Thermodynamics and information in linear quantum lattices

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Introduction

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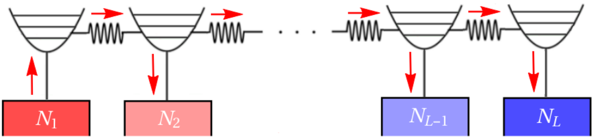
- Our focus: transport phenomena and quantum thermodynamics.



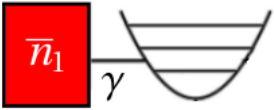
Temperature gradient will produce heat currents.

- The contribution of **decoherence** for the irreversibility.
- How to quantify **correlations** between different parts.

One bosonic mode

Lindblad Master Equation:

$$\mathscr{H} = \omega\, a^\dagger a, \tag{1}$$



$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[\mathscr{H},\rho] + \gamma(\bar{n}_1+1)D[a](\rho) + \gamma\bar{n}_1 D[a^\dagger](\rho), \tag{2}$$

$$D[a](\rho) = a\rho a^\dagger - \frac{1}{2}\{a^\dagger a,\rho\}. \tag{3}$$

$$\frac{\mathrm{d}\langle a^\dagger a\rangle_t}{\mathrm{d}t} = \gamma\left(\bar{n}_1 - \langle a^\dagger a\rangle_t\right) := J_1(t). \tag{4}$$

Steady State: $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0,$ and $D[a](e^{-\beta\omega a^\dagger a}) = 0.$

Multi-modes

A Gaussian state is completely characterized by the CM:

$$\boldsymbol{R} = (a_1, a_1^\dagger, \ldots, a_L, a_L^\dagger)^\top, \tag{5}$$

$$\Theta_{ij} = \frac{1}{2}\langle\{\delta R_i, \delta R_j^\dagger\}\rangle, \tag{6}$$

where $\delta R_i = R_i - \langle R_i\rangle$ and $i, j = 1, \ldots, 2L.$

Reduced Covariance Matrices:

$$C_{ij} = \langle a_j^\dagger a_i\rangle - \langle a_j^\dagger\rangle\langle a_i\rangle, \qquad i, j = 1, \ldots, L. \tag{7}$$

$$S_{ij} = \langle a_i a_j\rangle - \langle a_i\rangle\langle a_j\rangle, \qquad i, j = 1, \ldots, L. \tag{8}$$

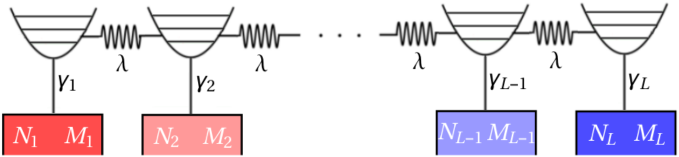
$$\boldsymbol{\Theta} = \frac{\boldsymbol{I}_{2L}}{2} + \boldsymbol{C}\otimes(\boldsymbol{\sigma}_+\boldsymbol{\sigma}_-) + \boldsymbol{C}^\top\otimes(\boldsymbol{\sigma}_-\boldsymbol{\sigma}_+) + \boldsymbol{S}\otimes\boldsymbol{\sigma}_+ + \boldsymbol{S}^*\otimes\boldsymbol{\sigma}_-. \tag{9}$$

Example: $L = 2$ and $\langle a_i \rangle = 0$

$$\Theta = \begin{pmatrix} \langle a_1^\dagger a_1 \rangle + 1/2 & \langle a_1 a_1 \rangle & \langle a_1 a_2^\dagger \rangle & \langle a_1 a_2 \rangle \\ \langle a_1^\dagger a_1^\dagger \rangle & \langle a_1^\dagger a_1 \rangle + 1/2 & \langle a_1^\dagger a_2^\dagger \rangle & \langle a_1^\dagger a_2 \rangle \\ \langle a_1^\dagger a_2 \rangle & \langle a_1 a_2 \rangle & \langle a_2^\dagger a_2 \rangle + 1/2 & \langle a_2 a_2 \rangle \\ \langle a_1^\dagger a_2^\dagger \rangle & \langle a_1 a_2^\dagger \rangle & \langle a_2^\dagger a_2^\dagger \rangle & \langle a_2^\dagger a_2 \rangle + 1/2 \end{pmatrix}, \tag{10}$$

$$C = \begin{pmatrix} \langle a_1^\dagger a_1 \rangle & \langle a_1 a_2^\dagger \rangle \\ \langle a_2 a_1^\dagger \rangle & \langle a_2^\dagger a_2 \rangle \end{pmatrix}, \qquad S = \begin{pmatrix} \langle a_1 a_1 \rangle & \langle a_1 a_2 \rangle \\ \langle a_2 a_1 \rangle & \langle a_2 a_2 \rangle \end{pmatrix}. \tag{11}$$

