Generalized coordinates

We have seen that a physical system is characterized by its Lagrangian $\mathcal{L}$, which is in general a function of the coordinates and velocities, $\mathcal{L} = \mathcal{L}(x, \dot{x})$.

From $\mathcal{L}$, if we want the equations of motion we then use the Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}} \tag{1}$$

But now take a moment and try to remember the derivation of this formula. If you recall, all we used was the fact that $\mathcal{L} = \mathcal{L}(x, \dot{x})$. Nowhere did we have to state how $\mathcal{L}$ depended on x or $\dot{x}$. More than that: nowhere did we have to specify what x <u>was</u>. It doesn't have to be a position; it may be any dynamical variable describing the system.

This is one of the big advantages of the Lagrangian formulation. Newton's law are defined exclusively for cartesian coordinates. But the Euler-Lagrange equations work for any type of coordinate you want.

Thus, in general the Lagrangian will depend on a set of <u>generalized coordinates</u> $q_1, q_2, \ldots q_N$. These can be anything you want: positions, angles, etc. The only point is that your system must be completely characterized by these coordinates.

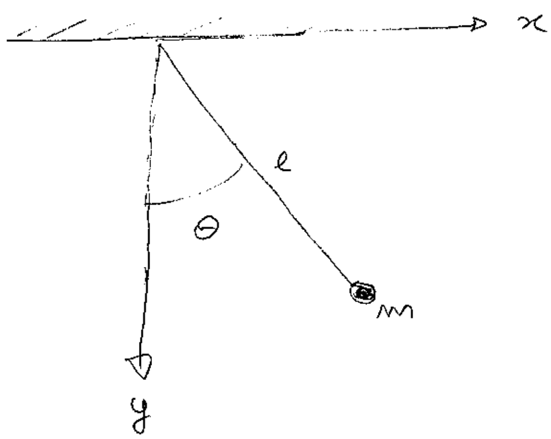
The Lagrangian will then in general depend on q_i, $\dot{q}_i$ and maybe on t explicitly. And for each q_i we will have an Euler - Lagrange equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}} \qquad (2)$$

If your q_i are cartesian coordinates, the Euler- Lagrange equations will give Newton's law. Otherwise it will give something else. The resuets may be quite weird equations.

Example: pendulum

The pendulum looks like this



The position of the mass m is described by two coordinates: x and y. But it is silly to use two numbers since the length l is fixed. For this reason the position of the pendulum is completely characterized by the angle θ. Based on my choice of coordinate system, we have

$$x = l \sin\theta \tag{3}$$
$$y = l \cos\theta$$

The force of gravity is $F = mg$, positive because I chose y downwards. Hence the potential will be

$$U = -mgy \tag{4}$$

and the Lagrangian will be

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy \tag{5}$$

Now we substitute x and y in terms of θ to obtain a Lagrangian which depends only on θ. We have

$$\begin{aligned} \dot{x} &= l\dot{\theta}\cos\theta \\ \dot{y} &= -l\dot{\theta}\sin\theta \end{aligned} \tag{6}$$

Thus

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta = l^2\dot{\theta}^2$$

Hence

$$\boxed{\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta} \tag{7}$$

This is the Lagrangian of the pendulum. Now, if you want the equations of motion, you use (2)

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}(ml^2\dot{\theta}) = ml^2\ddot{\theta}$$

$$\therefore \quad ml^2\ddot{\theta} = -mgl\sin\theta$$

or

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta} \tag{8}$$

which is the famous equation for the pendulum. We will not try to solve this equation now. That is another step, which actually can be done numerically quite easily nowadays. Instead, let us apply this same procedure to more complicated problems.

Example: particle moving in a Helix

The mathematical equation for a Helix is

$$(x, y, z) = (a\cos\phi,\ a\sin\phi,\ b\phi) \tag{9}$$

The radius is a and at every turn the Helix goes up by a factor of $2\pi b$. For DNA $a = 1$ mm and $b = 3.4$ mm.

Now suppose a particle is constrained to move in the Helix. This would be an approximate model for the transport of electrons in a DNA stripe.

