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MEASURES OF IRREVERSIBILITY IN QUANTUM PHASE SPACE

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ENTROPY PRODUCTION AND THE 2ND LAW

IRREVERSIBLE PROCESSES

- The basic laws of physics (Schrödinger's equation, Newton's law) are all *reversible*.
- However, the phenomena that *emerge* from them are generally not.

Special thanks to Eric Lutz, for showing me these videos.

HOW TO QUANTIFY IRREVERSIBILITY?

- *Understanding* the emergence of irreversibility is a fundamental problem in physics.

- *Quantifying* it is crucial for several applications:

  - Engines, power plants, biological motors, electronic devices, &c.

- In thermodynamics, irreversibility is quantified by the **entropy production**.

- Thermodynamic processes obey the *Clausius inequality:*

$$\Delta S \geq \frac{\delta Q}{T}$$

- The difference is called the entropy production:

$$\Sigma := \Delta S - \frac{\delta Q}{T} \geq 0$$

- It is always non-negative and zero iff the process is reversible.

- It is more common to work with *rates:*

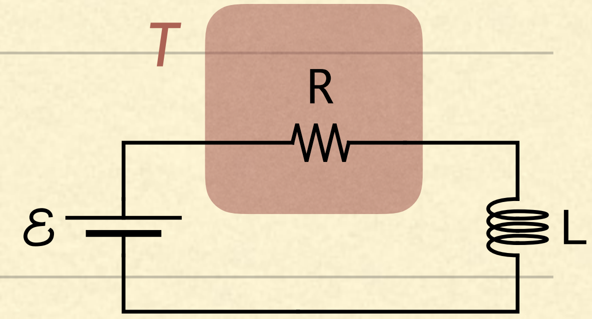
Rate of change of the entropy

$$\Pi = \frac{d\Sigma}{dt} \qquad \frac{dS}{dt} = \Pi - \Phi \qquad \Phi = -\frac{1}{T}\frac{dQ}{dt}$$

entropy production rate

entropy flux rate

EXAMPLE: ELECTRICAL CIRCUIT

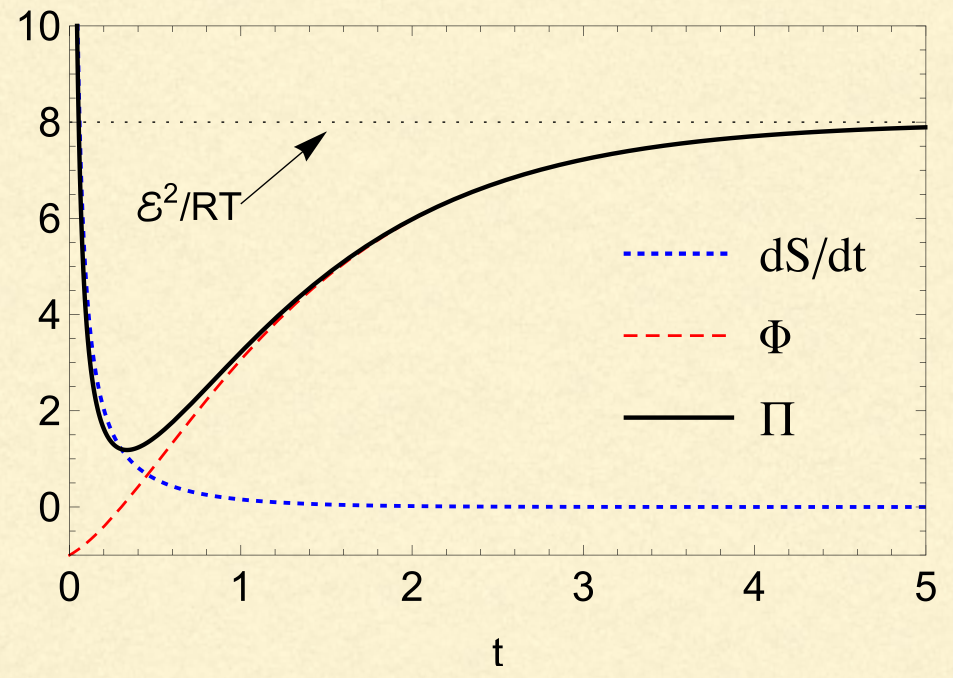


- The simplest example is an RL circuit connected to a bath and a battery.

$$\frac{dS}{dt} = \frac{R}{L}\frac{1}{e^{2Rt/L}-1}$$

$$\Pi(t) = \frac{\mathcal{E}^2}{RT}(1-e^{-Rt/L})^2 + \frac{R}{L}\frac{e^{-2Rt/L}}{e^{2Rt/L}-1}$$

$$\Phi = \Pi - \frac{dS}{dt}$$



- In the long-time limit, this system will reach a *non-equilibrium steady state:*

$$\frac{dS}{dt} = 0 \qquad \text{but} \qquad \Pi_{ss} = \Phi_{ss} = \frac{\mathcal{E}^2}{RT}$$

GTL, T. Tomé and M. J. de Oliveira, *J. Phys. A*, **46**, 395001 (2013)

WHY ENTROPY PRODUCTION MATTERS

- The entropy production directly influences the efficiency of a heat engine.