Time evolution of a linear quantum lattice:



$$\mathscr{H} = \omega \sum_{i=1}^{L} a_i^\dagger a_i + i\lambda \sum_{i=1}^{L-1} (a_i^\dagger a_{i+1} - a_{i+1}^\dagger a_i), \tag{12}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[\mathscr{H}, \rho] + \sum_{k=1}^{L} \mathscr{D}_k(\rho), \tag{13}$$

$$\mathscr{D}_k(\rho) = \gamma_i(N_i + 1)\left[a_i\rho a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i, \rho\}\right] + \gamma_i N_i\left[a_i^\dagger \rho a_i - \frac{1}{2}\{a_i a_i^\dagger, \rho\}\right]$$
$$- \gamma_i M_i\left[a_i^\dagger \rho a_i^\dagger - \frac{1}{2}\{a_i^\dagger a_i^\dagger, \rho\}\right] - \gamma_i M_i^*\left[a_i\rho a_i - \frac{1}{2}\{a_i a_i, \rho\}\right]. \tag{14}$$

A Lyapunov equation for $\Theta(t)$:

$$\frac{\mathrm{d}\mathbf{\Theta}}{\mathrm{d}t} = \mathbf{V}\mathbf{\Theta} + \mathbf{\Theta}\mathbf{V}^\dagger + \mathbf{G}, \tag{15}$$

where

$$\mathbf{V} = \mathbf{W} \otimes \mathbf{I}_2, \qquad W_{i,j} = -\frac{\gamma_i}{2}\delta_{i,j} + \lambda(\delta_{i+1,j} - \delta_{i,j+1}), \tag{16}$$

$$\mathbf{G} = \text{diag}(\mathbf{P}_1, \ldots, \mathbf{P}_L), \qquad \mathbf{P}_i = \gamma_i \begin{pmatrix} N_i + 1/2 & M_i \\ M_i & N_i + 1/2 \end{pmatrix}. \tag{17}$$

Due to our parameters' choice:

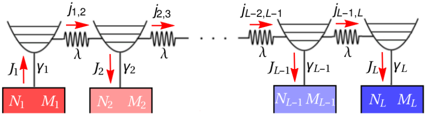
$$\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t} = \mathbf{W}\mathbf{C} + \mathbf{C}\mathbf{W}^\dagger + \mathbf{F}_N, \tag{18}$$

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t} = \mathbf{W}\mathbf{S} + \mathbf{S}\mathbf{W}^\dagger + \mathbf{F}_M, \tag{19}$$

$$\mathbf{F}_N = \text{diag}(\gamma_1 N_1, \ldots, \gamma_L N_L). \qquad \mathbf{F}_M = \text{diag}(\gamma_1 M_1, \ldots, \gamma_L M_L).$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \sum_{k=1}^{L} a_k^\dagger a_k \right\rangle = \sum_{k=1}^{L} \operatorname{tr}\left\{ a_k^\dagger a_k \frac{\mathrm{d}\rho}{\mathrm{d}t} \right\} = -i\sum_{k=1}^{L} \operatorname{tr}\left\{ a_k^\dagger a_k [\mathscr{H},\rho] \right\} + \sum_{k,i=1}^{L} \operatorname{tr}\left\{ a_k^\dagger a_k \mathscr{D}_i(\rho) \right\}$$
$$= \sum_{k=2}^{L-1} j_{k-1,k} - j_{k,k+1} + \sum_{k=1}^{L} J_k. \qquad (20)$$

