THERMODYNAMIC UNCERTAINTY RELATIONS FROM FLUCTUATION THEOREMS

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Quantum thermodynamics
quantum transport
IFUSP

Quantum Thermodynamics for Young Scientists
Bad Honnef, Feb 06, 2020

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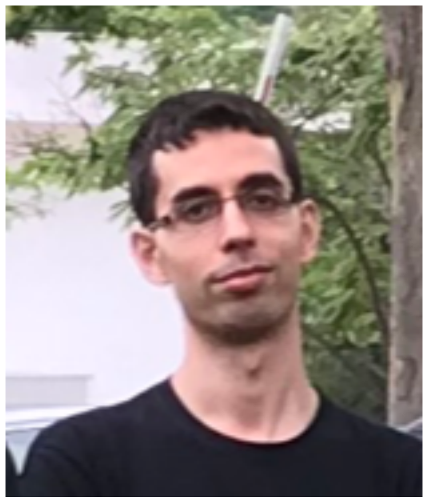
Summary

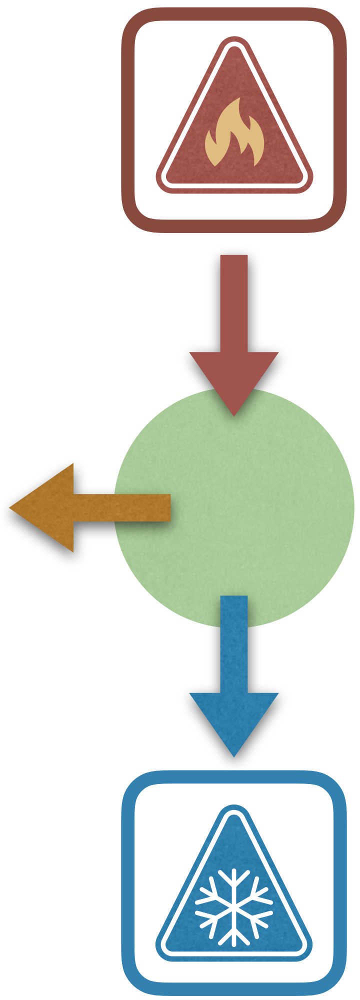
André M. Timpanaro, Giacomo Guarnieri, John Goold, GTL,
"**Thermodynamic uncertainty relations from exchange fluctuation theorems**".
Phys. Rev. Lett. **123**, 090604 (2019) (arXiv 1904.07574)

Why entropy production matters

- 1st and 2nd laws for a system coupled to two baths:

$$\frac{dU}{dt} = \dot{Q}_h + \dot{Q}_c + \dot{W} = 0$$

$$\frac{dS}{dt} = \dot{\Sigma} + \frac{\dot{Q}_h}{T_h} + \frac{\dot{Q}_c}{T_c} = 0$$



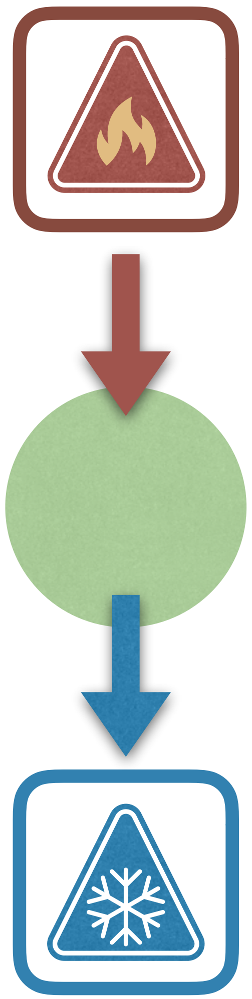
- The efficiency of the engine may then be written as