- The 1st and 2nd laws *in the steady-state*, read:

$$\frac{dU}{dt} = \dot{Q}_h + \dot{Q}_c + \dot{W} = 0$$

$$\frac{dS}{dt} = \Pi + \frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} = 0$$

- The efficiency in the steady-state reads:

$$\eta = -\frac{\dot{W}}{\dot{Q}_h} = 1 + \frac{\dot{Q}_c}{\dot{Q}_h} = 1 - \frac{T_c}{T_h} - \frac{T_c}{\dot{Q}_h}\Pi$$

- Entropy production is the reason why the efficiency is smaller than Carnot's.

THEORETICAL FRAMEWORKS FOR ENTROPY PRODUCTION

SCHNAKENBERG'S APPROACH

- Consider a discrete state system described by the classical master equation:

$$\frac{dp_n}{dt} = \sum_m \left\{ W(n|m)p_m - W(m|n)p_n \right\}$$

- We also assume detailed balance for simplicity (his result is actually more general):

$$\frac{W(n|m)}{W(m|n)} = \frac{p_n^{\mathrm{eq}}}{p_m^{\mathrm{eq}}} = e^{-\beta(E_n - E_m)}$$

- We now look at the evolution of the Shannon entropy:

$$S = -\sum_n p_n \ln p_n$$

Schnakenberg, Rev. Mod. Phys., **48**, 571 (1976)

- Schnakenberg showed that

$$\frac{dS}{dt} = \frac{1}{T}\frac{dQ}{dt} + \Pi$$

- where

$$\Pi = \frac{1}{2}\sum_{n,m}\left\{W(n|m)p_m - W(m|n)p_n\right\}\ln\frac{W(n|m)p_m}{W(m|n)p_n} \geq 0$$

- We can rewrite this formula in a neat way in terms of the *relative entropy (Kullback-Leibler divergence):*

$$\Pi = -\frac{d}{dt}S(\boldsymbol{p}(t)||\boldsymbol{p}^{\mathrm{eq}})$$

$$S(\boldsymbol{p}||\boldsymbol{q}) = \sum_n p_n \ln p_n/q_n$$

QUANTUM MASTER EQUATIONS

- The entropy production for *certain* master equations can be formulated in an analogous way.

- Consider, for instance

$$\frac{d\rho}{dt} = \gamma(1-f)\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right] + \gamma f\left[\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\right]$$

- This equation falls under a category known as Davies map.

- The entropy production can be shown to be:

$$\Pi = -\frac{dS(\rho||\rho_{\text{eq}})}{dt} \qquad S(\rho||\sigma) = \text{tr}\Big\{\rho\ln\rho - \rho\ln\sigma\Big\}$$

H. Spohn, *J. Math. Phys.*, **19**, 1227 (1978).

- Davies maps select the energy basis as a preferred basis (*einselection*).
- This means the populations $p_n = \langle n|\rho|n\rangle$ will evolve according to

$$\frac{dp_n}{dt} = \sum_m \Big\{ W(n|m)p_m - W(m|n)p_n \Big\}$$

- We can now split

$$S(\rho||\rho_{\text{eq}}) = S(\boldsymbol{p}||\boldsymbol{p}_{\text{eq}}) + \mathcal{C}(\rho)$$

(relative entropy of coherence)

$$\mathcal{C}(\rho) = S(\boldsymbol{p}) - S(\rho)$$

- This yields:

$$\Pi = -\frac{dS(\boldsymbol{p}||\boldsymbol{p}_{\text{eq}})}{dt} - \frac{d\mathcal{C}}{dt}$$

- Coherence does *not* affect the entropy flux.
- But decoherence is irreversible and thus affects the entropy production.

J. P. Santos, L. Céleri, GTL, M. Paternostro, to appear in *Nature Quantum Information* (arXiv 1707.08946)

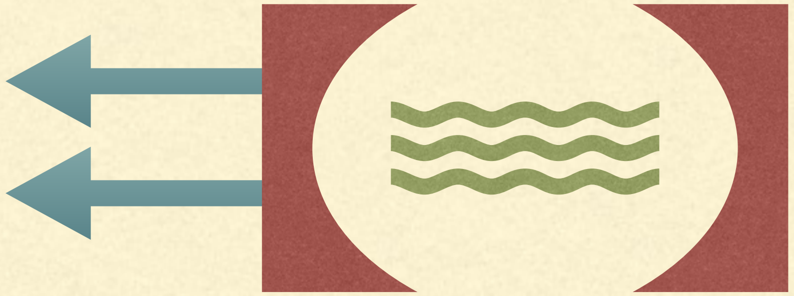
ENTROPY PRODUCTION IN QUANTUM PHASE SPACE

Jader P. Santos, GTL and Mauro Paternostro, Phys. Rev. Lett, **118**, 220601 (2017), (arXiv 1706.01145)

TROUBLE @T = 0

- An optical cavity with leaky photons is well described by a Lindblad dissipator of the form

$$D(\rho) = \gamma \left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\} \right]$$



- This is equivalent to a zero temperature dissipator, as it never creates excitations, but only destroys them.

$$D(\rho) = \gamma(N+1) \left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\} \right] + \gamma N \left[a^\dagger \rho a - \frac{1}{2}\{a a^\dagger, \rho\} \right]$$