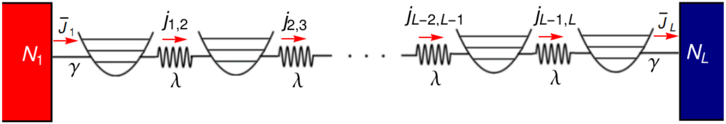
where $j_{k,k+1} = \lambda\langle a_k^\dagger a_{k+1} + a_{k+1}^\dagger a_k\rangle$ and $J_k = \gamma_k \left(N_k - \langle a_k^\dagger a_k\rangle \right)$.[1]



Repeated interaction model ⇒ Heat current = Quasi-particles current.[2]

[1] *arXiv:0904.3527 and arXiv:1204.0904.*
[2] *arXiv 1808.10450.*

Transport in linear quantum chains

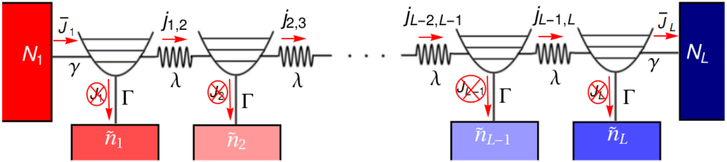


Diffusive transport: dephasing baths[3].

$$\mathcal{K}(\rho) = \frac{\Gamma}{2} \sum_{i=1}^{L} \left[a_i^\dagger a_i \rho a_i^\dagger a_i - \frac{1}{2} \{ (a_i^\dagger a_i)^2, \rho \} \right]. \tag{21}$$

[3] A. Asadian, D. Manzano, M. Tiersch, and H. J. Briegel. Heat transport through lattices of quantum harmonic oscillators in arbitrary dimensions. Phys. Rev. E, 87(1):012109. arXiv:1204.0904.

Self-Consistent reservoirs[4]



$$\tilde{n}_k = \langle a_k^\dagger a_k \rangle \qquad \Rightarrow \qquad J_k = \gamma_k (N_k - \langle a_k^\dagger a_k \rangle) = 0. \tag{22}$$

$$\bar{J}_1 = j_1 = \cdots = j_L = \bar{J}_L \equiv \mathscr{J}. \tag{23}$$

[4] *M. Bolsterli, M. Rich, and W. M. Visscher. Simulation of Nonharmonic Interactions in a Crystal by Self-Consistent Reservoirs. Phys. Rev. A, 1(4):1086–1088, 1970.*

Solution in the (NESS):

$$\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} = \boldsymbol{W}\boldsymbol{C} + \boldsymbol{C}\boldsymbol{W}^\dagger + \boldsymbol{F}_N \overset{\text{NESS}}{=} 0.$$

$$
\boldsymbol{C} = \begin{pmatrix}
A_1 & x & 0 & 0 & 0 & \dots & 0 \\
x & A_2 & x & 0 & 0 & \dots & 0 \\
0 & x & A_3 & x & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \dots & 0 & x & A_{L-2} & x & 0 \\
0 & \dots & 0 & 0 & x & A_{L-1} & x \\
0 & \dots & 0 & 0 & 0 & x & A_L
\end{pmatrix}
$$

Solution in the (NESS):

$$
\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} = \boldsymbol{W}\boldsymbol{C} + \boldsymbol{C}\boldsymbol{W}^\dagger + \boldsymbol{F}_N \overset{\text{NESS}}{=} 0.
$$

$$
\begin{aligned}
x &= \langle a_k^\dagger a_{k+1} \rangle \\
&= \frac{\gamma\lambda}{4\lambda^2 + \gamma^2 + \gamma\Gamma(L-1)}(N_L - N_1),
\end{aligned}
$$

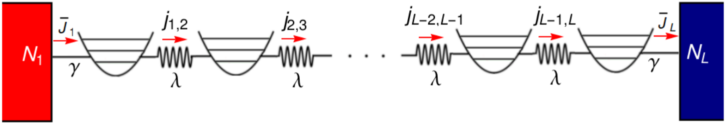
$$
A_k = \langle a_k^\dagger a_k \rangle = \frac{N_1 + N_L}{2} + \frac{1}{2}\frac{\gamma(N_1 - N_L)}{4\lambda^2 + \gamma^2 + \gamma\Gamma(L-1)}\left[\Gamma(L - 2k + 1) + \gamma(\delta_{k,1} - \delta_{k,L})\right].
$$

