Master equation: phenomenological vs microscopical approach

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1 Introduction

The dynamics of open quantum systems have been studied extensively in the fields of quantum optics, quantum information. The main goal is to describe the nonunitary behavior resulting from the fact that the system is not closed. Usually, such master equations are obtained by considering a microscopic model for the interaction of the system under study and the environment, and tracing out the environment variables in some exact or, most of the times, perturbative treatment. Usually, the presence of interactions among parts of the system or its subsystems is not taken into account in the derivation of the master equation. It is not uncommon to find examples of such a phenomenological approach where one considers the response of a system to the environment to be exactly the same regardless of whether it is coupled or not to another quantum system. As an illustrative example, let us consider the Jaynes-Cummings model describing the interaction of a single-mode quantized electromagnetic field in a cavity and a two-level atom in the rotating wave approximation. When considering cavity losses at a decay rate κ, one usually describes the open system by using a master equation in the form[1]

$$\partial_t \rho = -i[H_{JC}, \rho] + \kappa(a\rho a^\dagger - \{a^\dagger a, \rho\}/2) \tag{1}$$

As we know, the second term, the Lindblad dissipator, accounts for the losses. The potential problem of using Eq. (1) resides in the fact that the derivation of the Lindblad dissipator was realized in another different microscopic model, and not the Jaynes-Cummings plus environment. Actually, this dissipator is deduced for a cavity mode losing photons to the vacuum environment without the presence of the atom. For this reason, the use of Eq. (1) is a phenomenological approach. A microscopic derivation of the master equation for the Jaynes-Cummings model with cavity losses is found in [1].

Usually, this kind of phenomenological approach works well when the subsystems (in the above case, the mode and the atom) are weakly coupled.

1.1 Master equation for qubits in a common environment

In this section, we would like to discuss about the following system: an a priori non-interacting set of qubits which are interacting with a global bosonic reservoir. This system is been shown in Fig. 1(a).

As we have seen before, when we have just one qubit interacting with a bosonic reservoir, the master equation for the system can be written as [2]

$$\partial_t \rho = -i[H_1, \rho] + \mathcal{D}_1(\rho) \tag{2}$$

with $H_1 = \frac{\omega_1}{2}\sigma_1^z$ and

$$\mathcal{D}_1(\rho) = \gamma(1+\bar{n})(\sigma_1^- \rho \sigma_1^+ - \{\sigma_1^+\sigma_1^-, \rho\}/2) + \gamma\bar{n}(\sigma_1^+ \rho \sigma_1^- - \{\sigma_1^-\sigma_1^+, \rho\}/2) \tag{3}$$

A ingenuous way to describe a set of M two-level system interacting with a common global reservoir could be just by adding a new dissipator for each two-level system. In this case we would have,

$$\partial_t \rho = -i[H_S, \rho] + \sum_{i=1}^{M} \mathcal{D}_i(\rho) \tag{4}$$

where $H_S = \sum_{i=1}^M \omega_i \sigma_i^z/2$ and

$$\mathcal{D}_i(\rho) = \gamma(1+\bar{n})(\sigma_i^- \rho \sigma_i^+ - \{\sigma_i^+\sigma_i^-, \rho\}/2) + \gamma\bar{n}(\sigma_i^+ \rho \sigma_i^- - \{\sigma_i^-\sigma_i^+, \rho\}/2) \tag{5}$$

The Eq. (4) is an example of a phenomenological master equation for our system. This master equation would be correct if we were describe a collection of two-level system interacting individually with local baths, as shown in Fig. 1(b), but that is not the case. Next section we will derive what we call a microscopic master equation for the system of interest.

[1]Here H_{JC} stands for the well-know Jaynes-Cummings Hamiltonian $H_{JC} = \frac{\omega_0}{2}\sigma^z + \omega a^\dagger a + g(a^\dagger \sigma^- + \sigma^+ a)$.
[2]We mean, by using the traditional approach via the Born-Markov approximations.

