The principle of least action

I want to tell you about the most important principle in all of physics. It is called the principle of least action. The reason why it is so important is because all major physical theories can be written in terms of it: classical mechanics, relativity, electromagnetism, quantum mechanics, quantum field theory and so on. Look at any of the major paper that revolutionized physics in the last century and you will always see this principle. It is used by Bohr in his famous paper on the energy levels of the hydrogen atom. And also by Schrödinger in his 1926 paper where he first wrote down his famous equation.

Feynman's greatest contribution to physics was to recast quantum mechanics in terms of the principle of least action. This caused a revolution in the theories of elementary particles and interactions. They have now been unified into a single theory, called quantum field theory, which uses the principle of least action extensively.

Classical mechanics in 1D

I want to discuss the principle of least action in the simplest possible scenario: a particle moving in 1 dimension. In this case we describe the motion of the particle by specifying its position $x(t)$ as a function of time. But that is not enough. We also need to specify its velocity

$$v(t) = \frac{dx}{dt} = \dot{x} \tag{1}$$

If we know at each instant where a particle is (x) and where it is going (v), then we know all we need to know about its motion.

This is kinematics. Dynamics, on the other hand, requires specifying the equation of motion, which is Newton's second law:

$$F = ma \tag{2}$$

where F is the force acting on the particle, m is the mass and a is the acceleration

$$a = \frac{dv}{dt} = \dot{v} = \ddot{x} \tag{3}$$

Everyone writes $F = ma$. But that doesn't make sense. It is not the acceleration which causes a force. It is the force which causes the acceleration. The correct way to write it is

$$a = \frac{F}{m} \tag{4}$$

Now its ok: a force causes an acceleration.

It is also customary to define the momentum

$$p = m v \tag{5}$$

then Newton's second law may be written as

$$\boxed{\frac{dp}{dt} = m\frac{dv}{dt} = F} \tag{6}$$

This equation is called an equation of motion because it gives a recipe specifying how $x(t)$ and $v(t)$ will evolve in time.

For example, if the particle is subject to no force at all $(F=0)$ then we have

$$m\frac{dv}{dt} = 0 \tag{7}$$

which implies that

$$v(t) = v_0 = \text{constant} \tag{8}$$

So a particle under the influence of no forces (a free particle) moves with a constant velocity. This is called the law of inertia.

The position then follows a uniform motion

$$\frac{dx}{dt} = v = v_0$$

so

$$x(t) = x_0 + v_0 t \tag{9}$$

The motion depends on two constants: x_0 and v_0. They are determined by the initial conditions. Specify $x(0)$ and $v(0)$, then Newton's law tells you where you will be at any subsequent time.

Example : constant force

close to the surface of the earth the force due to gravity is roughly constant and given by

$$F = mg \tag{10}$$

where g is a constant (have you ever wondered why this m is the same m appearing in Eq (4) It is not obvious why that should be the case).

Eq (6) now gives

$$m\dot{v} = -mg \tag{11}$$

which may be integrated to give

$$v(t) = v_0 - gt \tag{12}$$

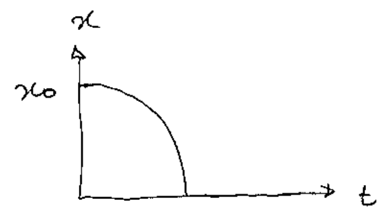
Integrating once again we find $x(t)$

$$x(t) = x_0 + v_0 t - \frac{gt^2}{2} \tag{13}$$

The fall under gravity is accelerated. Suppose the particle starts from rest ($v_0 = 0$). Then

$$x(t) = x_0 - \frac{gt^2}{2} \tag{14}$$

This is a parabola



xample : harmonic oscillator

Suppose you are in a bathtub with Donald, your yellow floating duck. If you press Donald down there will be a force upwards due to buoyancy. Donald will then oscillate up and down. If the push you gave was sufficiently gentle, the restoring force will be linearly proportional to the displacement. That is what we call a harmonic oscillator:

$$F = -kx \tag{15}$$

Newton's law gives

$$m\ddot{x} = -kx$$

or

$$\ddot{x} = -\omega^2 x \tag{16}$$

where

$$\omega^2 = k/m \tag{17}$$

The solution of (16) is

$$x(t) = A\cos(\omega t - \phi) \tag{18}$$

where A and ϕ are imposed by the initial conditions. If you want you can also expand the cosine and write

