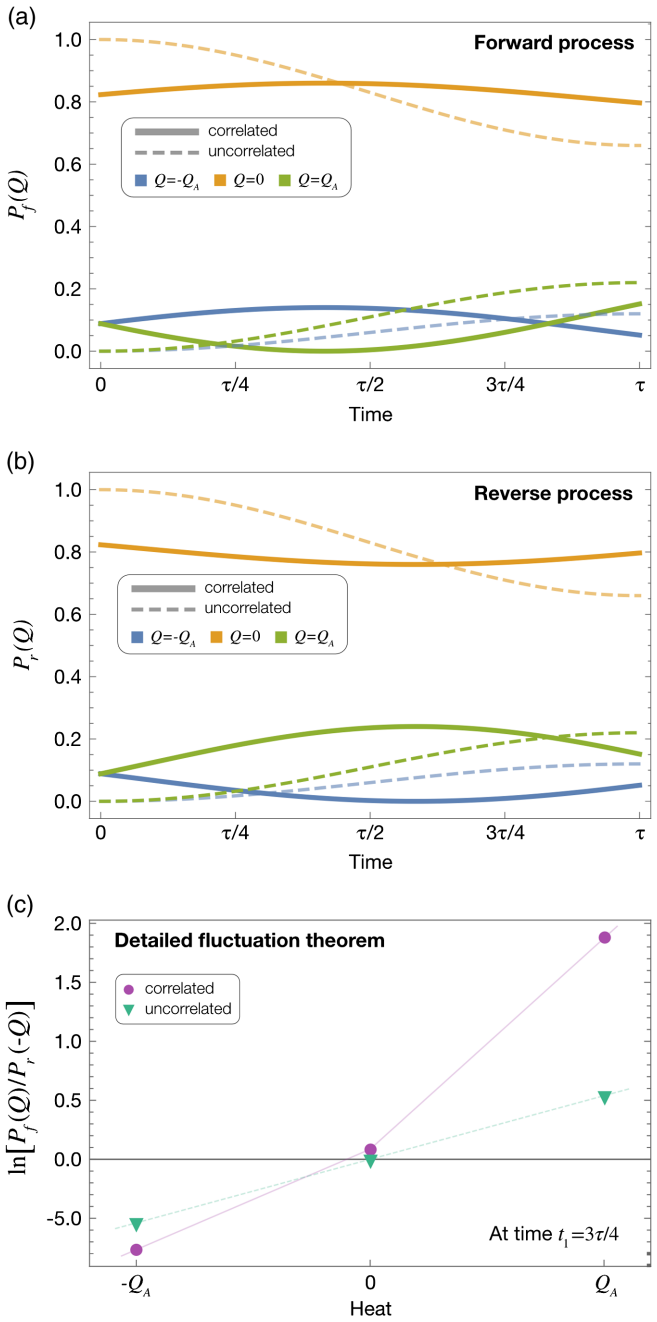

FIG. 3. Generalized quantum fluctuation theorem for heat for the two-spin-1/2 example. (a) Forward quantum heat distribution $P_f(Q)$ for the three values $(0, \pm Q_A)$ with (solid lines) and without (dashed lines) initial quantum correlations χ_{AB}, as a function of the thermal interaction time τ. (b) Corresponding reversed heat distribution $P_r(Q)$. (c) Without initial correlations ($\alpha = 0$), the Jarzynski-Wójcik relation $P_f(Q)/P_r(-Q) = \exp(Q\Delta\beta)$ (green triangles) holds. With initial quantum correlations ($\alpha \neq 0$), we have the generalized fluctuation theorem, $P_f(Q)/P_r(-Q) = \exp(Q\Delta\beta)/\Psi(Q)$ [Eq. (17)] (purple dots). The factor $\Psi(Q)$ encapsulates the quantum features of the correlations and modifies the Q dependence (solid and dashed lines are guide to the eye).