- If we apply the standard results of thermodynamics, both the entropy flux and the entropy production diverge:

$$\frac{dS}{dt} = \Pi + \frac{\dot{Q}}{T} \qquad \Pi = -\frac{d}{dt} S(\rho || \rho_{\text{eq}})$$

EXAMPLE: EVOLUTION OF A COHERENT STATE

- Consider the evolution of a harmonic oscillator subject only to:

$$\frac{d\rho}{dt} = \gamma\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right]$$

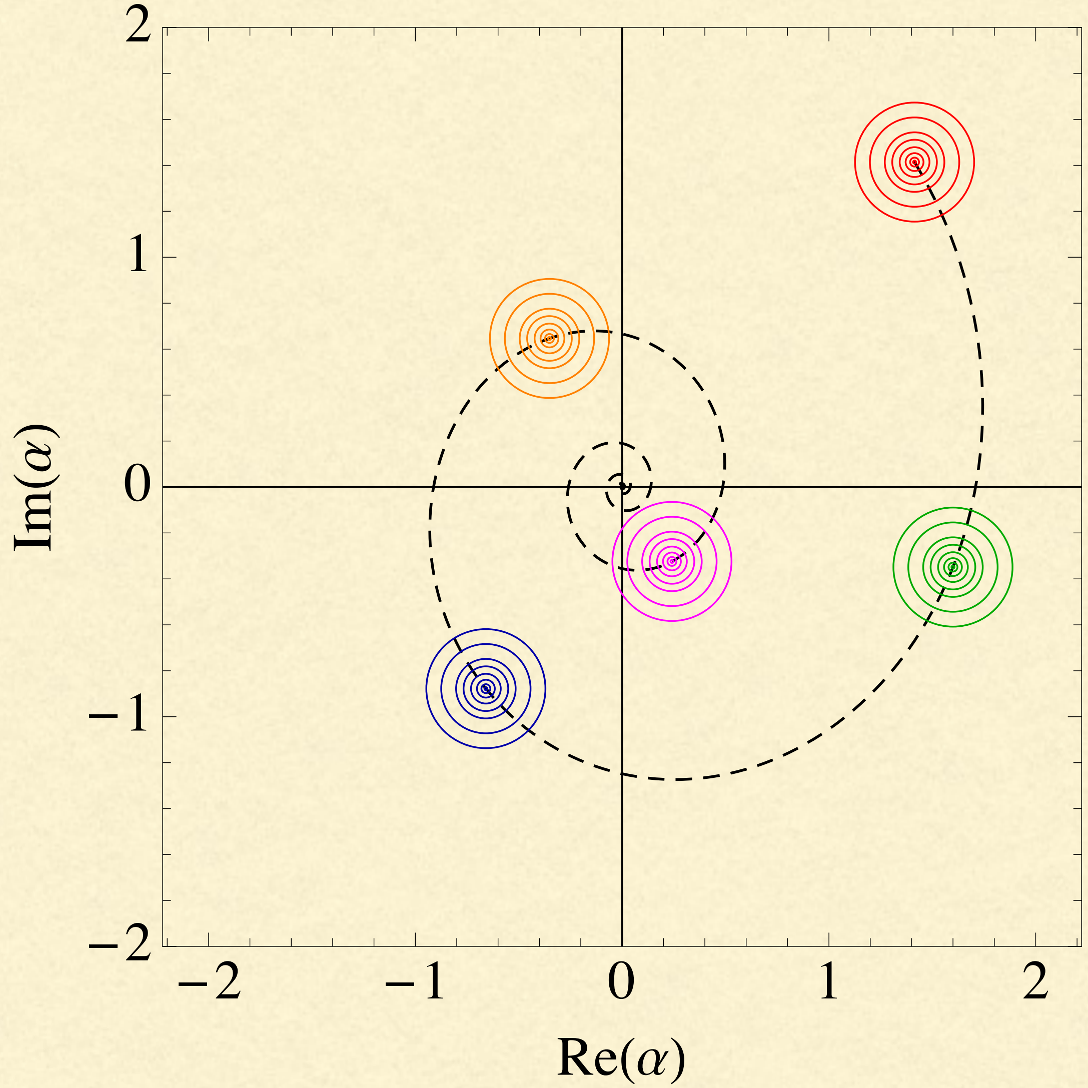
- And assume the initial state is a coherent state:

$$\rho(0) = |\mu\rangle\langle\mu|$$

- Then the state will remain a coherent state throughout:

$$\rho(t) = |\mu_t\rangle\langle\mu_t|$$

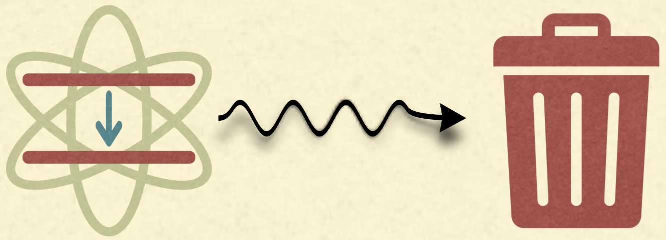
$$\mu_t = \mu e^{-(i\omega+\gamma/2)t}$$



- The entropy is always zero. But Π and Φ will be infinite.