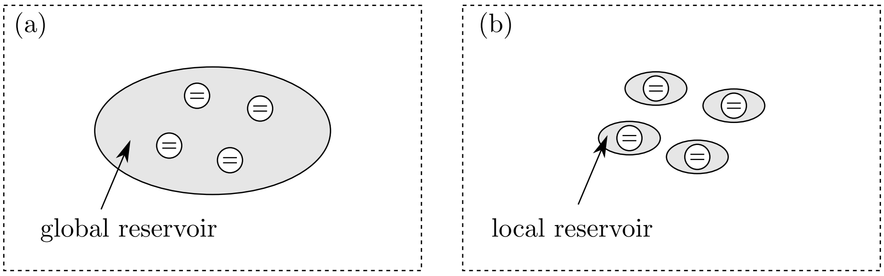

Figure 1: (a) Set of two-level system interacting with a global reservoir. (b) Set of two-level system interacting individually with local reservoirs.

Derivation of the master equation

Let us start by writing the system Hamiltonian of the total system

$$H_T = H_S + H_E + H_{SE} \tag{6}$$

where

$$H_S = \sum_{i=1}^{M} \omega_i \sigma_i^z/2 \tag{7}$$

$$H_E = \sum_k \nu_k a_k^\dagger a_k \tag{8}$$

$$H_{SE} = -i \sum_k \sum_{i=1}^{M} (g_{ik}\sigma_i^+ a_k - g_{ik}^* \sigma_i^- a_k^\dagger) \tag{9}$$

The density operator of the total system, ρ_T, obeys the von-Neumann equation,

$$\partial_t \rho_T = -i[H_T, \rho_T] \tag{10}$$

First, let us go to the interaction picture, $\tilde{\rho}_T = e^{iH_0 t}\rho_T e^{-iH_0 t}$, with $H_0 = H_S + H_E$. So we find

$$\partial_t \tilde{\rho}_T = -i[V(t), \tilde{\rho}_T] \tag{11}$$

where $V(t) = e^{iH_0 t} H_{SE} e^{-iH_0 t}$, can be rewritten as

$$V(t) = -i \sum_k \sum_{i=1}^{M} (g_{ik}\sigma_i^+ a_k e^{i\Delta_{ik}t} - g_{ik}^* \sigma_i^- a_k^\dagger e^{-i\Delta_{ik}t}) \tag{12}$$

where $\Delta_{ik} = \omega_i - \nu_k$. For the initial time $t = 0$, direct integration of Eq. (11) leads to the following first order solution in $V(t)$

$$\tilde{\rho}_T(t) = \tilde{\rho}_T(0) - i\int_0^t \mathrm{d}t' \, [V(t'), \tilde{\rho}_T(t')] \tag{13}$$

Substituting Eq. (13) back into the right side of Eq. (11), we obtain the integro-differential equation

$$\partial_t \tilde{\rho}_T = -i[V(t), \tilde{\rho}_T(0)] - \int_0^t \mathrm{d}t' [V(t), [V(t'), \tilde{\rho}_T(t')]] \tag{14}$$

We can continue the procedure to obtain an infinite series of integral terms, which can be regarded as an exact explicit solution for $\tilde{\rho}_T(t)$. The usual practice however is to introduce approximations into the exact second order equation. By tracing Eq.(14) over the bath, and noting that $\mathrm{Tr}_E\{\tilde{\rho}_T(t)\} = \tilde{\rho}_S(t)$, we obtain

$$\partial_t \tilde{\rho}_S = -i \,\mathrm{Tr}_E \left\{[V(t), \tilde{\rho}_T(0)]\right\} - \int_0^t \mathrm{d}t' \,\mathrm{Tr}_E \left\{[V(t), [V(t'), \tilde{\rho}_T(t')]]\right\} \tag{15}$$