Example.—Our findings are valid for arbitrary quantum systems. As an illustration, we now consider the case of an initially quantum correlated two-spin-1/2 system with Hamiltonians $H_A = H_B = (1-\sigma_z)/2$, where σ_z is the usual Pauli operator. This system has been recently investigated experimentally in a nuclear magnetic resonance setup in Ref. [40]. The correlation term in Eq. (1) is taken of the form $\chi_{AB} = \alpha|01\rangle\langle 10| + \alpha^*|10\rangle\langle 10|$ with parameter α [40]. The value $\alpha = 0$ corresponds to initially uncorrelated local systems. We choose $\alpha = -i\exp[-(\beta_A+\beta_B)/2]/(Z_A Z_B)$ for initial quantum correlations with nonzero geometric discord [40]. We let the two systems interact, and exchange the amount of heat Q, via the thermal operation $H_{\rm int} = g(\sigma_A^+\sigma_B^- + \sigma_A^-\sigma_B^+)$, up to a maximum time $\tau = \pi/(2g)$. Since $[H_A + H_B, H_{\rm int}] = 0$, the total energy is conserved. This exchange interaction can induce four possible transitions between the eigenstates of the two qubits, corresponding to $(a_0, b_0) \to (a_1, b_1)$. This leads to three stochastic values of the heat, $Q = 0$ (twice) and $Q = \pm Q_A$, where $Q_A = (E_{a_1} - E_{a_0})$ is the energy variation of spin A.

We analytically solve the respective global and local spin dynamics and determine the forward and reversed heat distributions, $P_f(Q) = \sum_\Gamma \delta(Q - Q[\Gamma])\mathcal{P}[\Gamma]$ and $P_r(-Q) = \sum_{\Gamma^*} \delta(Q + Q[\Gamma^*])\mathcal{P}[\Gamma^*]$ (see Ref. [37] for details). The results are presented in Fig. 3 for $\exp(-\beta_A)/Z_A = 0.2$, $\exp(-\beta_B)/Z_B = 0.3$ and $g = 1$. Figures 3(a) and 3(b) show the forward and reversed quantum heat distributions for the three values $(0, \pm Q_A)$, with (solid lines) and without (dashed lines) initial quantum correlations, as a function of the interaction τ. We observe that the heat distributions depend explicitly on time and that the forward and reversed distributions are identical in the absence of initial correlations [37]. Figure 3(c) displays the detailed quantum fluctuation theorems for heat exchange Eq. (17). Without initial correlations ($\alpha = 0$), we recover the Jarzynski-Wójcik relation which corresponds to $\Psi_{\rm JW}(Q) = \exp(Q\Delta\beta)P_r(-Q)/P_f(Q) = 1$ (green triangles). For $\alpha \neq 0$, the effect of quantum correlations is clearly visible (purple dots), modulating the Q dependence via the function $\Psi(Q) \neq 1$. The function $\Psi(Q)$ may equivalently be evaluated directly from the formula below Eq. (17). This is, however, more involved due to the conditional probability appearing in that equation.

Conclusions.—We have used a dynamic Bayesian network approach to derive detailed and integral heat exchange fluctuation theorems for initially quantum correlated thermal bipartite systems. These fluctuation relations fully account for both quantum correlations and quantum coherence, two central quantum features, at arbitrary times, in contrast to the two-projective-measurement scheme. They provide much refined formulations of the second law of thermodynamics for small interacting quantum systems, compared to existing ones. We thus expect them to be useful for the study of far from equilibrium quantum thermodynamic systems.

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Supplemental Material: Quantum fluctuation theorems beyond two-point measurements

A. INTEGRAL FLUCTUATION THEOREMS

In this section, we present the derivations of the individual integral fluctuation theorems given in Eq. (15) of the main text. Special care should be paid to the order with which sums are evaluated.