NON-EQUILIBRIUM BATHS

- The leaky cavity is not an equilibrium bath. It is engineered.
  - This divergence is not physical: we are applying a thermodynamic result to a non-thermodynamic scenario.
- **Motivation:** to extend the theory of entropy production to non-equilibrium environments. e.g.
  - leaky cavities
  - squeezed baths
  - dephasing baths
  - spontaneous emission.

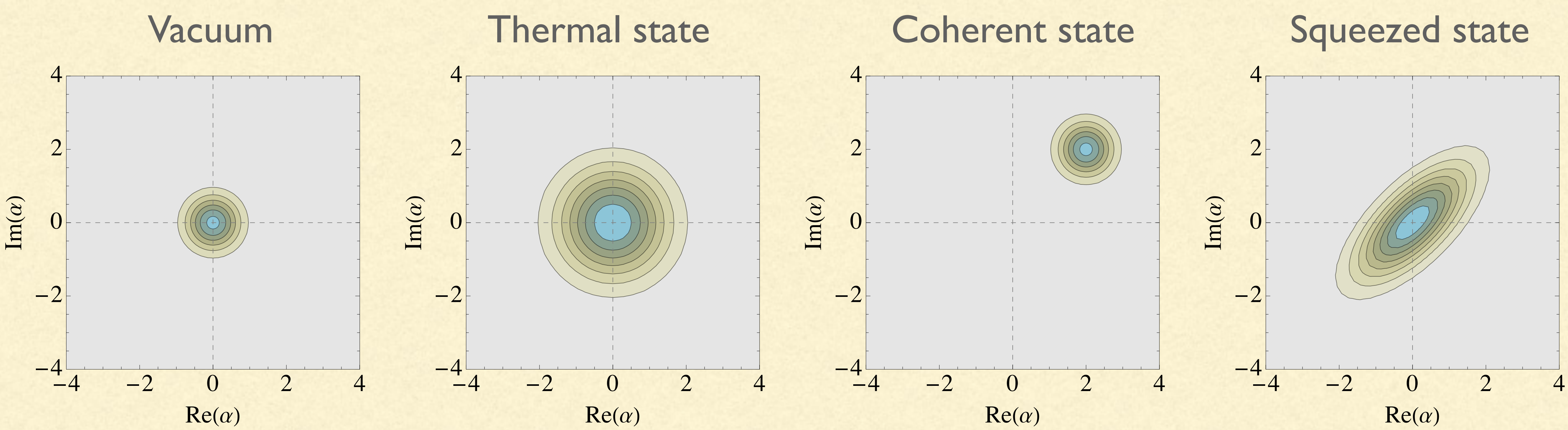


WIGNER FUNCTION

- In our approach we formulate the problem in quantum phase space

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int d^2\lambda e^{\lambda^*\alpha-\lambda\alpha^*}\mathrm{tr}\left\{\rho e^{\lambda a^\dagger-\lambda^* a}\right\}$$

$$\mathrm{Re}(\alpha) = q$$
$$\mathrm{Im}(\alpha) = p$$



QUANTUM FOKKER-PLANCK EQUATION

- The Lindblad equation (no Hamiltonian for simplicity),

$$\frac{d\rho}{dt} = D(\rho) = \gamma(N+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right] + \gamma N\left[a^\dagger \rho a - \frac{1}{2}\{a a^\dagger, \rho\}\right]$$

- then becomes the Fokker-Planck equation

$$\partial_t W = \frac{\gamma}{2}\left\{\partial_\alpha(\alpha W) + \partial_{\alpha^*}(\alpha^* W)\right\} + \gamma(N+1/2)\partial_\alpha \partial_{\alpha^*} W$$

- This has the typical form of a diffusion equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x)P(x,t)\right] + D\frac{\partial^2 P}{\partial x^2}$$

WIGNER CURRENTS

- The QFP may always be written as a *continuity equation*

$$\partial_t W = \partial_\alpha J(W) + \partial_{\alpha^*} J^*(W) \qquad (= \text{``} - \nabla \cdot \boldsymbol{J}\text{''})$$

- where *J(W)* are quasi-probability currents

$$J(W) = \frac{\gamma}{2}\left[\alpha W + (N + 1/2)\partial_{\alpha^*} W\right]$$

- Why this matters: *equilibrium is the only state where the currents are zero.*

$$J(W_{\text{eq}}) = 0 \qquad W_{\text{eq}} = \frac{1}{\pi(N+1/2)} e^{-\frac{|\alpha|^2}{N+1/2}}$$

- This is detailed balance.

WIGNER ENTROPY AND RÉNYI-2

- Instead of using the von Neumann entropy, we adopt the entropy of the Wigner function:

$$S = -\int d^2\alpha \, W \ln W$$

- This will be real for Gaussian states (because then W > 0).
- Moreover, for Gaussian states it turns out to coincide with the Rényi-2 entropy

$$S_2 = -\ln \mathrm{tr}\rho^2$$

- The Wigner entropy therefore has an operational interpretation, in terms of the purity of a Gaussian state.

Adesso, Girolami, Serafini, *PRL*, **109**, 190502 (2012)

- For non-Gaussian states, one may use instead the *Wehrl entropy* for the Husimi function:

$$S_Q = -\int d^2\alpha \, Q \ln Q \qquad\qquad Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

A. Wehrl, "General Properties of Entropy,", *Rev. Mod. Phys.* , **50**, 221 (1978)

- It can be interpreted as a sampling entropy, related to phase space measurements.

