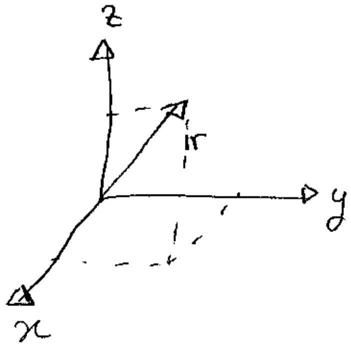
The principle of least action: part 2

classical mechanics in 3D

I will now discuss the mechanics of a particle in 3D. To describe a particle in 3D we need 3 coordinates: x, y, z. They form the components of the position vector

$$\mathbb{r} = (x, y, z) \tag{1}$$

I am lazy, so I write $\mathbb{r}$ with a bar on the side, instead of $\vec{r}$.



Sometimes it is more convenient to write x_1, x_2, x_3 instead of x, y, z. I will alternate constantly between the two notations so please keep your heads up.

$$\mathbb{r} = (x, y, z) = (x_1, x_2, x_3) \tag{2}$$

The velocity is also a vector

$$\mathbb{v} = \dot{\mathbb{r}} = (\dot{x}, \dot{y}, \dot{z}) \tag{3}$$

s is the acceleration $\mathbb{a} = \dot{\mathbb{v}}$

The motion of the particle is governed by Newton's law

$$m \frac{d\mathbf{v}}{dt} = \mathbb{F} \qquad (4)$$

where $\mathbb{F} = (F_x, F_y, F_z)$ is the force acting on the particle. Of course, Eq (4) is really 3 equations

$$\begin{aligned} m\dot{v}_x &= F_x \\ m\dot{v}_y &= F_y \\ m\dot{v}_z &= F_z \end{aligned} \qquad (4')$$

we just bundle them together for convenience. We may also write them in the x_1, x_2, x_3 notation as

$$m\dot{v}_i = F_i \qquad (5)$$

where $i = 1, 2, 3$.

The momentum of the particle is defined as

$$\mathbb{p} = m\mathbf{v} \qquad (6)$$

so that Newton's law may also be written as

$$\frac{d\mathbb{p}}{dt} = \mathbb{F} \qquad (7)$$

when no force acts on the particle, $\mathbb{F} = 0$, and we obtain

$$\frac{d\mathbb{p}}{dt} = 0 \qquad (8)$$

so that

$$\mathbb{p} = \text{constant} \qquad (8')$$

this is the <u>free particle</u>: it moves uniformly through space.

All forces we will deal with will be <u>conservative</u>. A conservative force may be written as the gradient of a <u>potential energy</u> $U(x, y, z)$:

$$F = -\nabla U \qquad (9)$$

written explicitly

$$\mathbb{F} = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \qquad (10)$$

or, if you prefer, we may write

$$F_i = -\frac{\partial U}{\partial x_i} \qquad (11)$$

I will also use another notation which I think is really nice:

$$\mathbb{F} = -\frac{\partial U}{\partial \mathbb{r}} \qquad (12)$$

It looks like you are differentiating with respect to a vector, but it is just a compact way of writing a vector whose i-th component is $-\frac{\partial U}{\partial x_i}$

thus

$$\frac{\partial}{\partial \mathbf{r}} = \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \tag{13}$$

Work and energy

The work done by an external force $\mathbb{F}$ in taking a particle from point $\mathbb{r}_1$ to point $\mathbb{r}_2$ is

$$W = \int_1^2 \mathbb{F} \cdot d\mathbb{r} \tag{14}$$

the work is the force you apply multiplied by the distance you pushed. Since $\mathbb{v} = \frac{d\mathbb{r}}{dt}$ we may write

$$d\mathbb{r} = \mathbb{v}\, dt \tag{15}$$

Together with Newton's law [Eq (4)] this gives

$$W = \int_1^2 m\frac{d\mathbb{v}}{dt} \cdot \mathbb{v}\, dt$$

But, by the product rule

$$\frac{d}{dt}(\mathbb{v} \cdot \mathbb{v}) = \frac{d\mathbb{v}}{dt} \cdot \mathbb{v} + \mathbb{v} \cdot \frac{d\mathbb{v}}{dt}$$

$$= 2\,\frac{d\mathbb{v}}{dt} \cdot \mathbb{v}$$

Thus

$$\frac{d\mathbb{v}}{dt} \cdot \mathbb{v} = \frac{1}{2}\frac{d}{dt}(v^2)$$

where I also used the fact that

$$\mathbb{v} \cdot \mathbb{v} = v^2 = v_x^2 + v_y^2 + v_z^2 \tag{16}$$

Thus

$$W = \frac{m}{2}\int_1^2 \frac{d}{dt}(v^2)\,dt$$

cancelling out the dt's we finally arrive at

$$W = \frac{1}{2}m(v_2^2 - v_1^2) \tag{17}$$

We define the kinetic energy as

$$T = \frac{1}{2}mv^2 \tag{18}$$

Eq (17) then shows that the change in the kinetic energy of a particle is the work performed by the force.