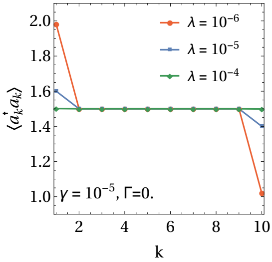
The Current:

$$
\mathscr{J} = 2\lambda x = \frac{2\gamma\lambda^2}{4\lambda^2 + \gamma^2 + \gamma\Gamma(L-1)}(N_L - N_1).
$$

Ballistic transport

$\Gamma \to 0$:

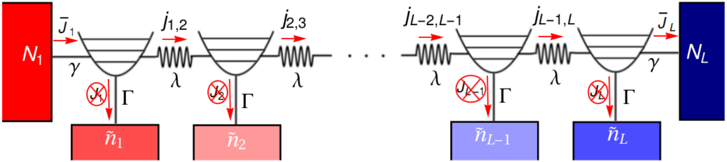


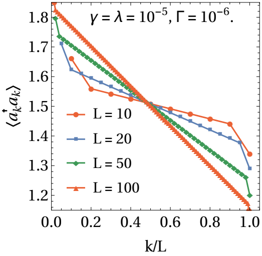


$$A_k = \langle a_k^\dagger a_k \rangle = \frac{N_1 + N_L}{2} + \frac{1}{2}\frac{\gamma^2 (N_1 - N_L)}{4\lambda^2 + \gamma^2}(\delta_{k,1} - \delta_{k,L}),$$

$$\mathscr{J} = \frac{2\gamma\lambda^2 \, (N_L - N_1)}{4\lambda^2 + \gamma^2}. \qquad (24)$$

Diffusive transport ($\Gamma \neq 0$)





Thermodynamic limit:

$$\langle a_k^\dagger a_k \rangle \to \frac{(L-k)N_1 + (k-1)N_L}{L-1}, \quad (25)$$

$$\mathscr{J} \to \frac{2\lambda^2}{\Gamma} \frac{(N_L - N_1)}{L-1}. \quad (26)$$

The entropy production/flux rate

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = \Pi(t) - \Phi(t), \tag{27}$$

$\Pi(t) \geq 0$ and $\Pi(t) = 0$ iff the system is in equilibrium.

NESS:

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = 0 \qquad \Rightarrow \qquad \Pi(t) = \Phi(t) > 0. \tag{28}$$

Von Neumann formulation:

$$S_{vN} = -\operatorname{tr}[\rho \ln \rho]. \tag{29}$$

$$\Pi_{vN} = \frac{\partial}{\partial t} K_{vN}(\rho||\rho^*), \qquad K_{vN}(\rho||\rho^*) = \operatorname{tr}[\rho \ln \rho - \rho \ln \rho^*]. \tag{30}$$

$$\Rightarrow \Phi_{vN} = -\frac{1}{T} \operatorname{tr}[\mathscr{H}\mathscr{D}(\rho)] := \frac{\Phi_E}{T}. \tag{31}$$

The divergence at $T = 0$ (when the target state is pure) is not physical. As an example, we can mention experiments in quantum optics where the limit $T \to 0$ is taken and the dynamics is well behaved.

Phase space: the Wigner function[5]

$$\mathcal{W}(\boldsymbol{\xi}) = \frac{1}{\pi^L \sqrt{|\boldsymbol{\Theta}|}} \exp\left\{-\frac{1}{2}(\boldsymbol{\xi}-\boldsymbol{\mu})^\dagger \boldsymbol{\Theta}^{-1}(\boldsymbol{\xi}-\boldsymbol{\mu})\right\}, \quad \boldsymbol{\xi} = (\alpha_1, \alpha_1^*, \ldots, \alpha_L, \alpha_L^*)^\top, \tag{32}$$

$$\boldsymbol{\mu} = \langle \boldsymbol{R} \rangle.$$

Quadratic Hamiltonian:

$$\mathscr{H} := \sum_{k,\ell=1}^{L} H_{k,\ell}\, a_k^\dagger a_\ell. \tag{33}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -i[\mathscr{H},\rho] + \sum_{k=1}^{L} \mathscr{D}_k(\rho) \quad \rightarrow \quad \frac{\partial \mathcal{W}}{\partial t} = \mathcal{U}(\mathcal{W}) + \sum_{k=1}^{L} \mathcal{D}_k(\mathcal{W}). \tag{34}$$