We first start with the final stochastic mutual information I_1. We have,

$$\begin{aligned}\langle e^{-I_1}\rangle &= \sum_{\Gamma^*} P[\Gamma^*]\,\exp\left(-\ln\frac{P_{s^*}}{P_{a_1}\,P_{b_1}}\right)\\ &= \sum_{s*,a_1,b_1}\sum_{a_0,b_0} |\langle a_1\,b_1|s^*\rangle|^2\,|\langle a_0\,b_0|U^\dagger(t)|s^*\rangle|^2\,P_{a_1}\,P_{b_1}\\ &= \sum_{a_1,b_1}\sum_{s^*} |\langle a_1\,b_1|s^*\rangle|^2\,P_{a_1}\,P_{b_1} = \sum_{a_1,b_1} P_{a_1}\,P_{b_1} = 1.\end{aligned} \tag{S1}$$

Replacing the reversed path Γ^* with the forward path Γ, a similar calculation shows that the initial stochastic mutual information I_0 satisfies $\langle e^{-I_0}\rangle = 1$. The classical component J_1 of the final stochastic mutual information verifies,

$$\begin{aligned}\langle e^{-J_1}\rangle &= \sum_{\Gamma^*} P[\Gamma^*]\,\exp\left(-\ln\frac{P(a_1,b_1)}{P_{a_1}P_{b_1}}\right)\\ &= \sum_{s^*,a_1,b_1}\sum_{a_0,b_0} P_{s^*}\,|\langle a_1\,b_1|s^*\rangle|^2\,|\langle a_0\,b_0|U^\dagger(t)|s^*\rangle|^2\,\frac{P_{a_1}P_{b_1}}{P(a_1,b_1)}\\ &= \sum_{a_1,b_1}\sum_{s^*} P_{s^*}\,|\langle a_1\,b_1|s^*\rangle|^2\,\frac{P_{a_1}P_{b_1}}{P(a_1,b_1)} = \sum_{a_1,b_1} P_{a_1}\,P_{b_1} = 1.\end{aligned} \tag{S2}$$

On the other hand, the calculation for the final stochastic relative entropy of coherence C_1 reads,

$$\begin{aligned}\langle e^{-C_1}\rangle &= \sum_{\Gamma^*} P[\Gamma^*]\,\exp\left(-\ln\frac{P_{s^*}}{P(a_1,b_1)}\right)\\ &= \sum_{s^*,a_1,b_1}\sum_{a_0,b_0} |\langle a_1\,b_1|s^*\rangle|^2\,|\langle a_0\,b_0|U^\dagger(t)|s^*\rangle|^2\,P(a_1,b_1)\\ &= \sum_{a_1,b_1}\sum_{s^*} |\langle a_1\,b_1|s^*\rangle|^2\,P(a_1,b_1) = \sum_{a_1,b_1} P(a_1,b_1) = 1.\end{aligned} \tag{S3}$$

As before, the integral fluctuation theorems for the initial stochastic classical mutual information J_0 and initial stochastic relative entropy coherence C_0 follow by taking the average over the forward path Γ.

We next turn to the local stochastic entropy productions, Σ_A and Σ_B, during the forward process Γ. We find,

$$\begin{aligned}\langle e^{-\Sigma_A}\rangle &= \sum_{\Gamma} P[\Gamma]\,\exp\left(-\ln\frac{\mathcal{P}_{a_1}}{P_{a_1}}\right)\\ &= \sum_{s,a_1,b_1}\sum_{a_0,b_0} P_s\,|\langle a_0\,b_0|s\rangle|^2\,|\langle a_1\,b_1|U(t)|s\rangle|^2\,\frac{P_{a_1}}{\mathcal{P}_{a_1}}\\ &= \sum_{a_1,b_1}\sum_{s} P_s\,|\langle a_1\,b_1|U(t)|s\rangle|^2\frac{P_{a_1}}{\mathcal{P}_{a_1}} = \sum_{a_1}\sum_{b_1}\mathcal{P}(a_1,b_1)\frac{P_{a_1}}{\mathcal{P}_{a_1}} = \sum_{a_1} P_{a_1} = 1\end{aligned} \tag{S4}$$