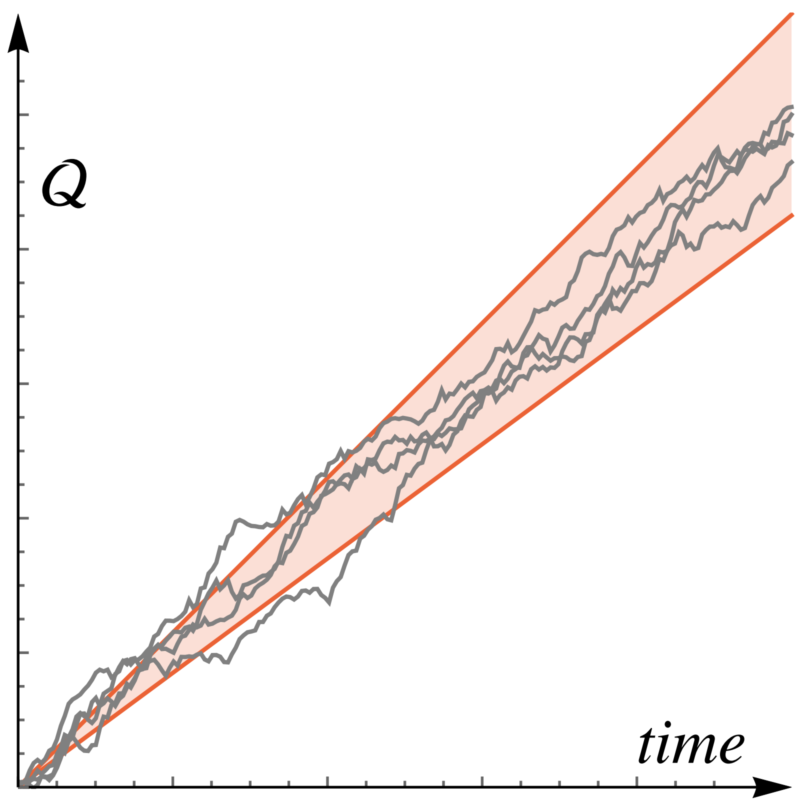
$$\eta = -\frac{\dot{W}}{\dot{Q}_h} = 1 + \frac{\dot{Q}_c}{\dot{Q}_h} = 1 - \frac{T_c}{T_h} - \frac{T_c}{\dot{Q}_h}\dot{\Sigma}$$

- *Entropy production is therefore the reason the efficiency is smaller than Carnot:*

$$\eta = \eta_C - \frac{T_c}{\dot{Q}_h}\dot{\Sigma}$$

Thermodynamic Uncertainty Relations (TURs)





$$\frac{\mathrm{var}(\dot{Q})}{\mathbb{E}(\dot{Q})^2} \geq \frac{2}{\mathbb{E}(\dot{\Sigma})}$$

$$\Sigma = \delta\beta\, Q \ \text{(in the simplest case)}$$

- Simple, elegant and powerful.
- Counterintuitive: To reduce the fluctuations, the process should be *more irreversible*.
- Derived only for the steady-state of classical Markov chains.
- Can be violated in many relevant scenarios (e.g. thermoelectrics).

A. C. Barato, U. Seifert, *Physical Review Letters*, **114**, 158101 (2015)

Implications for mesoscopic engines

- In an autonomous engine the output power is $\dot{W}$
- The TUR in this case then reads

$$\frac{\mathrm{var}(\dot{W})}{\mathbb{E}(\dot{W})^2} \geq \frac{2}{\mathbb{E}(\dot{\Sigma})}$$

- From our previously derived result:

$$\eta = \eta_C - \frac{T_c}{\dot{Q}_h}\dot{\Sigma} \quad \rightarrow \quad \mathbb{E}(\dot{\Sigma}) = \frac{\mathbb{E}(\dot{Q}_h)}{T_c}(\eta_C - \eta)$$

- Thus:

$$\frac{\mathrm{var}(\dot{W})}{\mathbb{E}(\dot{W})^2} \geq \frac{2T_c}{\mathbb{E}(\dot{Q}_c)}\frac{1}{\eta_C - \eta}$$

P. Pietzonka and U. Seifert, *Phys. Rev. Lett.*, **120**, 190602 (2017)

- Thus:

$$\frac{\mathrm{var}(\dot W)}{\mathbb{E}(\dot W)^2} \geq \frac{2T_c}{\mathbb{E}(\dot Q_c)} \frac{1}{\eta_C - \eta}$$

- Finally, we note that $\eta = \dfrac{\mathbb{E}(\dot W)}{\mathbb{E}(\dot Q)}$, so that

$$\boxed{\mathrm{var}(\dot W) \geq 2T_c |\mathbb{E}(\dot W)| \frac{\eta}{\eta_C - \eta}}$$

- If you wish to operate the engine close to Carnot efficiency, you pay the price that the fluctuations may become very large.

  - To curb fluctuations, the engine should be operated irreversibly!

  - Goes against everything we learn in undergraduate thermodynamics.

P. Pietzonka and U. Seifert, *Phys. Rev. Lett.*, **120**, 190602 (2017)

nature physics
PERSPECTIVE
https://doi.org/10.1038/s41567-019-0702-6

Thermodynamic uncertainty relations constrain non-equilibrium fluctuations

Jordan M. Horowitz[1,2,3] and Todd R. Gingrich[4]

Experimental study of the thermodynamic uncertainty relation

Soham Pal,[1] Sushant Saryal,[1] D. Segal,[2,3] T. S. Mahesh,[1] and Bijay Kumar Agarwalla[1,*]

1912.08391

Thermodynamic uncertainty relation in atomic-scale quantum conductors

Hava Meira Friedman,[1] Bijay K. Agarwalla,[2] Ofir Shein-Lumbroso,[3] Oren Tal,[3] and Dvira Segal[1,4,*]