[5] *Jader P. Santos, Gabriel T. Landi, and Mauro Paternostro. Wigner Entropy Production Rate. Phys. Rev. Lett., 118(22):220601. arXiv: 1706.01145.*

Quantum Fokker-Planck: a continuity equation

$$\frac{\partial \mathcal{W}}{\partial t} = \mathcal{U}(\mathcal{W}) + \sum_{k=1}^{L} \mathcal{D}_k(\mathcal{W}), \tag{35}$$

where:

$$\mathcal{U}(\mathcal{W}) = \sum_{k=1}^{L} \left[\frac{\partial \mathcal{A}_k(\mathcal{W})}{\partial \alpha_k} + \frac{\partial \mathcal{A}_k^*(\mathcal{W})}{\partial \alpha_k^*} \right], \quad \mathcal{A}_k(\mathcal{W}) = i \sum_{\ell=1}^{L} H_{k\ell} \alpha_\ell \mathcal{W}. \tag{36}$$

$$\mathcal{D}_k(\mathcal{W}) = \frac{\partial\, \mathcal{J}_k(\mathcal{W})}{\partial \alpha_k} + \frac{\partial\, \mathcal{J}_k^*(\mathcal{W})}{\partial \alpha_k^*}, \quad \mathcal{J}_k(\mathcal{W}) = \frac{\gamma_k}{2} \Big(\alpha_k \mathcal{W} + (\bar{n}_k + 1/2) \frac{\partial \mathcal{W}}{\partial \alpha_k^*} \Big). \tag{37}$$

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An alternative way to quantify equilibrium:

$$\mathcal{J}_k(\mathcal{W}) = 0 \qquad \Leftrightarrow \qquad \mathcal{W} \equiv \mathcal{W}_{eq}. \tag{38}$$

Wigner entropy production of each dissipation channel

$$S(\mathcal{W}) = -\int d\boldsymbol{\xi}\, \mathcal{W}(\boldsymbol{\xi}) \ln \mathcal{W}(\boldsymbol{\xi}).^{6} \tag{39}$$

$$\frac{dS}{dt} = \Pi - \Phi = \sum_k (\Pi_k - \Phi_k), \tag{40}$$

$$\Pi_k = \frac{4}{\gamma_k(\bar{n}_k + 1/2)} \int \frac{|\mathcal{J}_k(\mathcal{W})|^2}{\mathcal{W}} d\boldsymbol{\xi} \quad \text{and} \quad \Phi_k = \int \left\{ \mathcal{J}_k \partial_k \ln \mathcal{W}_{\text{eq}} + \mathcal{J}_k^* \partial_{k^*} \ln \mathcal{W}_{\text{eq}} \right\} d\boldsymbol{\xi}.$$

[6] G. Adesso, D. Girolami and A. Serafini. Measuring gaussian quantum information and correlations using the Rényi entropy of order 2. Phys. Rev. Lett., 109(19):190502. arXiv:1203.5116.

[7] Tânia Tomé and Mário J. de Oliveira. Entropy production in irreversible systems described by a Fokker-Planck equation. Phys. Rev. E, 82(2):021120, aug 2010.

[8] Richard E. Spinney and Ian J. Ford. Entropy production in full phase space for continuous stochastic dynamics. Phys. Rev. E, 85(5):051113. arXiv:1203.0485.

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- $\Pi \geq 0$, with the equality holding only in the equilibrium.
- The parity of Π and Φ in the irreversible current[7].
- Within a stochastic trajectories framework this expression for the entropy production satisfy integral fluctuation theorems[8].

[6] *G. Adesso, D. Girolami and A. Serafini. Measuring gaussian quantum information and correlations using the Rényi entropy of order 2. Phys. Rev. Lett., 109(19):190502. arXiv:1203.5116.*

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Analysis of the results:

$$\Pi_k = \frac{4}{\gamma_k(\bar{n}_k + 1/2)} \int \frac{|\mathcal{J}_k(\mathcal{W})|^2}{\mathcal{W}} \, d\xi, \tag{41}$$

$$\Phi_k = \int \left\{ \mathcal{J}_k \partial_k \ln \mathcal{W}_{\text{eq}} + \mathcal{J}_k^* \partial_{k^*} \ln \mathcal{W}_{\text{eq}} \right\} d\xi. \tag{42}$$

– Identification of the individual contribution of each dissipation channel to the total entropy production rate and entropy flux.

