

Introduction to the course

this course has two main goals:

Goals

- 1) To understand the interaction among the constituents of matter and the novel phases which they produce
- 2) To establish a common language that is capable of explaining a broad range of phenomena in condensed matter and beyond

the interaction among electrons may lead to ferromagnetism.
the electron-ion interactions may lead to superconductivity. And
so on. The most striking phenomena in condensed matter occur
due to the interactions among its basic constituents.

In this course we will go over a large number of such problems.

But during all our trajectory, our main focus will be on establishing a common language (ie, a way of thinking). You may eventually forget the specific calculations, but the language will stay with you forever (unless you smoke crack, then you will likely forget that too). Some of the concepts you will learn include

- Emergent properties and complexity
- Broken symmetries
- Excitations
- Second quantization

I will now try to explain some of these in detail.

Emergent phenomena

In CMF we usually know the laws of physics governing our system. what we want is to be able to extract meaningful information from these laws. For instance, a solid is made of protons and electrons interacting via coulomb forces. thus the Hamiltonian of a solid will be

$$H = \sum_i \frac{\dot{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i,j} \frac{q_i q_j}{|r_i - r_j|} \quad (1)$$

We can also complicate it further and add spin effects, relativistic corrections and so on. But the point is that writing down the Hamiltonian (the "law") is usually the easy part. The real question is what to do with it. Superconductivity, for instance, should somehow be contained in this Hamiltonian. But it is not at all obvious how to make that rule.

The reductionist view is therefore wrong: knowing the basic microscopic interactions tells you nothing about macroscopic properties. This would be the same as saying that to understand biology all you need is to know the laws of chemistry. That is not true. Even though deep down a cell is just a bunch of chemical reactions, the enormous number of such reactions leads to completely new phenomena (life) which cannot be predicted by analyzing the individual reactions.

We say that life, or superconductivity, are emergent phenomena. What this means is that the properties of the whole cannot be predicted from an analysis of the individual parts. We are talking here about collective behavior.

There are millions of examples of collective behavior, in science and in our daily lives. For instance, a city with 1 million inhabitants is not the same as 100 small villages with 10000 each. Another example is group behavior, like collective panic. When people do things in groups they act differently from how they would individually (and, more often than not, they do stupid things).

At first it may seem strange that new laws can emerge in this way. Starting with the Hamiltonian (\mathcal{H}), we may diagonalize it for 2 particles, then 3, then 4. In all cases we will never see any new phases, like a superconducting state for instance. But we can't diagonalize it for 10^{23} particles and that is where the argument fails. We therefore reach the idea of complexity.

Let us simplify the problem and consider spin systems. The dimension of the Hilbert space of a spin $\frac{1}{2}$ particle is 2. If we have N spin $\frac{1}{2}$ particles then the dimension will be 2^N . For instance

$$N = 16 \quad (4 \times 4 \text{ lattice}) : \quad 2^{16} \approx 65536 \quad \text{not bad}$$

$$N = 256 \quad (16 \times 16 \text{ lattice}) : \quad 2^{256} \approx 10^{77} \quad \text{soh!}$$

For a 16×16 lattice we can't even store the state of the system: 10^{77} is approximately the number of atoms in the universe! We are talking here about a fundamental problem. This is not merely a computational issue. It is out of this complexity that emergent properties arise.

This also shows that if we hope to make any progress, approximations will be essential.

Broken symmetries : the Landau paradigm.

Our understanding of the different phases of matter comes from identifying their underlying symmetries. For instance, consider a spin system. It can be found in one of two phases

$\uparrow \uparrow \downarrow \quad \uparrow \uparrow \quad \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow$	paramagnetic
$\uparrow \uparrow \uparrow \quad \uparrow \uparrow \quad \uparrow \uparrow \uparrow \uparrow$	ferromagnetic.

The symmetry in question here is up-down symmetry (also called \mathbb{Z}_2 symmetry): if you flip a spin system upside down you should get the same behavior. This is what you see in the paramagnetic phase, but it is no longer true in the ferromagnetic phase. In the FM phase the up-down symmetry is broken: the system chooses a specific orientation and stays there. The symmetries of a system characterize its phase. The FM has a lower symmetry than the PM phase. Low and high symmetry phases are separated by a phase transition. And we characterize them by an order parameter (See the table in the next page)

More recently we discovered topological phases of matter, which do not fit into this basic paradigm. That is, they cannot be described by a local order parameter.