$$x(t) = a\cos\omega t + b\sin\omega t \tag{19}$$

where

$$a = A\cos\phi \qquad b = A\sin\phi \tag{20}$$

In any case there are two initial conditions, as is always the case

A <u>conservative force</u> is one which may be written as

$$F = -\frac{\partial U}{\partial x} \qquad (21)$$

we call U the potential energy, or just <u>potential</u> for the intima

this is true for the gravitational force:

$$U = mgx \qquad (22)$$

and for the harmonic force

$$U = \frac{1}{2} k x^2 \qquad (23)$$

An example of a non-conservative force is <u>friction</u>.

99% of this course will be based on conservative forces. The reason is that usually non-conservative forces are associated with the interaction between the particle and some other body (think about friction). So they usually have to be put by hand in the theory. This leaves less room to explore symmetries of the system and other nice things, which we will be particularly interested in

Combining (6) and (21), we may write Newton's law as

$$\boxed{m\frac{dv}{dt} = -\frac{\partial U}{\partial x}} \qquad (24)$$

this is exactly the same as $F = ma$, but written in a more fancy way. Just by writing things differently we can feel like we are making big process.

The principle of least action

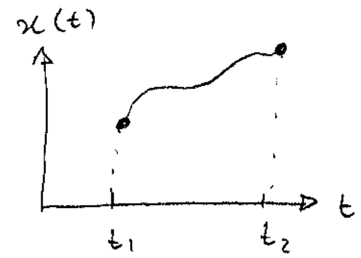
Now I want to show you a completely different way to formulate the principles of mechanics. We start with a new object, called the Lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2} m v^2 - U(x)} \tag{25}$$

It is the kinetic energy minus the potential energy. This object is absolutely central to the theory. As we will soon see, a system is completely specified by its Lagrangian. Note also that $\mathcal{L}$ is a function of x and $v = \dot{x}$, which makes sense since this is what we need to specify a system. In general we will write

$$\mathcal{L} = \mathcal{L}(x, \dot{x}) = \mathcal{L}(x, v) \tag{26}$$

We also define another object called the action. The action depends on the path of the particle $x(t)$, between two points



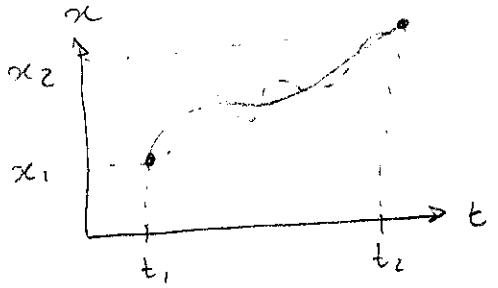
It is defined as

$$\boxed{S[x(t)] = \int_{t_1}^{t_2} \mathcal{L}(x, v)\, dt} \tag{27}$$

So the idea is: given a path $x(t)$, integrate the Lagrangian between t_1 and t_2. The result will be a number called the action.

the action depends on the entire path $x(t)$. So it is not really a function because a function $f(x)$ depends on a number x. The action depends on an entire function $x(t)$. It is called a functional and we write it with square brackets $S[x(t)]$ to emphasize this.

Now consider several paths which start and end at the same points at the same time



$$x(t_1) = x_1 \qquad (28)$$
$$x(t_2) = x_2$$

The principle of least action says the following:

"Given all the paths $x(t)$ with $x(t_1) = x_1$ and $x(t_2) = x_2$, the path which satisfies Newton's law is the path which minimizes the action"

Isn't this awesome!? It says that you may compute S for many paths and the true path (the one which obeys Newton's law) will give the least possible value for S.

I will demonstrate the principle of least action shortly. But before I do that, we need some math review

Taylor series; maxima and minima

Let $f(x)$ be an arbitrary function ($\sin x$, e^x, etc.). The Taylor series expansion of $f(x+\Delta x)$ is:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)}{2}\Delta x^2 + \frac{f'''(x)}{3!}\Delta x^3 + \dots \quad (29)$$

Maybe you have seen a different formula for the Taylor series; namely

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots \quad (30)$$

The two formulas are actually the same: set $x = x_0 + \Delta x$ in (30) and you get (29).