and

$$\begin{aligned}\langle e^{-\Sigma_B}\rangle &= \sum_{\Gamma} P[\Gamma]\,\exp\left(-\ln\frac{\mathcal{P}_{b_1}}{P_{b_1}}\right)\\ &= \sum_{s,a_1,b_1}\sum_{a_0,b_0} P_s\,|\langle a_0\,b_0|s\rangle|^2\,|\langle a_1\,b_1|U(t)|s\rangle|^2\,\frac{P_{b_1}}{\mathcal{P}_{b_1}}\\ &= \sum_{a_1,b_1}\sum_{s} P_s\,|\langle a_1\,b_1|U(t)|s\rangle|^2\frac{P_{b_1}}{\mathcal{P}_{b_1}} = \sum_{b_1}\sum_{a_1}\mathcal{P}(a_1,b_1)\frac{P_{b_1}}{\mathcal{P}_{b_1}} = \sum_{b_1} P_{b_1} = 1.\end{aligned} \tag{S5}$$

Finally, the stochastic entropy production γ satisfies an integral fluctuation theorem when averaging over the forward trajectory Γ,

$$\begin{aligned}\langle e^{-\gamma}\rangle &= \sum_{\Gamma} P[\Gamma]\, \exp\left(-\ln \frac{|\langle a_0\, b_0|s\rangle|^2\, |\langle a_1\, b_1|U(t)|s\rangle|^2}{|\langle a_1\, b_1|s^*\rangle|^2\, |\langle a_0\, b_0|U^\dagger(t)|s^*\rangle|^2}\right) \\ &= \left(\sum_s P_s\right)\left(\sum_{a_1,b_1} |\langle a_1\, b_1|s^*\rangle|^2\right)\left(\sum_{a_0,b_0} |\langle a_0\, b_0|U^\dagger(t)|s^*\rangle|^2\right) = 1.\end{aligned} \tag{S6}$$

B. DETAILED FLUCTUATION THEOREM

We next summarize the derivation of the detailed fluctuation theorem (16) of the main text. In order to evaluate the ratio $P_f(Q,K,\gamma)/P_r(-Q,-K,\bar\gamma)$, we need to consider that the forward trajectory Γ is a function of (s,a_0,b_0,a_1,b_1), while $Q[\Gamma]$ and K and γ are all functions of (Γ,s^*). We first define,

$$P_f(Q,K,\gamma|s^*) = \sum_{\Gamma} \delta(Q-Q[\Gamma])\,\delta(K-K[\Gamma,s^*])\,\delta(\gamma-\gamma[\Gamma,s^*])\,P(\Gamma) \tag{S7}$$

which gives the probability of having (Q,K,γ) when one starts the reverse process with a vector $|s^*\rangle$. We have,

$$P_f(Q,K,\gamma) = \sum_{s^*} P(s^*)\,P_f(Q,K,\gamma|s^*). \tag{S8}$$

It then follows that,

$$\begin{aligned}P_f(Q,K,\gamma) &= \sum_{\Gamma,s^*} \delta(Q-Q[\Gamma])\,\delta(K-K[\Gamma,s^*])\,\delta(\gamma-\gamma[\Gamma,s^*])\,P(\Gamma)\,P(s^*) \\ &= e^{Q\Delta\beta-K+\gamma}\sum_{\Gamma^*,s} \delta(Q+Q[\Gamma^*])\,\delta(K+K[\Gamma^*,s])\,\delta(\gamma-\bar\gamma[\Gamma^*,s])\,P(\Gamma^*)\,P(s) \\ &= e^{Q\Delta\beta-K+\gamma}\sum_{s} P(s)\,P(-Q,-K,\bar\gamma|s) = e^{Q\Delta\beta-K+\gamma}\,P(-Q,-K,\bar\gamma)\end{aligned} \tag{S9}$$

where $\bar\gamma[\Gamma,s^*] = -\ln \frac{|\langle a_0\, b_0|s\rangle|^2\,|\langle a_1\, b_1|U_t^\dagger|s\rangle|^2}{|\langle a_1\, b_1|s^*\rangle|^2\,|\langle a_0\, b_0|U_t|s^*\rangle|^2}$.