The phases of matter

Phase	Broken Sym.	Order para.	Excitations
Ferromagnetic crystals	up-down	magnetization m	magnons
Nematic (liquid crystals)	translation sym. (Galileo inv.) $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{R}$	$\rho - \rho_c$ density difference	phonons
Superfluid	Rotation symmetry	average angle between 2 molecules	—
Superconductors	Gauge invariance $\psi \rightarrow e^{i\theta} \psi$	$\langle \psi \rangle$	phonons and rotons
	Gauge invariance	$\langle \psi \rangle \sim \Delta = \text{energy gap}$	Electrons.

Excitations

A crystalline solid is a phase of matter. One thing is to understand how the crystal forms. The other is to understand how it vibrates. The vibration of a crystal is what we call an excitation.

Suppose you have a bar of metal and you kick it on one side.



This kick will cause a local distortion in the atoms which will propagate through the material. Understanding how it propagates is an important task.

But things are much more interesting. We will learn that these excitations are usually quantized and propagate through the material very much like a particle propagating through space. The quanta of vibration is called a photon. When we kick a material we generate a bunch of photons.

These ideas allow us to interpret a solid as a medium in which different excitations may exist. It is a world filled with particles moving in every direction. And, what is most interesting, interacting with each other. Superconductivity, for instance, is a direct consequence of the interaction between photons and electrons.

In the table of page 5 I list some of the excitations associated to each broken symmetry phase.

In all aspects, excitations behave like particles. This means that they have momentum p and energy ϵ . One of the most important aspects of CMB (and one we will pay a lot of attention to) is how energy and momentum are related. This is called a dispersion relation.

Non-relativistic particles usually have a dispersion relation of the form (quadratic)

$$\epsilon = \frac{p^2}{2m} \quad (2)$$

whereas massless particles, like photons or phonons, have dispersions of the form (linear)

$$\epsilon = cp \quad (3)$$

We always set $\hbar=1$ so energy ϵ and frequency ω are the same thing and similarly for momentum p and wave vector k :

$$\begin{aligned} \epsilon &= \hbar\omega \\ p &= \hbar k \end{aligned} \quad \boxed{\text{But } \hbar=1. \text{ Yay!}} \quad (4)$$

So I will usually write (2) as

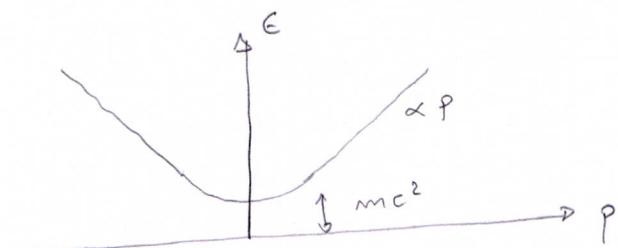
$$\epsilon = \frac{k^2}{2m} \quad (5)$$

The dispersion relation determines, to great extent, the physics of your system. For instance, electrons in solids have low energies and thus usually behave as (5). But in graphene the ions cause the dispersion relation to change and they end up behaving like (3); electrons in graphene thus move as massless relativistic particles (with a different value of c , of course). It is this change in dispersion relation which gives graphene all its amazing properties.

A more general dispersion relation is

$$\epsilon = \sqrt{m^2 c^4 + p^2 c^2} \quad (6)$$

This is a relativistic but massive dispersion. We always plot these dispersions close to $p=0$. Eq (6), for instance, look like



At high p we get $\epsilon \approx cp$. But at low p it bends and becomes parabolic.

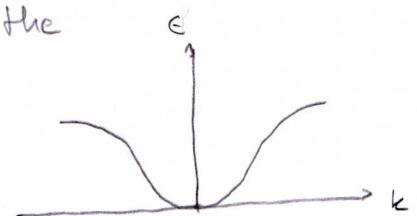
At $p=0$ there is a gap between the curve and the p-axis. This gap is labeled mc^2 . What you should look for in these plots is the energy gap at $p=0$. In this figure it has the value mc^2 . This value represents that minimum energy you need to create an excitation. If the gap is zero, creating excitations is easy. But if it's large than you need a lot of energy for that.

In CMF we also see more exotic dispersion relations, such as

$$\epsilon = 1 - \cos k \quad (7)$$

(see figure). At low momentum we may expand the cosine as $\cos k \approx 1 - k^2/2$. We then get

$$\epsilon \approx \frac{k^2}{2}$$



This is like the non-relativistic dispersion (5). We say this represents the long wavelength behavior (because $k = 2\pi/\lambda$ so small k means large λ). CMF gives very exotic dispersion relations but usually their long wavelength behavior look like (3) or (5).