If you want to demonstrate (29), here is how you do it. You assume that $f(x+\Delta x)$ may be written as a power series in Δx

$$f(x+\Delta x) = a_0(x) + a_1(x)\Delta x + a_2(x)\Delta x^2 + \dots \quad (31)$$

where the $a_i(x)$ are functions of x which we must try to determine.

If you set $\Delta x = 0$ in (31) you get

$$f(x) = a_0(x)$$

so this determines a_0. To find a_1 we differentiate (31) with respect to Δx, keeping x constant. We then get

$$f'(x+\Delta x) = a_1(x) + 2a_2(x)\Delta x + \dots$$

Again, setting $\Delta x = 0$ yields

$$f'(x) = a_1(x)$$

This is the procedure. Next we differentiate again with respect to x and obtain a_2, and so on. Continuing in this way we then obtain (29).

Now consider a function of two variables : $f(x, y)$. The Taylor series expansion is done in a very similar way. The linear terms are particularly intuitive :

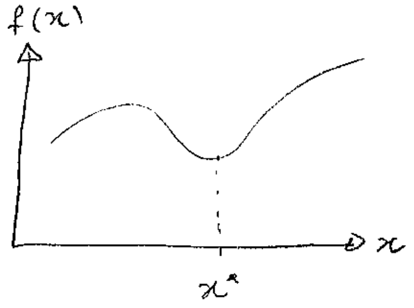
$$f(x+\Delta x, y+\Delta y) \simeq f(x, y) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \qquad (32)$$

The higher order terms start involving combinations of Δx and Δy. For instance, the second order terms are

$$\frac{1}{2} \left\{ \frac{\partial^2 f}{\partial x^2} \Delta x^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \Delta x \Delta y + \frac{\partial^2 f}{\partial y^2} \Delta y^2 \right\} \qquad (33)$$

and so on for the higher order terms. Lucky for us, we will usually only need the linear term.

Eq (29) can be used as an intuitive way to understand the concepts of maxima and minima. Consider the following function



The point x^* is a minimum of $f(x)$ because all points in its neighborhood have a higher $f(x)$ value.

Let us use (29) with $x = x^*$. Then

$$f(x^*+\Delta x) = f(x^*) + f'(x^*)\Delta x + \frac{f''(x^*)}{2}\Delta x^2 + \dots \qquad (34)$$

We are thinking about Δx as a tiny change. So, looking at the linear term in (34), we can imagine changing Δx a little bit so as to make $f(x^*+\Delta x) < f(x^*)$. Well, if that is possible then x^* is not actually a minimum.

We therefore conclude that for x^* to be a minimum we must have

$$f'(x^*) = 0 \qquad (35)$$

Well, actually, this is the same condition required for a maximum, because the same reasoning applies. What differentiate maxima and minima is the second derivative. With $f'(x^*)=0$, Eq (34) becomes

$$f(x^*+\Delta x) \simeq f(x^*) + \frac{f''(x^*)}{2}\Delta x^2 \qquad (36)$$

$\Delta x^2 > 0$ so that, if $f''(x') \geq 0$ then $f(x'+\Delta x) > f(x')$ for sure. This would then be a minimum. Similarly, if $f''(x') < 0$ then $f(x'+\Delta x) < f(x')$ for sure, which means its a maximum.

We will soon be dealing with more complicated objects and we won't care about whether its a maximum or a minimum. We will simply try to <u>extremise</u> the object.

Going back to (29) let us write

$$\Delta f = f(x+\Delta x) - f(x) \tag{37}$$

To first order in Δx we then have

$$\Delta f \approx f'(x)\,\Delta x \tag{38}$$

This is a nice formula: a perturbation Δx causes a response Δf. <u>A point x will be an extremum when a perturbation Δx does not affect Δf.</u> In the case of the Taylor series this means $f'(x) = 0$. But the idea is actually quite more general: <u>an extremum is a point insensitive to small changes.</u>

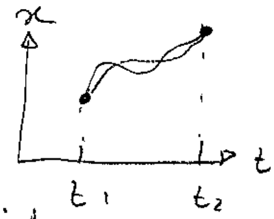
This is what we will use to demonstrate the principle of least action.