V. Buzek, C. Keitel and. P. Knight, *Phys. Rev. A.*, **51**, 2575 (1995)

- Can also be extended to spin systems using spin coherent states.

$$|\theta,\phi\rangle = e^{-i\phi S_z} e^{-i\theta S_y}|S,S\rangle$$

J. P. Santos, L. Céleri, F. Brito, GTL, M. Paternostro, *Phys. Rev. A.* **97**, 052123 (2018)

WIGNER Π AND Φ

- We can now exploit the extensive literature in entropy production for *classical* Fokker-Planck equations.

$$\partial_t W = \partial_\alpha J(W) + \partial_{\alpha^*} J^*(W)$$

$$S = -\int d^2\alpha \, W \ln W$$

- A straightforward calculation shows that $$\frac{dS}{dt} = \frac{\dot{Q}}{\omega(N+1/2)} + \Pi$$

- where

$$\Pi = \frac{4}{\gamma(N+1/2)} \int d^2\alpha \frac{|J(W)|^2}{W}$$

U. Seifert, *Rep. Prog. Phys.*, **75**, 126001 (2012)

- The entropy production may also be written as

$$\Pi = -\frac{dS(W||W_{\rm eq})}{dt}$$

$$S(W||W_{\rm eq}) = \int d^2\alpha \, W \ln \frac{W}{W_{\rm eq}}$$

- The entropy flux is

$$\Phi = -\frac{\dot{Q}}{\omega(N+1/2)} = \frac{\gamma}{N+1/2}\left[\langle a^\dagger a\rangle - N\right]$$

- At high temperatures $\omega(n+1/2) \simeq T$ which leads to $\Phi = -\frac{\dot{Q}}{T}$

- But now bot remain finite at $T = 0$.

- The extension to other reservoirs is now identical: *squeezing, dephasing, &c.*

Jader P. Santos, GTL and Mauro Paternostro, Phys. Rev. Lett, **118**, 220601 (2017), (arXiv 1706.01145)

NON-EQUILIBRIUM STEADY-STATES (NESS)

William B. Malouf, Jader P. Santos, Luis A. Correa, Mauro Paternostro and GTL
Wigner entropy production and heat transport in linear quantum lattices
arXiv 1901.03127

BOUNDARY DRIVEN SYSTEMS (LOCAL M-EQUATION)

- We now consider a chain of bosonic modes coupled to 2 baths at the end points:

$$\frac{d\rho}{dt} = -i[H,\rho] + D_1(\rho) + D_L(\rho)$$

- The corresponding Fokker-Planck equation is

$$\partial_t W = \mathcal{U}(W) + \sum_{i=1,L} \partial_{\alpha_i} J_i(W) + \partial_{\alpha_i^*} J_i^*(W)$$

- The entropy balance equations now read

$$\frac{dS}{dt} = \sum_{i=1,L} (\Pi_i - \Phi_i)$$

- Allows us to quantify flux and production to *each* individual dissipation channel.

- The entropy flux rate is now trivially generalized:

$$\Phi_i = -\frac{\dot{Q}_i}{\omega_i(N_i+1/2)} = \frac{\gamma_i}{N_i+1/2}\left[\langle a_i^\dagger a_i\rangle - N_i\right]$$

- The entropy production rate is also apparently trivial:

$$\Pi_i = \frac{4}{\gamma_i(N_i+1/2)}\int d^2\boldsymbol{\alpha}\,\frac{|J_i(W)|^2}{W}$$

- There is a subtlety, however: it is no longer simply a relative entropy,

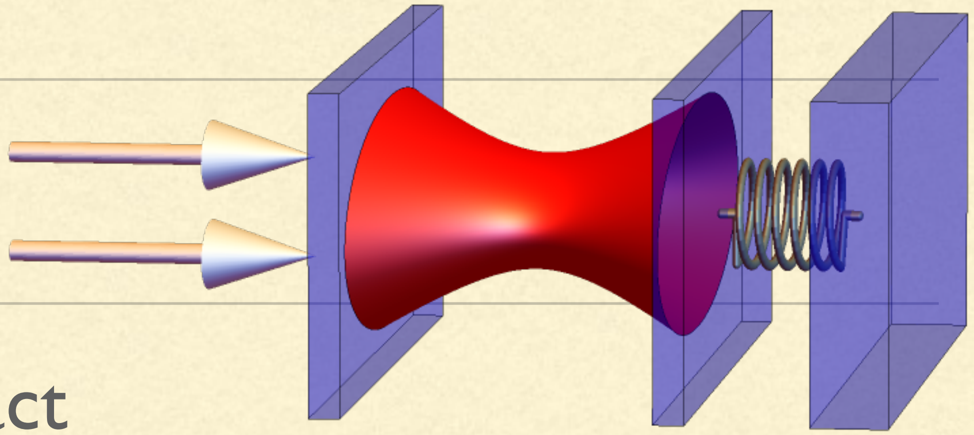
$$\Pi = \sum_i \Pi_i = -\frac{dS(W||W_{\text{eq}})}{dt} - \int d^2\boldsymbol{\alpha}\,\mathcal{U}(W)\ln W_{\text{eq}}$$

- Unitary terms now contribute. In fact, in the NESS they are absolutely essential:
  - They are the terms that allow for a current to flow from one bath to the other.