[9] *Udo Seifert. Stochastic thermodynamics, fluctuation theorems and molecular machines. Reps. Prog. Phys., 75(12):126001. arXiv:1205.4176.*

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– Identification of the individual contribution of each dissipation channel to the total entropy production rate and entropy flux.

– The entropy production rate may be interpreted as an average of the phase space velocities[9]:

$$v_k := \frac{\mathcal{J}_k(\mathcal{W})}{\mathcal{W}} \qquad \Rightarrow \qquad \Pi_k = \frac{4}{\gamma_k(\bar{n}_k + 1/2)} \int v_k^2 \mathcal{W} \, d\boldsymbol{\xi}. \tag{43}$$

[9] *Udo Seifert. Stochastic thermodynamics, fluctuation theorems and molecular machines. Reps. Prog. Phys., 75(12):126001. arXiv:1205.4176.*

Relation with the Covariance Matrix:

$$\Pi_k = \Phi_k - \gamma_k + \gamma_k(\bar{n}_k + 1/2)(\Theta^{-1})_{2k,2k}. \tag{44}$$

$$\Phi_k = \frac{\gamma_k}{\bar{n}_k + 1/2}\left(\langle a_k^\dagger a_k\rangle - \bar{n}_k\right), \tag{45}$$

If $\langle a_k^\dagger a_k\rangle > \bar{n}_k \quad \Rightarrow \quad \Phi_k > 0$ and entropy flows from the system to the environment.
If $\langle a_k^\dagger a_k\rangle < \bar{n}_k \quad \Rightarrow \quad \Phi_k < 0$ and entropy flows from the environment to the system.
Thermodynamic limit: $T_k >> \omega_k$

$$\Phi_k = \frac{\Phi_E^k}{\omega_k(\bar{n}_k + 1/2)} \quad \rightarrow \quad \frac{\Phi_E^k}{T_k}. \tag{46}$$

The role of unitary dynamics

Using the definition of the Wigner relative entropy

$$K(\mathcal{W}||\mathcal{W}_{\text{eq}}) := \int \mathcal{W} \ln\left(\mathcal{W}/\mathcal{W}_{\text{eq}}\right) \, d\xi. \tag{47}$$

we obtain that

$$\boxed{\Pi = -\frac{dK(\mathcal{W}||\mathcal{W}_{\text{eq}})}{dt} - \int \mathcal{U}(\mathcal{W}) \ln \mathcal{W}_{\text{eq}} \, d\xi = \Pi_{\text{trans}} + \Pi_{\text{NESS}}.} \tag{48}$$

Recovering Onsager's theory of irreversible thermodynamics

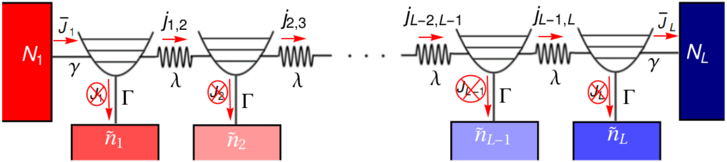
We can write the Π_{NESS} in terms of the currents:

$$\Pi_{\text{NESS}} = \frac{1}{2} \sum_{k,\ell} j_{k,\ell} \left(\frac{1}{\bar{n}_k + 1/2} - \frac{1}{\bar{n}_\ell + 1/2} \right). \tag{49}$$

- This result holds in the limit of $T \to 0$.
- For high temperatures $(\bar{n}_k + 1/2 \to T_k)$ we recover Onsager's result:

$$\Pi = j_{AB} \left(\frac{1}{T_A} - \frac{1}{T_B} \right). \tag{50}$$

Total entropy production



From Eq. (49), we obtain that

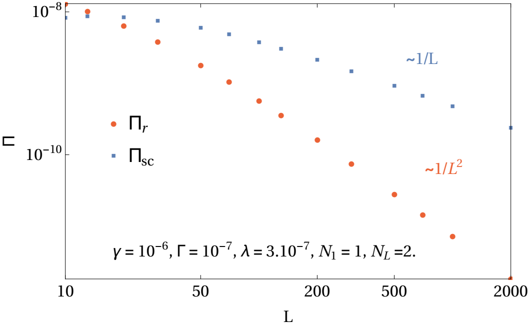
$$\Pi_{\text{NESS}} = \frac{2\lambda^2\gamma(N_L - N_1)}{4\lambda^2 + \gamma^2 + \gamma\Gamma(L-1)} \left(\frac{1}{N_1 + 1/2} - \frac{1}{N_L + 1/2} \right). \tag{51}$$

– Always non-negative and zero if and only if $N_1 = N_L$.