Demonstration of the principle of least action

The action is

$$S[x(t)] = \int_{t_1}^{t_2} \left[\tfrac{1}{2} m v^2 - U(x) \right] dt \qquad (39)$$

So the recipe is: given a path $x(t)$ between two instants of time t_1 and t_2, plug into this formula and you get the number S.



Now consider two paths: $x(t)$ and $x(t)+\eta(t)$, where $\eta(t)$ is a tiny variation over the original path. I will consider only paths which start and end at the same point so I assume that

$$\eta(t_1) = \eta(t_2) = 0 \qquad (40)$$

Let us compute the action of this modified path

$$S[x(t)+\eta(t)] = \int_{t_1}^{t_2} \left[\tfrac{1}{2} m (v+\dot\eta)^2 - U(x+\eta) \right] dt \qquad (41)$$

We are assuming that η is a tiny variation so we may expand in a Taylor series, just like in (29):

$$U(x+\eta) \simeq U(x) + U'(x)\,\eta \qquad (42)$$

and

$$(v+\dot\eta)^2 = v^2 + 2v\dot\eta + \dot\eta^2 \approx v^2 + 2v\dot\eta \qquad (43)$$

since $\dot\eta^2$ is of second order and we are only interested in first order variations.

We therefore have

$$S[x(t)+\eta(t)] \approx \int_{t_1}^{t_2} \left\{ \frac{1}{2} m v^2 - U(x) + m v \dot{\eta} - U'(x)\eta \right\} dt$$

The variation of the action is therefore (I use δ instead of Δ just because I'm fancy)

$$\delta S = S[x(t)+\eta(t)] - S[x(t)]$$
$$= \int_{t_1}^{t_2} \left\{ m v \dot{\eta} - U'(x)\eta \right\} dt \qquad (44)$$

I am looking at something of the sort of (38), with η playing the role of Δx and δS playing the role of Δf.

The last term is ok because it has the structure

$$(\text{something}) \times \eta \leftarrow \text{the small guy.}$$

But the first term is weird because it depends on $\dot{\eta}$ which is not necessarily small (you may have a function which is very tiny but has a huge slope).

We can get around this problem integrating by parts

$$m \int_{t_1}^{t_2} v \dot{\eta} \, dt = m \int_{t_1}^{t_2} v \frac{d\eta}{dt} dt$$

I may use the following trick based on the chain rule

$$\frac{d}{dt}(v\eta) = \frac{dv}{dt}\eta + v\frac{d\eta}{dt}$$

I then get

$$m\int_{t_1}^{t_2} v\frac{d\eta}{dt}\,dt = m\int_{t_1}^{t_2}\left\{\frac{d}{dt}(v\eta) - \frac{dv}{dt}\eta\right\}dt$$

The first integral is rather easy because the dt's cancel

$$\int_{t_1}^{t_2}\frac{d}{dt}(v\eta)\,dt = \int_{t_1}^{t_2} d(v\eta) = v\eta\Big|_{t_1}^{t_2}$$

This is where (40) comes in: $\eta(t_1) = \eta(t_2) = 0$ so this term is zero. We then get

$$m\int_{t_1}^{t_2} v\frac{d\eta}{dt}\,dt = -m\int_{t_1}^{t_2}\frac{dv}{dt}\eta\,dt \tag{45}$$

when you integrate by parts you simply throw the derivative from one guy to the other (with a bonus minus sign). Sometimes there are the annoying cross terms. But if you are lucky enough they cancel. Fortunately that is always true when dealing with the principle of least action

In any case, going back to (44) and using (45) we get

$$\delta S = -\int_{t_1}^{t_2}\left\{m\frac{dv}{dt} + U'(x)\right\}\eta(t)\,dt \tag{46}$$

This has exactly the structure of Eq (38)

Response = (...) × Stimulus

According to the principle of least action we should now set $\delta S = 0$:

$$\boxed{\text{Principle of least action : } \delta S = 0} \tag{47}$$

But we want this to be true for any $\eta(t)$ so the only possibility is that

$$m\frac{dv}{dt} + U'(x) = 0 \tag{48}$$

But this is exactly Newton's law because

$$F = -U'(x)$$

We have therefore demonstrated what we wanted: the path which minimizes the action is that which obeys Newton's second law.