EXPERIMENTAL DETERMINATION OF THE ENTROPY PRODUCTION IN MESOSCOPIC QUANTUM SYSTEMS

M. Brunelli, L. Fusco, R. Landig, W. Wieczorek, J. Hoelscher-Obermaier, GTL, F Semião, A. Ferraro, N. Kiesel, T. Donner, G. De Chiara, and M. Paternostro.
Phys. Rev. Lett., **121**, 160604 (2018)

OPTOMECHANICS



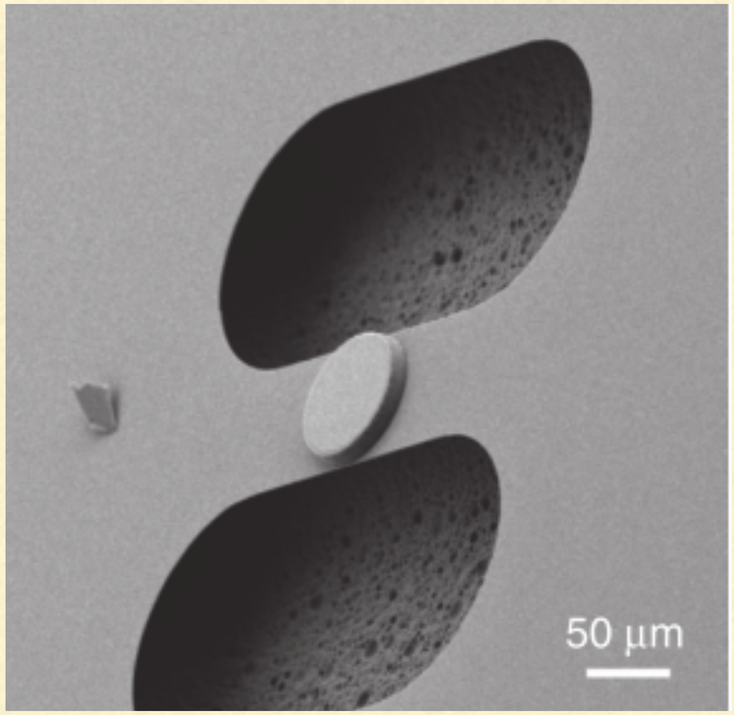
- A thin membrane is allowed to vibrate in contact with radiation trapped in a cavity:

$$H = \omega_c a^\dagger a + \left(\frac{p^2}{2m} + \frac{1}{2}m\omega_m^2 x^2\right)$$

$$-g a^\dagger a x + \epsilon(a^\dagger e^{-i\omega_p t} + a e^{i\omega_p t})$$

$$\frac{d\rho}{dt} = -i[H,\rho] + D_c(\rho) + D_m(\rho)$$

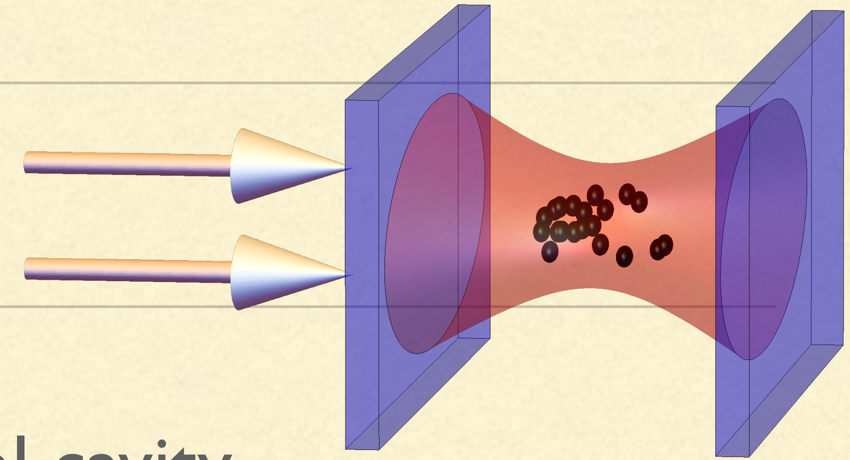
$$D_c(\rho) = 2\kappa\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right]$$



Aspelmeyer group
Viena

Groeblacher, *et. al.*, Nature Communications, **6**, 7606 (2015)