Second quantization

The quantum mechanics you learned in your undergraduate courses is best suited for working with one or a few particles. Using it for 10^{23} particles is way too messy. Second quantization is a method for writing down the rules of quantum mechanics in a way that is convenient for working with a large number of particles. It is still quantum mechanics so everything you learned continues to hold. But the way of thinking is completely different.

Second quantization is developed specifically for dealing with identical particles. When particles are identical, it makes no sense to say something like "this particle is in this state and that particle is in that state". We don't know which particle is which. All we can say is "one particle is in this state and another particle is in that state". Second quantization is about "how many" not about "which one".

The theory is constructed in terms of creation and annihilation operators, almost exactly like in the quantum harmonic oscillator. But now these operators create and destroy actual particles. If a_i represents some single particle state, then a_i^\dagger is the operator which creates a particle in state i , whereas a_i^\dagger annihilates a particle in state i . The number operator is $\hat{n}_i = a_i^\dagger a_i$ and it counts the number of particles in state i . Everything is constructed in terms of these operators and their commutation relations (their algebra).

This entire course will be based on second quantization so we will come back to this theory over and over again. I guarantee you that by the end of this course this will seem like the most natural language in the world.

Some thoughts about the importance of language

Having a common language to discuss physics is absolutely essential. But unfortunately, I think that in condensed matter the idioms are fragmented. Each research sub-sub-sub-area uses different jargons, making communication difficult. Moreover, many of the ideas that could be used to unify condensed matter, like second quantization, are usually postponed to advanced courses. I think that is outrageous.

As mentioned in the beginning of these notes, one of the goals of this course is precisely to develop a broadly applicable language. Now I want to try to explain why, in my view, having such a language is so essential.

This is actually related to how our brain works. Every time you hear or read a word or expression, your brain does two things. First, it gives you a mental picture of that word. And second, it lists a series of properties related to that word. This, of course, is all done subconsciously. You don't even realize it.

Here is a silly example: "chair". As soon as you read this, your brain associated a picture of a chair and also listed a series of properties expected from a chair, such as "you can sit on it", "it is rigid", "it is usually found indoors" and so on. The advantage of having such a list of properties, is that it makes it easier to interpret sentences which use the word chair.

For instance, a sentence such as "the chair he was sitting on, broke" makes sense, whereas "He ate a delicious chair" sounds like non-sense. More interesting, consider the sentence: "She used a chair to write an equation in the blackboard". If you didn't know what the meaning of "chair" was, this sentence would have been confusing: "chair" could very well be a weird slang for "chalk". But since you know that one of the properties of "chair" is "you can stand on it", you realize that the above sentence probably means she stood on the chair to write the equation in the black board.

Ok. Enough about chairs. Here is a more interesting word: "wavefunction". If you know quantum mechanics (and I hope you do!) then you probably associate with this word properties such as "It's modulus square is a probability", "is normalized", "appear often as eigenfunctions" and so on.

Now consider the following, much more difficult sentence:

"this variational wavefunction clearly improves the ground-state estimate". This sentence is packed with jargons. If you remember well the variational principle in quantum mechanics, then this sentence is very easy to understand. If you don't, then it may sound like crazy talk.

The reason why the above sentence makes sense is because each key term has associated with it a property that links with other terms. For instance, "variational" refers to the variational principle, which deals precisely with the ground-state. If we replace "variational" by "perturbed" the sentence would no longer make much sense.

In summary, language enables us to talk efficiently with your peers. Of course, the vocabulary one can learn is limited, so we cannot hope to teach a billion different terms. Instead, we need to focus on certain key terms which appear over and over again.

In this sense, particle physicists are way ahead of us. Look at any book on quantum field theory and you will see that all share a common backbone. Solid state physics used to be like this, many decades ago. Concepts such as "Bloch wavefunctions", "band gap", " 1^{st} Brillouin zone", and so on, were taught to literally everyone in the field. But as time passed, some of these concepts became less important and many new ones came of relevance (like the ones we discussed in these notes).

Ashcroft and Mermin, for example, which is a bible in the field, does not mention second quantization at all. But nowadays, with the exception of the DFT community, every one else publishes their research in this language.

The language used in condensed matter physics is fragmented and I think that is very bad for the field. I think it is time for the community to recognize the value of language and start working together toward unifying it.