Second demonstration of the principle of least action

The mathematics involved with the principle of least action is not very intuitive. But since it is really important, I want you to get used to it by looking at it from different angles.

So now I will carry out the same derivation but in a slightly more general way. I will assume that the system is described by a Lagrangian $\mathcal{L}(x, v) = \mathcal{L}(x, \dot{x})$, whose exact form I don't really know (this is not just a silly exercise. Very soon we will start finding Lagrangians which are not $\frac{1}{2}mv^2 - U$).

So, repeating the same steps, I have

$$\delta S = S[x(t) + \eta(t)] - S[x(t)]$$
$$= \int_{t_1}^{t_2} \mathcal{L}(x+\eta, \dot{x}+\dot{\eta})\,dt - \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x})\,dt \qquad (49)$$

Now I expand $\mathcal{L}$ in a Taylor series treating x and $\dot{x}$ as independent variables (cf Eq (32)):

$$\mathcal{L}(x+\eta, \dot{x}+\dot{\eta}) \simeq \mathcal{L}(x, \dot{x}) + \frac{\partial \mathcal{L}}{\partial x}\eta + \frac{\partial \mathcal{L}}{\partial \dot{x}}\dot{\eta} \qquad (50)$$

You should not be frightened with $\frac{\partial \mathcal{L}}{\partial \dot{x}}$. Remember that $\dot{x} = v$ so

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial v}$$

With this expansion I then obtain for δS:

$$\delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial \mathcal{L}}{\partial x} \eta + \frac{\partial \mathcal{L}}{\partial \dot{x}} \dot{\eta} \right\} dt \tag{51}$$

Again I integrate the $\dot{\eta}$ term by parts: remember, all you have to do is pass the derivative along

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{x}} \frac{d\eta}{dt} \, dt = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \eta \, dt \tag{52}$$

Thus

$$\delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \right\} \eta(t) \, dt \tag{53}$$

Again, setting $\delta S = 0$ according to the principle of least action, I get

$$\boxed{\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = 0} \tag{54}$$

This is called the <u>Euler-Lagrange equation</u>. Note that it holds for <u>any</u> Lagrangian, provided it depends only on x and $\dot{x}$.

To obtain Newton's law, we now evaluate Eq (54) for

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U(x)$$

we have the following :

$$\frac{\partial \mathcal{L}}{\partial x} = -U'(x) = F$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = mv$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\frac{dv}{dt}$$

thus (54) gives

$$F - m\frac{dv}{dt} = 0$$

which is Newton's law.

While we are on the subject, let me tell you something cool. Remember the Taylor series expansion :

$$f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \cdots$$

We could try to do something similar with $S[x(t)+\eta(t)]$.

It looks like this

$$S[x(t)+\eta(t)] = S[x(t)] + \int_{t_1}^{t_2} \frac{\delta S}{\delta x(t)}\,\eta(t)\,dt + \cdots \qquad (55)$$

This is not really a derivative because S is a functional, not a function. But the idea remains. We call $\frac{\delta S}{\delta x(t)}$ a <u>functional derivative</u>. In our case we already found it : just look at (53).

$$\boxed{\frac{\delta S}{\delta x(t)} = \frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)} \qquad (56)$$

I personally think that this is quite beautiful.

Example: harmonic oscillator (Hooke's law)

The potential energy of a harmonic oscillator is

$$U(x) = \frac{1}{2} k x^2 \tag{57}$$

where k is the spring constant. The force is

$$F = -\frac{\partial U}{\partial x} = -kx \tag{58}$$

which is Hooke's law. So Newton's equation reads

$$ma = m\ddot{x} = -kx$$

It is convenient to define

$$\omega^2 = \frac{k}{m} \tag{59}$$

so that

$$\ddot{x} = -\omega^2 x \tag{60}$$

Now let us practice with the Lagrangian formulation:

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \tag{61}$$

The Lagrangian completely characterizes the system. Once we have it we may obtain the equations of motion using the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \tag{62}$$

In our case we have

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2\right] = -kx$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k x^2\right] = m\dot{x}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}$$

Thus (62) gives

$$m\ddot{x} = -kx$$

which is the same as (60).