– Scaling with L like the current in the ballistic (diffusive) regime.

Entropy production rate from the physical and the self-consistent reservoirs[10]

$$\boxed{\Pi = \Pi_r + \Pi_{sc}} \tag{52}$$



[10] *William T. B. Malouf, Jader P. Santos, Luis A. Correa, Mauro Paternostro, and Gabriel T. Landi. Wigner entropy production and heat transport in linear quantum lattices. Phys. Rev. A, 99(5):052104, arXiv:1901.03127.*

Motivation

How to quantify the correlation between different parts of a chain in a NESS?

Two main assumptions:

– The reservoirs act only locally on the end points.

– The interaction Hamiltonian is short ranged.

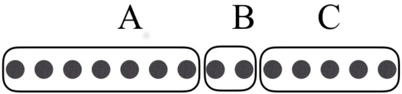
MI vs. CMI

Mutual Information (MI):

$$S(\rho) = -\mathrm{tr}(\rho \ln \rho). \tag{53}$$

$$\mathscr{I}(A:C) = S(\rho_A) + S(\rho_C) - S(\rho_{AC}) \geq 0, \tag{54}$$

Conditional Mutual Information (CMI):



$$\begin{aligned} \mathscr{I}(A:C|B) &:= S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B) \\ &= \mathscr{I}(AB:C) - \mathscr{I}(B:C) \geq 0.^{11} \end{aligned} \tag{55}$$

[11] *Consequence of the strong-subadditivity of the entropy.*

Markovian chains

- Discrete time chain: $X \to Y \to Z$.
- Markovian process: The past and the future are independents when we have knowledge of the present:

$$P(X,Z|Y) = P(X|Y)P(Z|Y) \tag{56}$$

- Thus, we can redefine markovianity as a process that the variable Z and X are **conditionally** independent.
- But, it does not mean that the variables X and Z are **unconditionally** independent:

$$P(X,Z) = \sum_Y P(X|Y)P(Z|Y)P(Y) \neq P(X)P(Z). \tag{57}$$

- Then, in general, $\mathscr{I}(X:Z) \neq 0$! This correlation just appears due the lack of information about Y and doesn't reflect the ability of X and Z to share information.

– Conversely, $\mathscr{I}(X:Z|Y) \equiv 0$ for Markovian systems[12].

[12] *Kohtaro Kato and Fernando G. S. L. Brandao. Quantum Approximative Markov Chains are Thermal. arXiv:1609.06636.*

[13] The dependence of the CMI on the size of the middle partition directly quantifies the degree of non-Markovianity:*M. Papapetrou and D. Kugiumtzis. Markov chain order estimation with conditional mutual information. Phys. A Stat. Mech. its Appl., 392(7):1593-1601, arXiv:1301.0148.*

- Conversely, $\mathscr{I}(X:Z|Y) \equiv 0$ for Markovian systems[12].
- Replacing the concept of time by the site index $i = 1, \ldots, L$.
- **We conclude that the shared information between two disconnected parts of the chain is better quantified by the CMI rather then the MI:**
  - The dependence of the CMI on the size of B quantifies the robustness of the chain in sharing information.[13]

[12] *Kohtaro Kato and Fernando G. S. L. Brandao. Quantum Approximative Markov Chains are Thermal. arXiv:1609.06636.*

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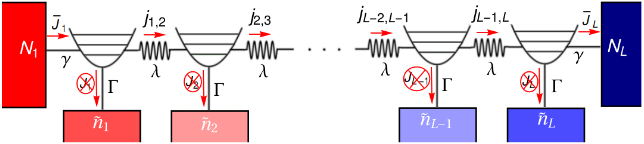
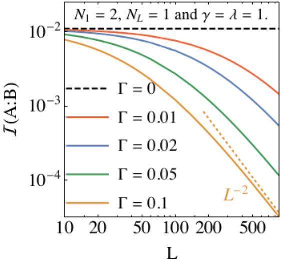
Application