We choose an initial state of the system that has no correlations between the two-level systems and the environment, $\tilde{\rho}_T(0) = \tilde{\rho}_S(0) \otimes \rho_E(0)$. We will assume the interaction between the two-level system and bath to be very weak, and then there is no back reaction of the two-level system on the bath. Thus, we can make our first approximation, the weak coupling or Born approximation, in which we assume that the effect of the system on the bath is very small, so that the state of the bath, appearing as a large reservoir to the two-level systems, does not change in time. Therefore, we can rewrite Eq.(15)

$$\partial_t\tilde{\rho}_S = -i\,\mathrm{Tr}_E\left\{[V(t),\tilde{\rho}_S(0)\otimes\rho_E(0)]\right\} - \int_0^t \mathrm{d}t'\,\mathrm{Tr}_E\left\{[V(t),[V(t'),\tilde{\rho}_S(t')\otimes\rho_E]\right\} \tag{16}$$

Where here we are using a shorter notation $\rho_E = \rho_E(0)$. If we change the time variable to $t' = t-\tau$, we obtain

$$\partial_t\tilde{\rho}_S = -i\,\mathrm{Tr}_E\left\{[V(t),\tilde{\rho}_S(0)\otimes\rho_E(0)]\right\} - \int_0^t \mathrm{d}\tau\,\mathrm{Tr}_E\left\{[V(t),[V(t-\tau),\tilde{\rho}_S(t-\tau)\otimes\rho_E]\right\} \tag{17}$$

Let us consider the trace over the field modes of the double commutator in the second term of Eq. (17), which can be written as

$$\begin{aligned}\mathrm{Tr}_E\left\{[V(t),[V(t-\tau),\tilde{\rho}_S(t-\tau)\otimes\rho_E]\right\} &= \mathrm{Tr}_E\left\{V(t)V(t-\tau)\tilde{\rho}_S(t-\tau)\otimes\rho_E]\right\}\\ &- \mathrm{Tr}_E\left\{V(t)\tilde{\rho}_S(t-\tau)\otimes\rho_E V(t-\tau)]\right\}\\ &- \mathrm{Tr}_E\left\{V(t-\tau)\tilde{\rho}_S(t-\tau)\otimes\rho_E V(t)]\right\}\\ &+ \mathrm{Tr}_E\left\{\tilde{\rho}_S(t-\tau)\otimes\rho_E V(t-\tau)V(t)]\right\}\end{aligned} \tag{18}$$

To calculate the first term on the right-hand side of Eq. (18), we substitute the explicit expression for the interaction Hamiltonian Eq. (12), and obtain

$$\begin{aligned}&\mathrm{Tr}_E\left\{V(t)V(t-\tau)\tilde{\rho}_S(t-\tau)\otimes\rho_E]\right\} = -\sum_{k,k'}\sum_{i,j=1}^{M}\mathrm{Tr}_E\Big\{\\ &(g_{ik}\sigma_i^+a_k e^{i\Delta_{ik}t} - g_{ik}^*\sigma_i^- a_k^\dagger e^{-i\Delta_{ik}t})(g_{jk'}\sigma_j^+a_{k'}e^{i\Delta_{jk'}(t-\tau)} - g_{jk'}^*\sigma_j^- a_{k'}^\dagger e^{-i\Delta_{jk'}(t-\tau)})\tilde{\rho}_S(t-\tau)\otimes\rho_E\Big\}\end{aligned} \tag{19}$$