On the other hand, if the force is conservative

$$W = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = -\int_1^2 \frac{\partial U}{\partial \mathbf{r}}\cdot d\mathbf{r} \tag{19}$$

cancelling the dr's then gives

$$W = -(U_2 - U_1) \tag{20}$$

Comparing with (17) we find that

$$\frac{1}{2}m(v_2^2 - v_1^2) = -U_2 + U_1$$

or

$$\frac{1}{2}mv_1^2 + U_1 = \frac{1}{2}mv_2^2 + U_2 \tag{21}$$

This motivates us to define a quantity called the energy as

$$\boxed{E = \frac{1}{2} m v^2 + U} \tag{22}$$

Our calculations then show that when the force is conservative, the energy is a conserved quantity. But of course, this is only true if U does not depend on time. If that happens E will not be conserved.

The Lagrangian of a particle in 3D

As in 1D, the Lagrangian is the kinetic energy minus the potential energy

$$\boxed{\mathcal{L} = \frac{1}{2} m v^2 - U(\mathbf{r})} \tag{23}$$

It is sometimes important to write the kinetic energy explicitly as

$$\boxed{\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(\mathbf{r})} \tag{24}$$

Before we start using the Lagrangian, let us think about its structure. This is very important because the Lagrangian completely characterizes a system so that by learning what is its formal structure, we learn a lot about the symmetries experienced by the system.

So let us consider a free particle; ie, one where no forces act on it. This means that $U=0$ so that

$$\mathcal{L} = \frac{1}{2} m v^2 \tag{25}$$

But suppose we didn't know that. Suppose we had no idea what the Lagrangian is supposed to be. All we know is that the Lagrangian completely characterizes the system. So, in general, it may be a function of $\mathbf{r}$ and $\mathbf{v}$:

$$\mathcal{L} = \mathcal{L}(\mathbf{r}, \mathbf{v}) \tag{26}$$

But if the particle is free, the space around it must be homogeneous. That is no part of space may differ from any other. This means that $\mathcal{L}$ cannot depend on $\mathbf{r}$. Because, if it did, one region of space would behave differently from another.

We therefore conclude that the homogeneity of space implies that

$$\mathcal{L}(\mathbf{r}, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \tag{27}$$

Next we invoke the isotropy of space. Suppose that $\mathcal{L}$ had the form

$$\mathcal{L} \propto (v_x^2 + 2 v_y^2 + v_z^2)$$

That would be completely unacceptable: it would favor the y direction over the x and z. But that cannot happen because space is isotropic

So $\mathcal{L}$ may only depend on $\mathbf{v}$ in a way which treats all direction in equal footing. This means that $\mathcal{L}$ must depend only on the modulus of $\mathbf{v}$, since the modulus is independent of the orientation:

$$\mathcal{L}(\mathbf{v}) = \mathcal{L}(v) \tag{28}$$

The last question now is how $\mathcal{L}$ depend on v? Is it like $\mathcal{L} \propto v^2$ or $\mathcal{L} \propto \sin(v)$ or what. That we cannot say for now. But we will be able to do so later as a consequence of another symmetry: Galileo invariance. Similarly, Lorentz invariance implies that the Lagrangian of a relativistic system is

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} \tag{29}$$

All other definitions are quite similar to the 1D case. The action is now a functional of the path $\mathbf{r}(t)$

$$S[\mathbf{r}(t)] = \int_{t_1}^{t_2} \mathcal{L}\, dt \tag{30}$$

and if you minimize the action you obtain one Euler-Lagrange equation for each coordinate x_i

$$\boxed{\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = 0} \quad , \quad i = 1, 2, 3 \tag{31}$$

Let us check that this indeed gives Newton's law. We have

$$\mathcal{L} = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U(x_1, x_2, x_3) \tag{32}$$

Thus

$$\boxed{\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{\partial U}{\partial x_i} = F_i} \tag{33}$$

So the derivative of $\mathcal{L}$ with respect to x_i is the force. Next

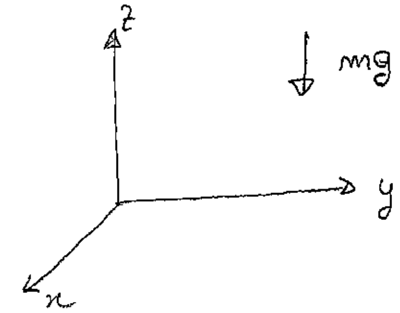
$$\boxed{\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m \dot{x}_i = p_i} \tag{34}$$

The derivative of $\mathcal{L}$ with respect to $\dot{x}_i$ is the momentum p_i. Thus Eq (31) indeed gives

$$\frac{dp_i}{dt} = F_i \tag{35}$$

The momentum p_i is a constant when $F_i = 0$. Thinking in terms of Lagrangians, we see that F_i will only be zero if $\mathcal{L}$ does not depend on x_i.

Take as an example a particle under the effect of gravity.



The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m v^2 - mgz \qquad (36)$$

The equations of motion are

$$\frac{dp_x}{dt} = F_x = \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{dp_y}{dt} = F_y = \frac{\partial \mathcal{L}}{\partial y} = 0 \qquad (37)$$

$$\frac{dp_z}{dt} = F_z = \frac{\partial \mathcal{L}}{\partial z} = -mg$$

So we see that p_x and p_y will be conserved. But we could have anticipated this from the Lagrangian (36), since it does not depend on x and y.

When a coordinate does not appear explicitly in the Lagrangian we call it *cyclic*. The momentum associated to a cyclic coordinate is a constant of the motion.