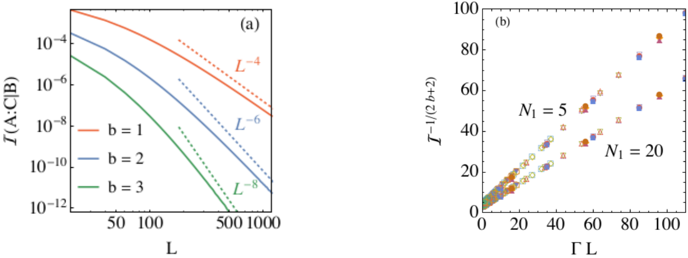
Mutual Information of symmetric bipartitions:



- Size independent in the ballistic regime.
- Decays with L^{-2} in the diffusive regime.

Conditional Mutual Information

Considering A and C symmetric and b = length(B):



In the figure (a) it was fixed $N_1 = 15$, $N_L = 1$, $\Gamma = 0.1$ and $\lambda = \gamma = 1$. Already in the figure (b) it was used multiples values of Γ, L and b with $N_L = \lambda = \gamma = 1$ fixed.

$$\boxed{\mathcal{I}(A:C|B) = \frac{u}{(v+\Gamma L)^{2b+2}}},$$

where u and v are constants.

Local equilibration[14]

Let us consider a tripartition:
$A = \{1, \dots, k-1\},\ B = \{k\}$ and $C = \{k+1, \dots, L\}$. If

$$\mathscr{I}(A : C|B) < \epsilon \tag{58}$$

then there exists a *local* Hamiltonian $H = \sum_i h_{i,i+1}$, $[h_{i-1,i}, h_{i,i+1}] = 0$, acting only on sites $i, i+1$, such that

$$K_{vN}\left(\rho_{\mathscr{S}} \middle\| \frac{e^{-H}}{\operatorname{tr} e^{-H}}\right) < \epsilon L. \tag{59}$$

[14] *Kohtaro Kato and Fernando G. S. L. Brandao. Quantum Approximative Markov Chains are Thermal. arXiv:1609.06636.*

[15] *William T. B. Malouf, John Goold, Gerardo Adesso, and Gabriel T. Landi. Quantifying shared information in quantum non-equilibrium steady-states. arXiv:1809.09931v2.*

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Applying in our model $(b = 1)$:

Ballistic $\rightarrow \mathscr{I}(A:C|B) \sim L^0$
–Far from local equilibrium.

Diffusive $\rightarrow \mathscr{I}(A:C|B) \sim L^{-4}$
– Local thermal equilibrium.[15]
– Like the temperature profile in a bar.

[14] *Kohtaro Kato and Fernando G. S. L. Brandao. Quantum Approximative Markov Chains are Thermal. arXiv:1609.06636.*

[15] *William T. B. Malouf, John Goold, Gerardo Adesso, and Gabriel T. Landi. Quantifying shared information in quantum non-equilibrium steady-states. arXiv:1809.09931v2.*

Conclusions

Thermodynamics[16]:

- Closed-form expressions for Π_k and Φ_k.
- Unitary contribution for the irreversibility in the NESS.
- Solved a transport model (Ballistic-Diffusive).
- Calculated the entropic cost of diffusivity.

Information[17]:

- CMI vs MI.
- Study of correlations in the ballistic-diffusive model.
- Local thermalization.

[16] *William T. B. Malouf, Jader P. Santos, Luis A. Correa, Mauro Paternostro, and Gabriel T. Landi. Wigner entropy production and heat transport in linear quantum lattices. Phys. Rev. A, 99(5):052104, arXiv:1901.03127.*

[17] *William T. B. Malouf, John Goold, Gerardo Adesso, and Gabriel T. Landi. Quantifying shared information in quantum non-equilibrium steady-states. arXiv:1809.09931v2.*

Some questions to be answered...

- What happens with Π_k, Φ_k and Π_{NESS} if one consider a GME rather then a LME?
- Working with the Rényi-2 entropy rather than von-Neumman's, one can obtain an analytical expression for the CMI.
- How these results would be affected by anomalous diffusion (that is, in which $J \sim 1/L^{\alpha}$ for some exponent α)?
- What would be, in this case, the critical value of α for which local equilibration breaks down?

Thank you!