Let us define $a_k(t) = a_k e^{-i\nu_k t}$. Then we can write

$$\begin{aligned}\mathrm{Tr}_E\left\{V(t)V(t-\tau)\tilde{\rho}_S(t-\tau)\otimes\rho_E]\right\} = -\sum_{k,k'}\sum_{i,j=1}^{M}\Big[&\\ +g_{ik}g_{jk'}\sigma_i^+\sigma_j^+ e^{+i\omega_i t+i\omega_j(t-\tau)}\,&\mathrm{Tr}_E\left\{a_k(t)a_{k'}(t-\tau)\rho_E\right\}\\ -g_{ik}^*g_{jk'}\sigma_i^-\sigma_j^+ e^{-i\omega_i t+i\omega_j(t-\tau)}\,&\mathrm{Tr}_E\left\{a_k^\dagger(t)a_{k'}(t-\tau)\rho_E\right\}\\ -g_{ik}g_{jk'}^*\sigma_i^+\sigma_j^- e^{+i\omega_i t-i\omega_j(t-\tau)}\,&\mathrm{Tr}_E\left\{a_k(t)a_{k'}^\dagger(t-\tau)\rho_E\right\}\\ +g_{ik}^*g_{jk'}^*\sigma_i^-\sigma_j^- e^{-i\omega_i t-i\omega_j(t-\tau)})\,&\mathrm{Tr}_E\left\{a_k^\dagger(t)a_{k'}^\dagger(t-\tau)\rho_E\right\}\Big]\tilde{\rho}_S(t-\tau)\end{aligned} \tag{20}$$

In order to calculate the second order correlations functions of the field operator we assume that are in a finite temperature thermal state. In this case, the field correlation functions are given by

$$\mathrm{Tr}_E\left\{a_k(t)\rho_E\right\} = 0 \tag{21}$$

$$\mathrm{Tr}_E\left\{a_k(t)a_{k'}(t-\tau)\rho_E\right\} = 0 \tag{22}$$

$$\mathrm{Tr}_E\left\{a_k^\dagger(t)a_{k'}(t-\tau)\rho_E\right\} = \bar{n}_k\delta_{k,k'}e^{+i\nu_k t}e^{-i\nu_{k'}(t-\tau)} \tag{23}$$

$$\mathrm{Tr}_E\left\{a_k(t)a_{k'}^\dagger(t-\tau)\rho_E\right\} = [\bar{n}_k+1]\delta_{k,k'}e^{-i\nu_k t}e^{i\nu_{k'}(t-\tau)} \tag{24}$$

$$\mathrm{Tr}_E\left\{a_k^\dagger(t)a_{k'}^\dagger(t-\tau)\rho_E\right\} = 0 \tag{25}$$

where $\bar{n}_k = \mathrm{Tr}\{a_k^\dagger a_k\rho_E\}$ is the average number of quanta in the mode k. If we use the definition

$$D(t) = \sum_{i=1}^{M} g_{ik}\sigma_i^\dagger e^{i\omega_i t} \tag{26}$$

and the correlation functions, we can write

$$\mathrm{Tr}_E\left\{V(t)V(t-\tau)\tilde{\rho}_S(t-\tau)\otimes\rho_E]\right\} = \sum_k\left[D^\dagger(t)D(t-\tau)\bar{n}_k e^{+i\nu_k\tau} + D(t)D^\dagger(t-\tau)[\bar{n}_k+1]e^{-i\nu_k\tau}\right]\tilde{\rho}_S(t-\tau) \tag{27}$$

Proceeding in a similar manner, we calculate the remaining three terms in Eq. (18).

$$\mathrm{Tr}_E\left\{V(t)\tilde{\rho}_S(t-\tau)\otimes\rho_E V(t-\tau)]\right\} = \sum_k\left[D(t)\tilde{\rho}_S(t-\tau)D^\dagger(t-\tau)\bar{n}_k e^{-i\nu_k\tau} + D^\dagger(t)\tilde{\rho}_S(t-\tau)D(t-\tau)[\bar{n}_k+1]e^{i\nu_k\tau}\right] \quad (28)$$

$$\mathrm{Tr}_E\left\{V(t-\tau)\tilde{\rho}_S(t-\tau)\otimes\rho_E V(t)]\right\} = \sum_k\left[D(t-\tau)\tilde{\rho}_S(t-\tau)D^\dagger(t)\bar{n}_k e^{i\nu_k\tau} + D^\dagger(t-\tau)\tilde{\rho}_S(t-\tau)D(t)[\bar{n}_k+1]e^{-i\nu_k\tau}\right] \quad (29)$$

$$\mathrm{Tr}_E\left\{\tilde{\rho}_S(t-\tau)\otimes\rho_E V(t-\tau)V(t)]\right\} = \sum_k\tilde{\rho}_S(t-\tau)\left[D(t-\tau)D^\dagger(t)[\bar{n}_k+1]e^{i\nu_k\tau} + D^\dagger(t-\tau)D(t)\bar{n}_k e^{-i\nu_k\tau}\right] \quad (30)$$