DRIVEN-DISSIPATIVE BEC



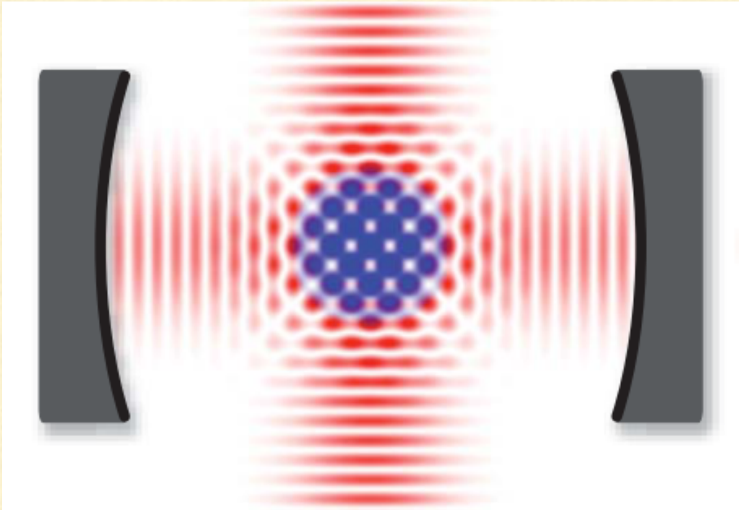
- BEC in an optical lattice, inside a high-finesse optical cavity

$$H = \omega_c a^\dagger a + \frac{\omega_0}{2}(b_1^\dagger b_1 - b_0^\dagger b_0) + \frac{2\lambda}{\sqrt{N}}(a + a^\dagger)(b_0^\dagger b_1 + b_1^\dagger b_0)$$

b_0 and b_1 are bosonic operators of the ground-state and first excited state of the BEC

$$\frac{d\rho}{dt} = -i[H,\rho] + D_c(\rho) + D_{\mathrm{BEC}}(\rho)$$

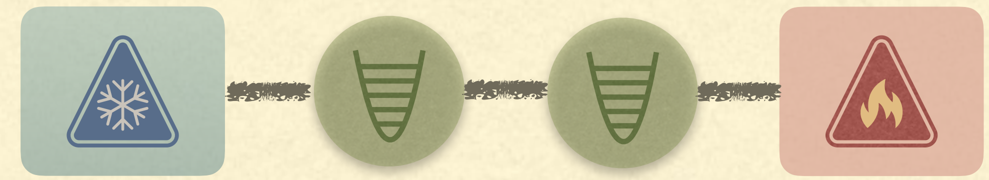
$$D_c(\rho) = 2\kappa\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right]$$



Esslinger group
ETH

Baumann, *et. al.*,Nature, **464**,1301 (2010)

GAUSSIANIZATION



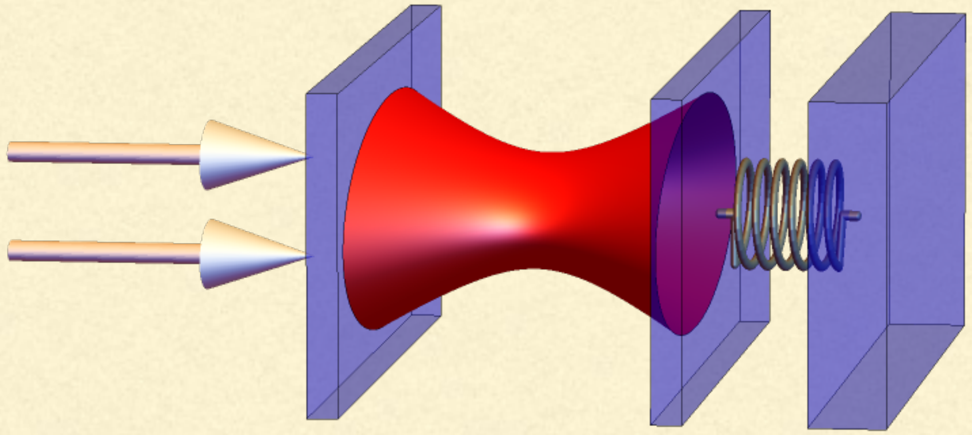
- Both models can be Gaussianized for large drive and converted into an effective system of two harmonic oscillators

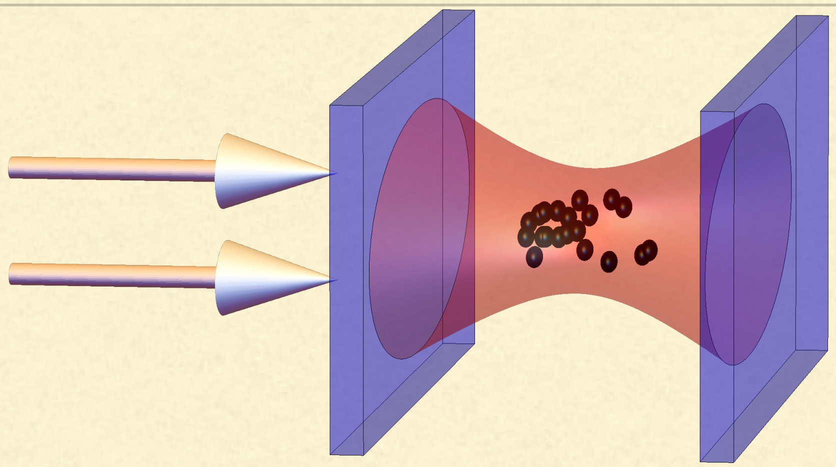
$$H = \omega_a a^\dagger a + \omega_b b^\dagger b + g(a + a^\dagger)(b + b^\dagger)$$