C. PATH PROBABILITY FOR THE LOCAL TRAJECTORY

The physics behind expression (6) of the main text for the path probability for the unconditional local trajectory can be made more transparent by introducing a transformation akin to the Choi matrix used in the theory of quantum operations [S1]. We introduce an auxiliary Hilbert space $A'B'$ and consider

$$\Omega = \sum_s p_s\, |s\rangle\langle s|_{AB} \otimes |s\rangle\langle s|_{A'B'}. \tag{S10}$$

We then construct the Choi matrix,

$$\Lambda(t) = (I_{AB}\otimes \mathcal{E}_{A'B'})(\Omega), \tag{S11}$$

where $\mathcal{E}(\rho) = U(t)\,\rho\,U^\dagger(t)$. With simple rearrangements, Eq. (6) of the main text may then be written as,

$$\mathcal{P}(a,b,a',b') = \langle a,b,a',b'|\Lambda(t)|a,b,a',b'\rangle, \tag{S12}$$

which is in the form of a standard quantum mechanical expectation value. Since $\Lambda(t)$ is both Hermitian and positive semi-definite, the probabilities $\mathcal{P}(a,b,a',b')$ are guaranteed to be positive and normalized.

D. ANALYTICAL SOLUTION OF THE TWO-QUBIT EXAMPLE

In this section, we provide the analytical solution for the two-spin example presented in the main text. For $\alpha = 0$ the global initial state is $\rho_{AB}(0) = \mathrm{diag}\big(1, e^{-\beta_B}, e^{-\beta_A}, e^{-\beta_A-\beta_B}\big)/(Z_A Z_B)$ where the diagonal is with respect to the $\sigma_z \otimes \sigma_z$ basis. From Eq. (7) in the main text, the probability $P_f(Q)$ is given in this case by $P_f(Q) = \sum_{b,b'}^{a,a} \delta(Q - \Delta E)\, \mathcal{P}(a,b)\, |\langle a', b'|U_t|a\, b\rangle|^2$. Under the action of the unitary $U_t = e^{-itH_{int}}$, the basis changes as follows,

$$U|00\rangle = |00\rangle, \tag{S13}$$

$$U|01\rangle = \cos\Big(t\frac{\pi}{2\tau}\Big)\,|01\rangle - i\sin\Big(t\frac{\pi}{2\tau}\Big)\,|10\rangle, \tag{S14}$$

$$U|10\rangle = -i\sin\Big(t\frac{\pi}{2\tau}\Big)\,|01\rangle + \cos\Big(t\frac{\pi}{2\tau}\Big)\,|10\rangle, \tag{S15}$$

$$U|11\rangle = |11\rangle. \tag{S16}$$

Since initially system A is colder than system B, $Q = +Q_A$ when $|01\rangle \to |10\rangle$ and $Q = -Q_A$ when $|10\rangle \to |01\rangle$. We have, therefore,

$$P_f(Q = +Q_A) = \mathcal{P}(0,1)\,|\langle 10|U_t|01\rangle|^2 = \frac{e^{-\beta_B}}{Z_A Z_B}\sin^2\Big(t\frac{\pi}{2\tau}\Big), \tag{S17}$$

$$P_f(Q = -Q_A) = \mathcal{P}(1,0)\,|\langle 01|U_t|10\rangle|^2 = \frac{e^{-\beta_A}}{Z_A Z_B}\sin^2\Big(t\frac{\pi}{2\tau}\Big), \tag{S18}$$

$$P_f(Q = 0) = \sum a, b\mathcal{P}(a,b)\,|\langle a\, b|U_t|a\, b\rangle|^2 = \frac{1 + e^{-\beta_A-\beta_B}}{Z_A Z_B} + \frac{e^{-\beta_A} + e^{-\beta_B}}{Z_A Z_B}\cos^2\Big(t\frac{\pi}{2\tau}\Big). \tag{S19}$$

For the reversed path Γ^*, the replacement $U_t \to U_t^\dagger$ implies the replacement $t \to -t$. In the uncorrelated case, this has no effect on the heat distribution and we have accordingly $P_f(Q) = P_r(Q)$.