By using these results, the master equation Eq. (17) simplifies to

$$\partial_t\tilde{\rho}_S = \int_0^t \mathrm{d}\tau\sum_k\Big\{[D(t-\tau)\tilde{\rho}_S(t-\tau), D^\dagger(t)]\bar{n}_k e^{i\nu_k\tau} + [D^\dagger(t), \tilde{\rho}_S(t-\tau)D(t-\tau)][\bar{n}_k+1]e^{i\nu_k\tau}$$
$$+[D(t), \tilde{\rho}_S(t-\tau)D^\dagger(t-\tau)]\bar{n}_k e^{-i\nu_k\tau} + [D^\dagger(t-\tau)\tilde{\rho}_S(t-\tau), D(t)][\bar{n}_k+1]e^{-i\nu_k\tau}\Big\} \quad (31)$$

Note also that we can write

$$\int_0^t \mathrm{d}\tau\sum_k[D^\dagger(t-\tau)\tilde{\rho}_S(t-\tau), D(t)][\bar{n}_k+1]e^{-i\nu_k\tau} =$$
$$\sum_{i,j=1}^M\left[\sigma_i^-\underbrace{\left(\sum_k e^{-i(\omega_i-\omega_j)t}[\bar{n}_k+1]g_{ik}^*g_{jk}\int_0^t \mathrm{d}\tau e^{-i(\nu_k-\omega_i)\tau}\tilde{\rho}_S(t-\tau)\right)}_{=X_{ij}(t)}, \sigma_j^+\right] \quad (32)$$

and

$$\int_0^t \mathrm{d}\tau\sum_k[D(t-\tau)\tilde{\rho}_S(t-\tau), D^\dagger(t)]\bar{n}_k e^{i\nu_k\tau} =$$
$$\sum_{i,j=1}^M\left[\sigma_i^+\underbrace{\left(\sum_k e^{i(\omega_i-\omega_j)t}\bar{n}_k g_{ik}g_{jk}^*\int_0^t \mathrm{d}\tau e^{i(\nu_k-\omega_i)\tau}\tilde{\rho}_S(t-\tau)\right)}_{=Y_{ij}(t)}, \sigma_j^-\right] \quad (33)$$

By using the definitions X_{ij} and Y_{ij}, we can rewrite the Eq. (31) as

$$\partial_t\tilde{\rho}_S = \sum_{i,j=1}^M\Big\{[\sigma_j^- X_{ij}(t), \sigma_i^+] + [\sigma_j^+ Y_{ij}(t), \sigma_i^-] + [\sigma_i^-, X_{ij}^\dagger(t)\sigma_j^+] + [\sigma_i^+, Y_{ij}^\dagger(t)\sigma_j^-]\Big\} \quad (34)$$

Continues Limit and Markov approximation

The functions $X_{ij}(t)$ and $Y_{ij}(t)$ involve a summation over the reservoir oscillators. We change this summation to an integration by introducing a density of states $G(\nu_k)$ such that $G(\nu_k)\,\mathrm{d}\nu_k$ gives the number of oscillators with frequency in the interval ν_k to $\nu_k + \mathrm{d}\nu_k$.

$$X_{ij}(t) = \int \mathrm{d}\nu_k\; G(\nu_k)e^{-i(\omega_i-\omega_j)t}[\bar{n}_k+1]g_{ik}^*g_{jk}\int_0^t \mathrm{d}\tau e^{-i(\nu_k-\omega_i)\tau}\tilde{\rho}_S(t-\tau) \quad (35)$$

$$Y_{ij}(t) = \int \mathrm{d}\nu_k\; G(\nu_k)e^{i(\omega_i-\omega_j)t}\bar{n}_k g_{ik}g_{jk}^*\int_0^t \mathrm{d}\tau e^{i(\nu_k-\omega_i)\tau}\tilde{\rho}_S(t-\tau) \quad (36)$$

Here we can make our third approximation, the Markov approximation, in which we replace $\tilde{\rho}_S(t-\tau)$ by $\tilde{\rho}_S(t)$ and extend the integral to infinity.