2002.00284

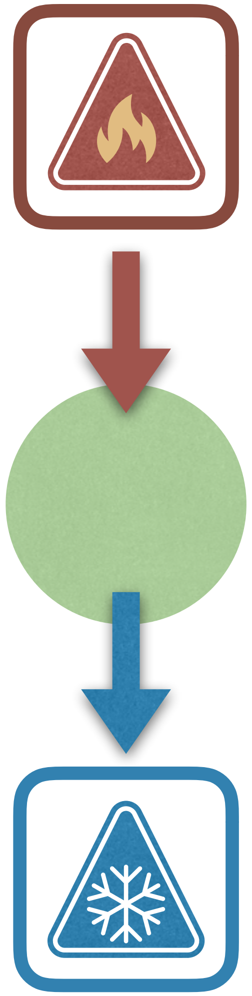
TUR from fluctuation theorems

André M. Timpanaro, Giacomo Guarnieri, John Goold, GTL,
"**Thermodynamic uncertainty relations from exchange fluctuation theorems**".
Phys. Rev. Lett. **123**, 090604 (2019) (arXiv 1904.07574)

EXCHANGE FLUCTUATION THEOREM

- Fluctuation theorems for thermodynamic processes usually have the form

$$\frac{P_F(\Sigma)}{P_B(-\Sigma)} = e^{\Sigma}$$



- e.g. Crooks theorem for work: $\Sigma = \beta(W - \Delta F)$
- FTs, however, compare a *forward* with a *backward* process.
- In some systems, both coincide. These are called *Exchange FTs:*

$$\frac{P(\Sigma)}{P(-\Sigma)} = e^{\Sigma}$$

- This is *much stronger*: it is a symmetry on a single probability distribution.
- Example: direct heat exchange: $\Sigma = \delta\beta Q$

- Motivated by this, we proved the following theorem:

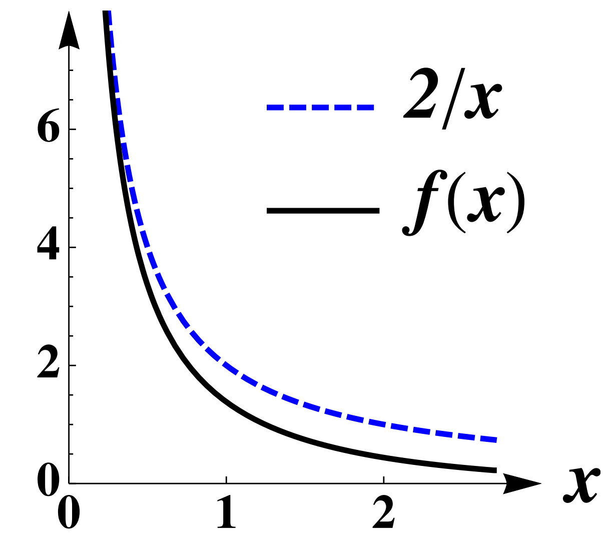
Theorem (*"TUR de force"*)**.** *For fixed finite $\mathbb{E}(\Sigma)$, the probability distribution $P(\Sigma)$ satisfying $P(\Sigma)/P(-\Sigma) = e^{\Sigma}$, with the smallest possible variance (the minimal distribution) is*

$$P_{min}(\Sigma) = \frac{1}{2\cosh(a/2)}\left\{e^{a/2}\delta(\Sigma - a) + e^{-a/2}\delta(\Sigma + a)\right\},$$

where the value of a is fixed by $\mathbb{E}(\Sigma) = a\tanh(a/2)$. For this distribution

$$\mathrm{Var}(\Sigma)_{min} = \mathbb{E}(\Sigma)^2 f(\mathbb{E}(\Sigma)),$$

where $f(x) = csch^2(g(x/2))$, $csch(x)$ is the hyperbolic cosecant and $g(x)$ is the function inverse of $x\tanh(x)$.



For any other distribution we must then have:

$$\frac{\mathrm{var}(\Sigma)}{\mathbb{E}(\Sigma)^2} \geq f(\mathbb{E}(\Sigma))$$

TUR de force IS TIGHT

- Our TUR is the tighest (saturable) bound for this scenario.
- And we know which thermodynamic process saturates it.
- This is relevant because, around the same time, similar papers appeared.
  - But all derived a looser bound with

$$f(x) = \frac{2}{e^x - 1}$$

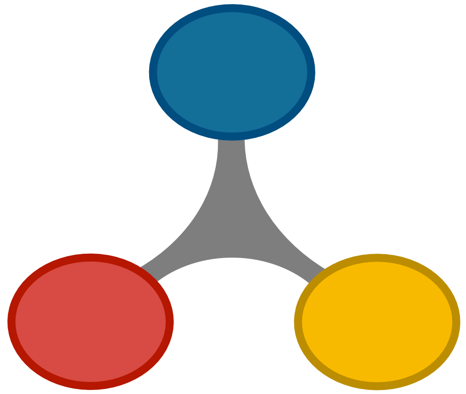
- This bound, however, is never tight.