On the other hand, in the correlated case when $\alpha = -i\exp\big[-(\beta_A + \beta_B)/2\big]/Z_A Z_B$, the initial state reads,

$$\rho_{AB}(0) = \frac{1}{Z_A Z_B}|00\rangle\langle 00| + \frac{e^{-\beta_A} + e^{-\beta_B}}{Z_A Z_B}|\phi\rangle\langle\phi| + \frac{e^{-\beta_A-\beta_B}}{Z_A Z_B}|11\rangle\langle 11|, \tag{S20}$$

with $|\phi\rangle = \Big(e^{-\frac{\beta_B}{2}}|01\rangle + i\,e^{-\frac{\beta_A}{2}}|10\rangle\Big)\big/\sqrt{e^{-\beta_A} + e^{-\beta_B}}$.

We have again, $Q = +Q_A$ when $|01\rangle \to |10\rangle$ and $Q = -Q_A$ when $|10\rangle \to |01\rangle$. As a result,

$$P_f(Q = +Q_A) = \mathcal{P}(\phi)\,|\langle 01|\phi\rangle|^2\,|\langle 10|U_t|\phi\rangle|^2 = \frac{e^{-\beta_B}}{Z_A Z_B}\frac{\Big[e^{-\frac{\beta_A}{2}}\cos\Big(t\frac{\pi}{2\tau}\Big) - e^{-\frac{\beta_B}{2}}\sin\Big(t\frac{\pi}{2\tau}\Big)\Big]^2}{e^{-\beta_A} + e^{-\beta_B}}, \tag{S21}$$

$$P_f(Q = -Q_A) = \mathcal{P}(\phi)\,|\langle 10|\phi\rangle|^2\,|\langle 01|U_t|\phi\rangle|^2 = \frac{e^{-\beta_A}}{Z_A Z_B}\frac{\Big[e^{-\frac{\beta_B}{2}}\cos\Big(t\frac{\pi}{2\tau}\Big) + e^{-\frac{\beta_A}{2}}\sin\Big(t\frac{\pi}{2\tau}\Big)\Big]^2}{e^{-\beta_A} + e^{-\beta_B}}, \tag{S22}$$

$$\begin{aligned} P_f(Q = 0) &= \mathcal{P}(0,0) + \mathcal{P}(1,1) + \mathcal{P}(\phi)\Big(|\langle 01|\phi\rangle|^2\,|\langle 01|U_t|\phi\rangle|^2 + |\langle 10|\phi\rangle|^2\,|\langle 10|U_t|\phi\rangle|^2\Big) \\ &= \frac{1 + e^{-\beta_A-\beta_B}}{Z_A Z_B} + \frac{e^{-\beta_A}}{Z_A Z_B}\frac{\Big[e^{-\frac{\beta_A}{2}}\cos\Big(t\frac{\pi}{2\tau}\Big) - e^{-\frac{\beta_B}{2}}\sin\Big(t\frac{\pi}{2\tau}\Big)\Big]^2}{e^{-\beta_A} + e^{-\beta_B}} \\ &\quad + \frac{e^{-\beta_B}}{Z_A Z_B}\frac{\Big[e^{-\frac{\beta_B}{2}}\cos\Big(t\frac{\pi}{2\tau}\Big) + e^{-\frac{\beta_A}{2}}\sin\Big(t\frac{\pi}{2\tau}\Big)\Big]^2}{e^{-\beta_A} + e^{-\beta_B}}. \end{aligned} \tag{S23}$$

In general, except for $t = (0, \tau)$, $P_f(Q) \neq P_r(Q)$.

[S1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, (Cambridge University Press, Cambridge, 2000).