Under the Markov approximation we can evaluate the integral over τ to obtain

$$\lim_{t\to\infty}\int_0^t \mathrm{d}\tau\tilde{\rho}_S(t-\tau)e^{ix\tau} \simeq \tilde{\rho}_S(t)\left[\pi\delta(x) + i\mathcal{P}\frac{1}{x}\right] \quad (37)$$

Now we can write,

$$X_{ij}(t) = \int \mathrm{d}\nu_k\; g_{ik}^*g_{jk}G(\nu_k)e^{-i(\omega_i-\omega_j)t}[\bar{n}_k+1]\left[\pi\delta(\nu_k-\omega_i) - i\mathcal{P}\frac{1}{\nu_k-\omega_i}\right]\tilde{\rho}_S(t) \quad (38)$$

$$X_{ij}(t) = \tilde{\rho}_S(t)e^{-i(\omega_i-\omega_j)t}\left[\bar{n}+1\right]\left(\frac{\gamma_{ij}}{2} - i\Lambda_{ij}\right) \quad (39)$$

where we are using the definitions

$$\gamma_{ij} = 2\pi \int \mathrm{d}\nu_k \; g_{ik}^* g_{jk} G(\nu_k)\delta(\nu_k - \omega_i) \tag{40}$$

$$\Lambda_{ij} = \int \mathrm{d}\nu_k \; \frac{g_{ik}^* g_{jk} G(\nu_k)}{\nu_k - \omega_i} \tag{41}$$

And for Y_{ij} we have (again using the definitions Eq.(40)) and (41)),

$$Y_{ij}(t) = \int \mathrm{d}\omega \; G(\omega) e^{i(\omega_i - \omega_j)t} \bar{n}_k g_{ik} g_{jk}^* \left[\pi\delta(\nu_k - \omega_i) + i\mathcal{P}\frac{1}{\nu_k - \omega_i}\right]\tilde{\rho}_S(t) \tag{42}$$

$$Y_{ij}(t) = \tilde{\rho}_S(t) e^{i(\omega_i - \omega_j)t} \; \bar{n} \left(\frac{\gamma_{ij}}{2} + i\Lambda_{ij}\right) \tag{43}$$

Substituting the Eq. (39) and Eq. (43) in the master equation Eq. (34) we find

$$\begin{aligned}\partial_t \tilde{\rho}_S = \sum_{i,j=1}^{M} \Big\{ &\Big(\frac{\gamma_{ij}}{2} - i\Lambda_{ij}\Big)(\bar{n}+1)[\sigma_j^- \tilde{\rho}_S(t), \sigma_i^+] e^{-i(\omega_i-\omega_j)t} + \Big(\frac{\gamma_{ij}}{2} + i\Lambda_{ij}\Big)\bar{n}[\sigma_j^+ \tilde{\rho}_S(t), \sigma_i^-] e^{i(\omega_i-\omega_j)t} \\ &+ \Big(\frac{\gamma_{ij}^*}{2} + i\Lambda_{ij}^*\Big)(\bar{n}+1)[\sigma_i^-, \tilde{\rho}_S(t)\sigma_j^+] e^{+i(\omega_i-\omega_j)t} + \Big(\frac{\gamma_{ij}^*}{2} - i\Lambda_{ij}^*\Big)\bar{n}[\sigma_i^+, \tilde{\rho}_S(t)\sigma_j^-] e^{-i(\omega_i-\omega_j)t} \Big\}\end{aligned} \tag{44}$$