Hasegawa & Vu 1902.06376.
Proesman & Horowitz 1902.07008.
Potts & Samuelsoon 1904.04913.

EXTENSION TO MULTIPLE CHARGES

- We can also generalize our framework to Exchange FTs involving multiple charges:

$$\frac{P(\mathcal{Q}_1,\ldots,\mathcal{Q}_n)}{P(-\mathcal{Q}_1,\ldots,-\mathcal{Q}_n)} = e^{\sum_i A_i \mathcal{Q}_i}$$



- The entropy production in this case is $\Sigma = \sum_i A_i \mathcal{Q}_i$

- ex: heat engine FT:

$$\frac{P(Q_h, W)}{P(-Q_h, -W)} = e^{(\beta_h - \beta_c) Q_h + \beta_c W}$$

- In this case we obtain the matrix bound

$$\boxed{\mathcal{C} - f(\mathbb{E}(\Sigma))\boldsymbol{q}\boldsymbol{q}^{\mathrm{T}} \geq 0}$$

$$q_i = \mathbb{E}(\mathcal{Q}_i)$$

$$\mathcal{C}_{ij} = \mathrm{cov}(\mathcal{Q}_i, \mathcal{Q}_j)$$

M. Campisi, J. Pekola, R. Fazio, NJP, **17**, 035012 (2015)

$$\mathcal{C} - f(\mathbb{E}(\Sigma))\boldsymbol{q}\boldsymbol{q}^{\mathrm{T}} \geq 0$$

$$q_i = \mathbb{E}(\mathcal{Q}_i)$$

$$\mathcal{C}_{ij} = \mathrm{cov}(\mathcal{Q}_i, \mathcal{Q}_j)$$

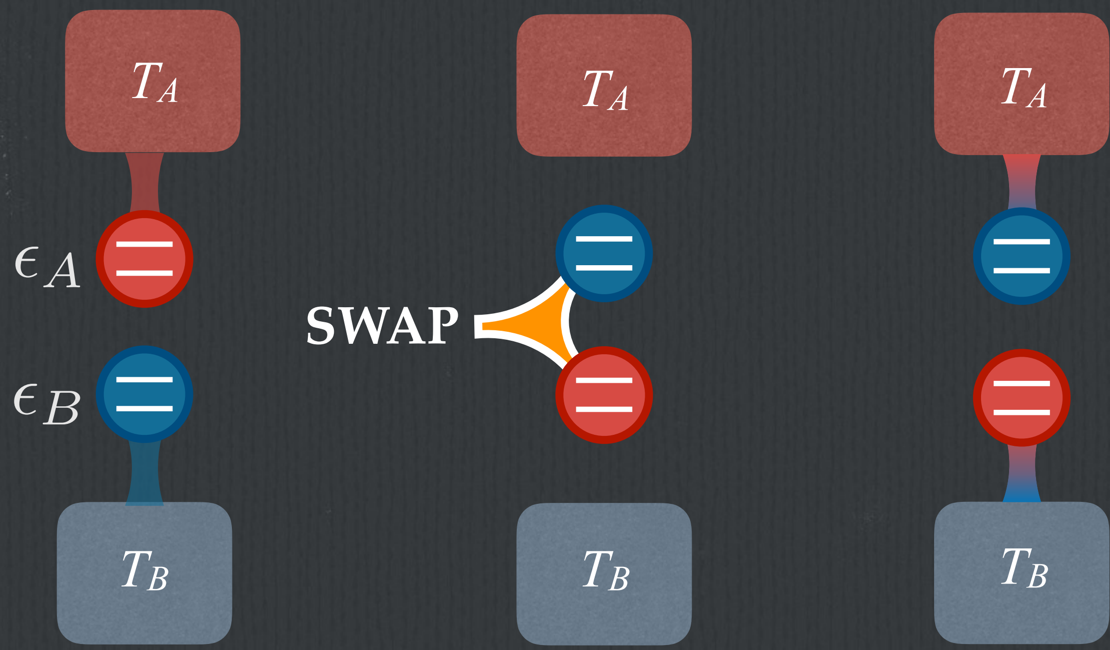
- This says that the matrix above is positive semi-definite.
- As a consequence, all diagonal entries must be positive, which implies an individual TUR for each charge:

$$\frac{\mathrm{var}(\mathcal{Q}_i)}{\mathbb{E}(\mathcal{Q}_i)^2} \geq f(\mathbb{E}(\Sigma))$$