The Lagrangian contains Newton's law in it, but it also contains additional information concerning the symmetries of the system.

Let me give you an example. Suppose you have two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$, such that

$$\mathcal{L}' = \alpha \mathcal{L} \tag{63}$$

where α is a constant. If we look at the Euler-Lagrange equation (62) for $\mathcal{L}'$ we see that the constant α cancels on both sides. So $\mathcal{L}$ and $\mathcal{L}'$ will have the same equations of motion. This means that

$$\boxed{\text{" Two Lagrangians differing by a multiplicative constant are physically equivalent "}}$$

Let us look again at the harmonic oscillator

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

we may write $k = m\omega^2$ so that

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 - \omega^2 x^2 \right) \tag{64}$$

The factor of $1/2\, m$ is just a multiplicative constant, so it will not affect the motion. Thus, even if we do not know the equations of motion, we may already predict that the two parameters m and k will only enter as the ratio $\omega^2 = k/m$.

Next let me show you something which is very peculiar of the harmonic oscillator. Suppose we rescale x as

$$x \to \lambda x$$

where λ is a constant. Eq (64) then remains unchanged, up to a constant:

$$\mathcal{L} = \frac{1}{2} m \lambda^2 \left(\dot{x}^2 - \omega^2 x^2 \right) \qquad (65)$$

so a rescalling of the coordinates does not affect the physics of the harmonic oscillator. This means that if you push the spring twice as much, you simply double the amplitude of the oscillations.

In practice we know that this will only be true in a real system if the displacement is small. If you push it too much, you will eventually deform the spring. this means that a more realistic potential will not share the rescalling property (65).

Feynman's formulation of quantum mechanics

We have just seen that to each path $x(t)$ there is an associated action $S[x(t)]$. But in practice we only observe one path: that which obeys Newton's law. Let us call it x_c, where c stands for classical.

In the 1950s R. P. Feynman invented an alternative formulation of quantum mechanics based on the principle of least action.

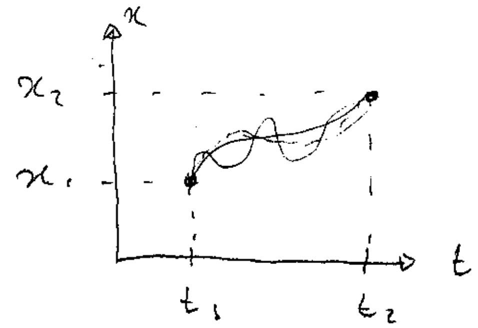
The big result of his theory is the following

$$\boxed{\langle x_2, t_2 | x_1, t_1 \rangle = \sum_{\substack{\text{all}\\ \text{paths}}} e^{iS/\hbar}} \qquad \hbar = \text{Planck's constant} \qquad (66)$$

Please don't run away! Let me explain what this means. The quantity $\langle \alpha | \beta \rangle$ is called a probability amplitude; it is a complex number such that its modulus squared is the actual probability (this is similar to the wave-function ψ and its modulus $|\psi|^2$):

$$|\langle \alpha | \beta \rangle|^2 = \text{prob. of finding a particle in } \alpha \text{ given that it was in } \beta.$$

So Eq (66) is the amplitude to find the particle at x_2 in time t_2, given that it was in x_1 at time t_1.



Eq (66) says that to find the probability to go from one point (t_1, x_1) to another (t_2, x_2), we must consider all possible paths between these two points (there is an infinite number of them). We then average $e^{iS/\hbar}$ for all paths.

The constant $\hbar$ gives you the definition of "small." Suppose we have a classical, macroscopic system. Then $S \gg \hbar$. In this case $e^{iS/\hbar}$ will oscillate violently between each path, and there will be a lot of destructive interference (cancelations). The only exception is the classical path x_c. According to the principle of least action, the classical path minimizes S so that for it $\boxed{\delta S = 0}$. This means if you stick to the classical path there will not be many destructive interferences.

Conclusion: when $S \gg \hbar$ (a classical system), the particle will follow the classical path $x_c(t)$. But when the system is not macroscopic, there may be a substantial contribution from other paths.

This explains the wave nature of particles and phenomena such as the diffraction of electrons.