In order to simplify Eq. (44), we will assume that:

- the coupling g_{ik} in such way that γ_{ij} and Λ_{ij} are real,

$$\{\gamma_{ij}, \Lambda_{ij}\} \in \Re \tag{45}$$

- the frequency ω_i are such that $\omega_i - \omega_j \ll \omega_0$ where $\omega_0 = \sum_{i=1}^{M} \omega_i / M$. By using this, we can make the approximation,

$$\gamma_{ij} = \gamma_{ji} \quad \text{and} \quad \Lambda_{ij} = \Lambda_{ji} \tag{46}$$

By using this approximations, we can write

$$\begin{aligned}\partial_t \tilde{\rho}_S = &\sum_{i,j=1}^{M} i\Lambda_{ij} \Big\{(\bar{n}+1)[\sigma_i^+ \sigma_j^- \tilde{\rho}_S - \tilde{\rho}_S \sigma_i^+ \sigma_j^-] + \bar{n}[\tilde{\rho}_S \sigma_j^- \sigma_i^+ - \sigma_j^- \sigma_i^+ \tilde{\rho}_S]\Big\} e^{-i(\omega_i-\omega_j)t} \\ &+ \sum_{i,j=1}^{M} \gamma_{ij} \Big\{(\bar{n}+1)[\sigma_j^- \tilde{\rho}_S \sigma_i^+ - \{\sigma_i^+ \sigma_j^-, \tilde{\rho}_S\}/2] e^{-i(\omega_i-\omega_j)t} + \bar{n}[\sigma_j^+ \tilde{\rho}_S \sigma_i^- - \{\sigma_i^- \sigma_j^+, \tilde{\rho}_S\}/2] e^{+i(\omega_i-\omega_j)t}\Big\}\end{aligned} \tag{47}$$

Note that we can rewrite the first term of Eq. (49) as

$$\begin{aligned}\sum_{i,j=1}^{M} i\Lambda_{ij} \Big\{(\bar{n}+1)[\sigma_i^+ \sigma_j^- \tilde{\rho}_S - \tilde{\rho}_S \sigma_i^+ \sigma_j^-] + \bar{n}[\tilde{\rho}_S \sigma_j^- \sigma_i^+ - \sigma_j^- \sigma_i^+ \tilde{\rho}_S]\Big\} e^{-i(\omega_i-\omega_j)t} = &\sum_{\substack{i \neq j \\ i,j=1}}^{M} i\Lambda_{ij} [\sigma_i^+ \sigma_j^-, \tilde{\rho}_S] e^{-i(\omega_i-\omega_j)t} \\ &+ \sum_{i=1}^{M} i\Lambda_{ii} (\bar{n} + 1/2)[\sigma_i^z, \tilde{\rho}_S]\end{aligned} \tag{48}$$

By substituting Eq.(48) in Eq. (49) and going back to the Schröedinger picture we have,

$$\partial_t \rho_S = -i[H_{\text{eff}}, \rho_S] + \sum_{i,j=1}^{M} \gamma_{ij} (\bar{n}+1)[\sigma_j^- \rho_S \sigma_i^+ - \{\sigma_i^+ \sigma_j^-, \rho_S\}/2] + \bar{n}[\sigma_j^+ \rho_S \sigma_i^- - \{\sigma_i^- \sigma_j^+, \rho_S\}/2] \tag{49}$$

with

$$H_{\text{eff}} = \sum_{i=1}^{M} \frac{1}{2}(\omega_i - \Lambda_{ii}(2\bar{n}+1))\sigma_i^z - \sum_{\substack{i>j \\ i,j=1}}^{M} \Lambda_{ij} (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+) \tag{50}$$

References

[1] M. Scala, B. Militello, A. Messina, J. Piilo, and S. Maniscalco, "Microscopic derivation of the jaynes-cummings model with cavity losses," *Phys. Rev. A*, vol. 75, p. 013811, Jan 2007.