- In addition, it also places restrictions on the covariances:
  - If *G* is psd then $G_{ij}^2 \leq G_{ii}G_{jj}$
- This also imposes a constraint on the *sign* of the covariances

$$\frac{\mathbb{E}(\mathcal{Q}_i)^2}{\mathrm{var}(\mathcal{Q}_i)} + \frac{\mathbb{E}(\mathcal{Q}_j)^2}{\mathrm{var}(\mathcal{Q}_j)} \geq \frac{1}{f(\mathbb{E}(\Sigma))} \quad \rightarrow \quad \mathrm{sign}(C_{ij}) = \mathrm{sign}(\mathbb{E}(\mathcal{Q}_i)\mathbb{E}(\mathcal{Q}_j))$$

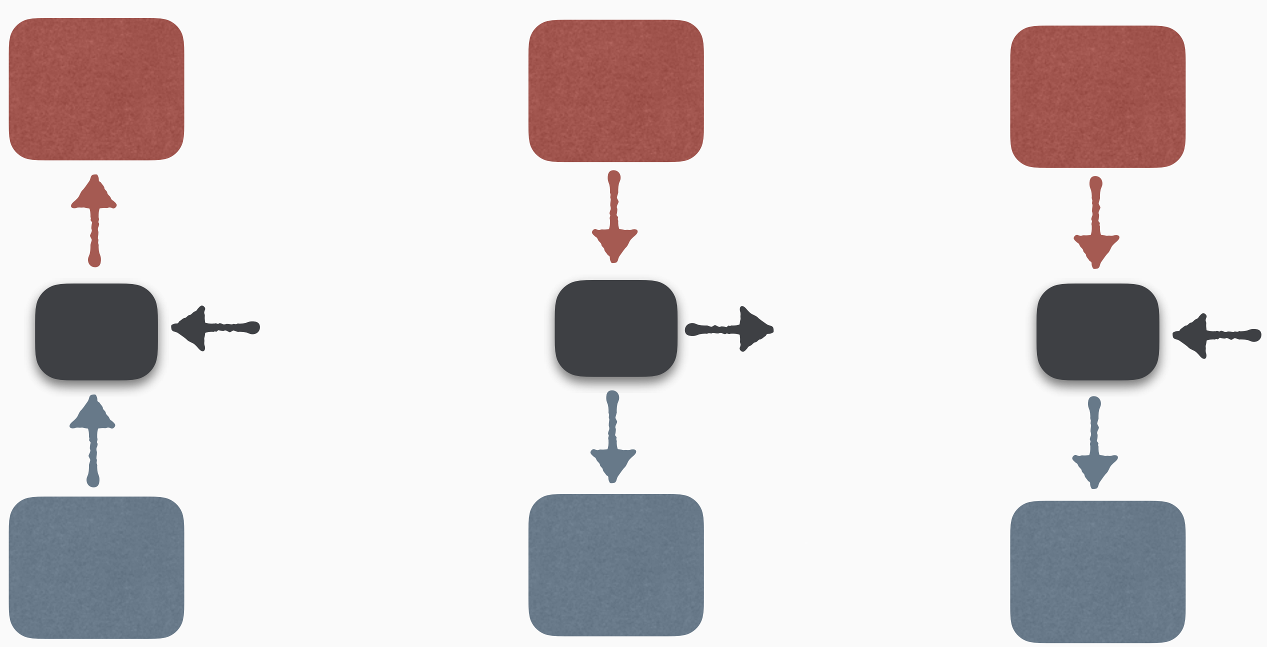
SWAP engine



$$\langle Q_h \rangle = \epsilon_A (f_A - f_B)$$

$$\langle Q_c \rangle = -\epsilon_B (f_A - f_B) \qquad f_i = \frac{1}{e^{\beta_i \epsilon_i} + 1}$$

$$\langle W \rangle = -(\epsilon_A - \epsilon_B)(f_A - f_B)$$



Refrigerator

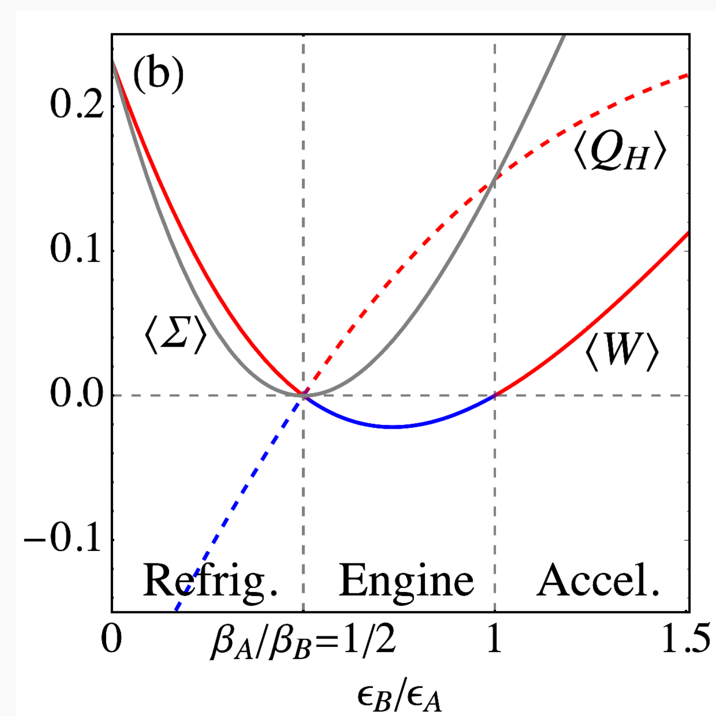
$$\frac{\epsilon_B}{\epsilon_A} < \frac{\beta_A}{\beta_B}$$