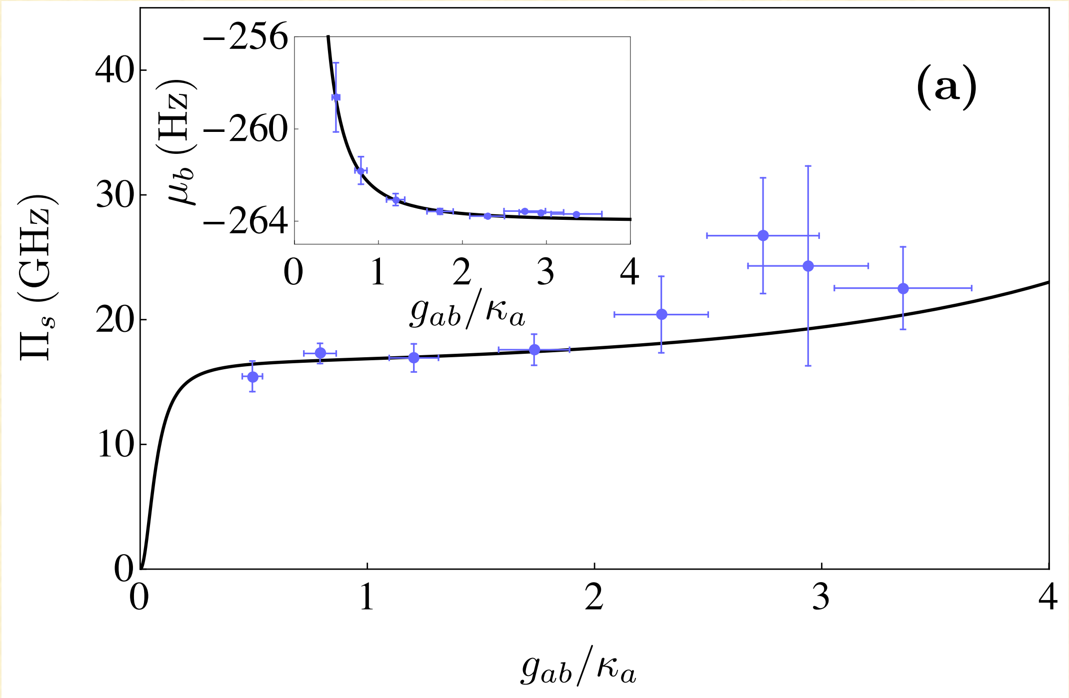
- For the BEC, the mode *b* is a collective (Schwinger) mode of the atomic system.

- von Neumann formalism gives diverging results due to the leaky cavity dissipator.
  - The Wigner framework is therefore the *only* framework capable of describing entropy production in these systems.

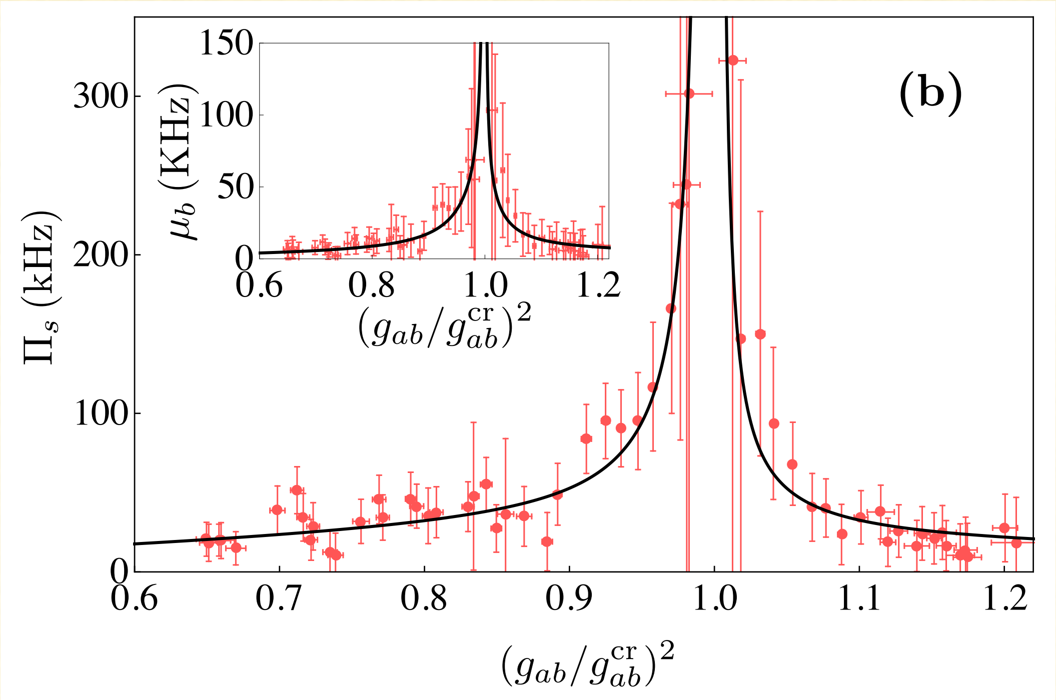
RESULTS







optomechanics



BEC

Π increases with optomechanical coupling.

Π diverges at the Dicke critical point.

CONCLUSIONS

- Irreversibility in quantum systems: *Understand and quantify.*
  - Bridge between non-equilibrium physics and quantum information.
- Phase space formalism provides new physical insight (*understand*).
  - And is the only available alternative at the moment for certain experimentally relevant problems (*quantify*).

Quantum thermodynamics
QT^2
quantum transport
IFUSP

Thank you!

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