Engine

$$\frac{\beta_A}{\beta_B} < \frac{\epsilon_B}{\epsilon_A} < 1$$

Accelerator

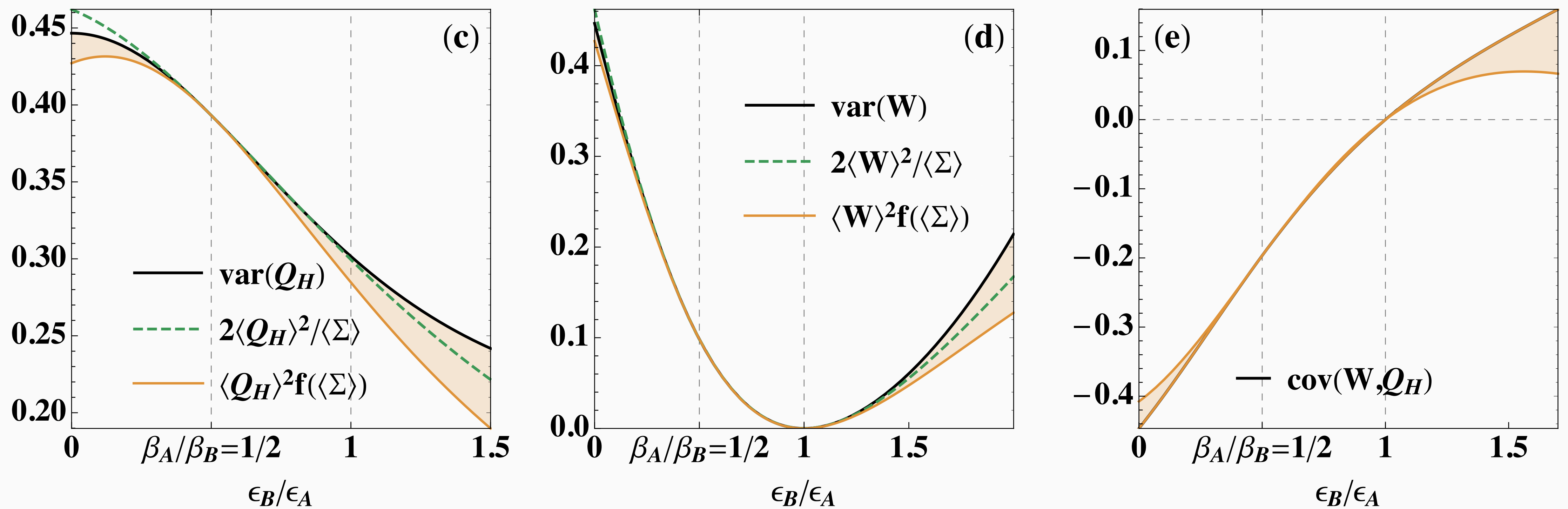
$$1 < \frac{\epsilon_B}{\epsilon_A}$$



M. Campisi, J. Pekola, R. Fazio, NJP, 17, 035012 (2015)

SWAP engine

$$\frac{P(Q_H, W)}{P(-Q_H, -W)} = e^{(\beta_B - \beta_A)Q_H + \beta_B W}$$



EXPERIM

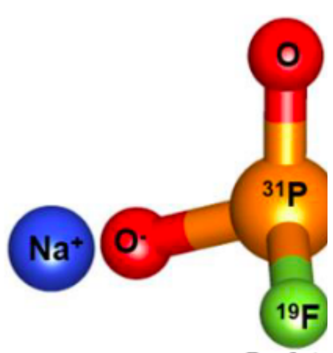
- The above r ... y.

Experim ... relation

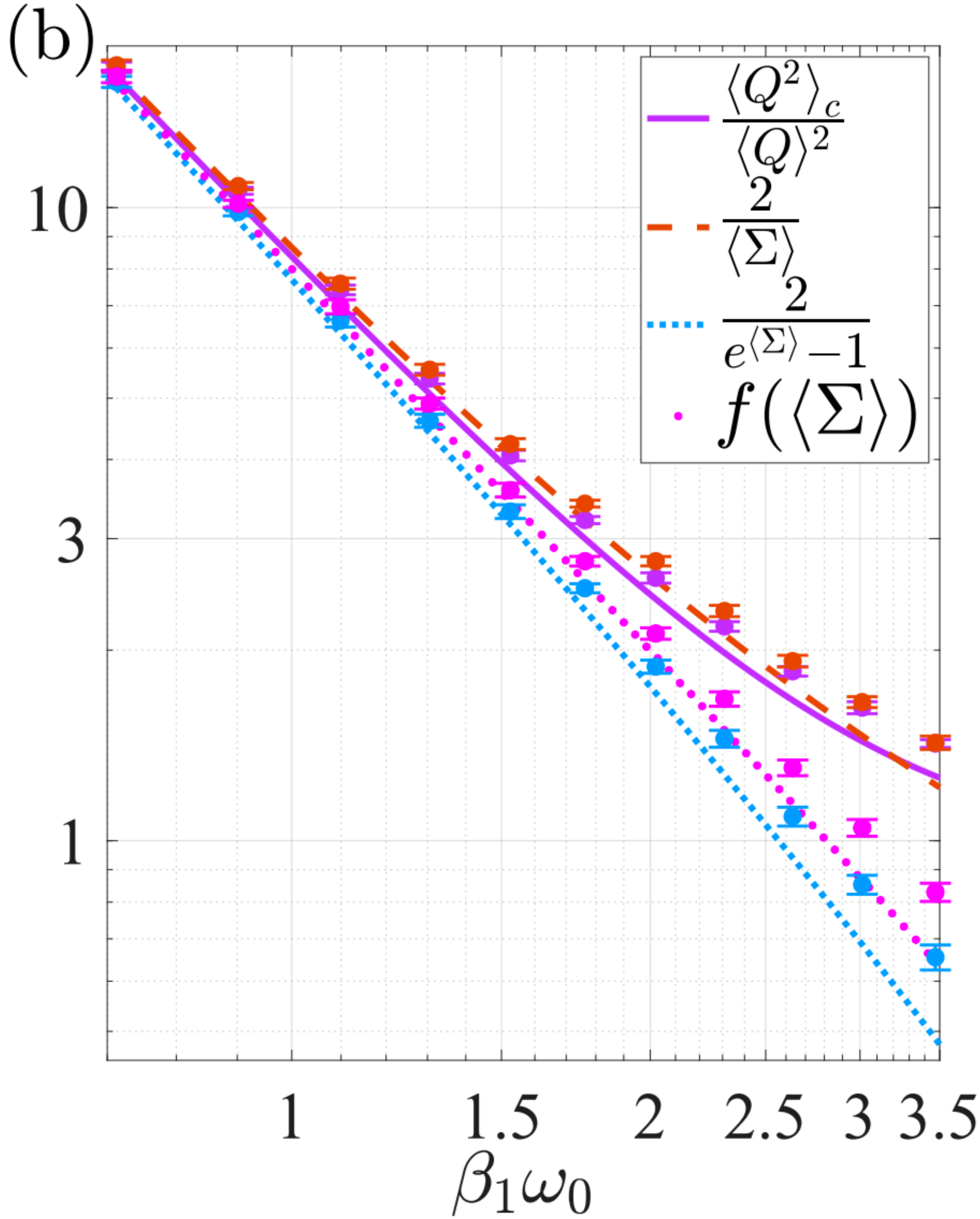
Soham Pal,[1] ... Agarwalla[1,*]

1912.08391

(a)



Sodium fluorop



ACHIEVABILITY OF THE OPTIMAL PROCESS

Theorem ("*TUR de force*"). *For fixed finite $\mathbb{E}(\Sigma)$, the probability distribution $P(\Sigma)$ satisfying $P(\Sigma)/P(-\Sigma) = e^{\Sigma}$, with the smallest possible variance (the minimal distribution) is*

$$P_{min}(\Sigma) = \frac{1}{2\cosh(a/2)}\left\{e^{a/2}\delta(\Sigma - a) + e^{-a/2}\delta(\Sigma + a)\right\},$$

where the value of a is fixed by $\mathbb{E}(\Sigma) = a\tanh(a/2)$. For this distribution

$$\mathrm{Var}(\Sigma)_{min} = \mathbb{E}(\Sigma)^2 f(\mathbb{E}(\Sigma)),$$

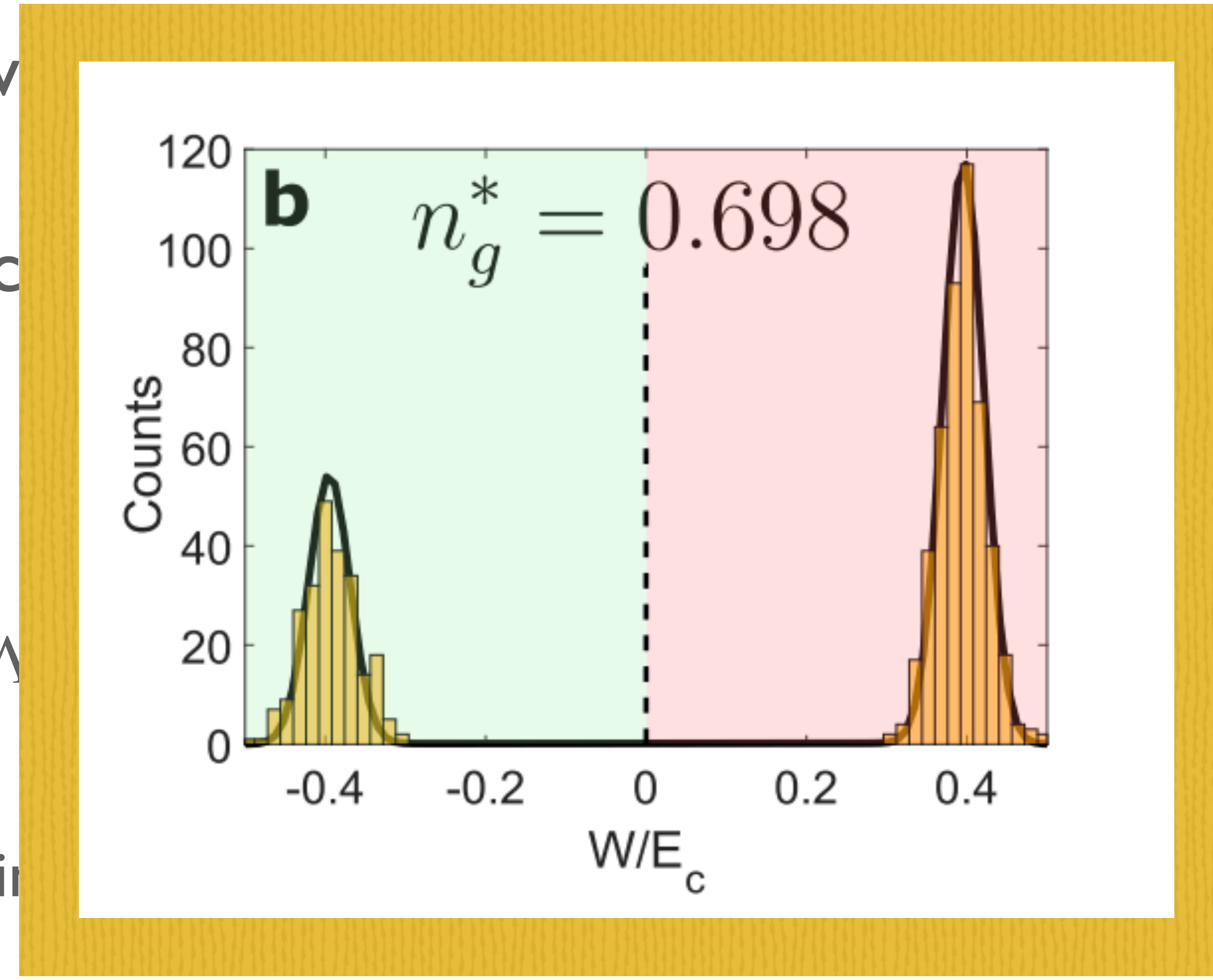
where $f(x) = \mathrm{csch}^2(g(x/2))$, $\mathrm{csch}(x)$ is the hyperbolic cosecant and $g(x)$ is the function inverse of $x\tanh(x)$.

The *minimal process* is one which has only 2 points in the support.

But is this achievable in practice?

i.e., is the bound saturable?

- A beautiful illustration of this was giv
- They were interested in work extrac
- The question they posed was:
  - which process maximizes $P(W \geq \Lambda$
    - $\mathbb{E}(e^{-\beta W}) = e^{-\beta \Delta F}$ and $P(W < W_{\text{mi}}$



- Answer: $P(W) = p\delta(W - \Lambda) + (1-p)\delta(W - W_{\text{min}})$

where. $$p = P(W \geq \Lambda) = \frac{e^{-\beta \Delta F} - e^{\beta W_{\text{min}}}}{e^{\beta \Lambda - e^{\beta W_{\text{min}}}}}$$

V. Cavina, A. Mari and V. Giovannetti, *Scientific Reports*, **6**, 29282 (2016).
O. Maillet, *PRL*, **122**, 150604 (2019)

Conclusions

Acknowledgements:
FAPESP, Heraeus foundation

- In this talk I discussed how TURs can be viewed as a consequence of Fluctuation Theorems.

- I believe that this is important because:

  a. It sheds light on the phy

  b. Shows that FTs not only
  additional constraints o

  c. Introduces the idea of a
  optimizes a given therm

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