In this case the only generalized coordinate will be ϕ. So let us compute the corresponding Lagrangian, starting with the kinetic energy:

$$\dot{x} = -a\dot{\phi}\sin\phi$$

$$\dot{y} = a\dot{\phi}\cos\phi$$

$$\dot{z} = b\dot{\phi}$$

thus

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m(a^2 + b^2)\dot{\phi}^2 = \frac{1}{2}\mu\dot{\phi}^2$$

where

$$\mu = m(a^2 + b^2)$$

Note that μ does not have units of mass, just like ϕ does not have units of position.

The Lagrangian will then be

$$\mathcal{L} = \frac{1}{2}\mu\dot{\phi}^2 - U(\phi) \qquad (10)$$

The potential will of course depend on the physical problem you have. But we may consider some interesting cases.

Suppose

$$U = \frac{k}{2} z^2$$

This is a harmonic potential, which traps the particle around $z = 0$. Since $z = b\phi$, this would give

$$\mathcal{L} = \frac{1}{2}\mu\dot{\phi}^2 - \frac{kb^2}{2}\phi^2$$

This is exactly the Lagrangian of a harmonic oscillator, but in the ϕ variable.

Alternatively, suppose that

$$U = -kx$$

This is like applying a constant force in the x direction. Since $x = a\cos\phi$, we obtain in this case

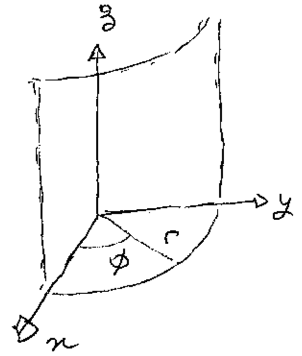
$$\mathcal{L} = \frac{1}{2}\mu\dot{\phi}^2 + ka\cos\phi$$

This is exactly the Lagrangian of the pendulum, Eq (7). So applying a constant force in the x direction has, due to the constraint that the particle should move in a helix, the same physical effect as a pendulum.

Cylindrical and spherical coordinates

Cylindrical coordinates are defined by

$$x = r\cos\phi$$
$$y = r\sin\phi \qquad (11)$$
$$z = z$$



Let us find the kinetic energy.

$$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$$
$$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$$

$$\therefore\ \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2 + \text{stuff which cancel out}$$

Thus, we conclude that the Lagrangian of a particle in cylindrical coordinates is

$$\boxed{\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - U(r,\phi,z)} \qquad (12)$$

Going to cylindrical coordinates places no constraint on the motion of the particle (like we did in the Helix problem). We just changed 3 degrees of freedom, x, y, z, to 3 new ones, r, θ, ϕ.

Additional constraints may be imposed if the problem calls for it. For instance, if it happens that the motion should be constrained to the xy plane, then $\ddot{z} = 0$.

Or, if the motion is to happen over the shell of a cylinder, then $\dot{r} = 0$, and so on.

If you are more used to vector calculus, there is an easier way to arrive at (12). By definition

$$v^2 = \left(\frac{d\mathbf{r}}{dt}\right)^2 = \frac{(d\mathbf{r})^2}{(dt)^2} \qquad (13)$$

In cartesian coordinates

$$(d\mathbf{r})^2 = dx^2 + dy^2 + dz^2$$

So

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

In cylindrical coordinates

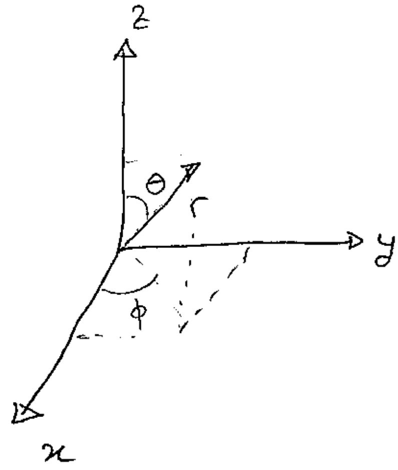
$$(d\mathbf{r})^2 = dr^2 + r^2 d\phi^2 \qquad (14)$$

So

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

Spherical coordinates are defined as

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad (15)$$
$$z = r\cos\theta$$



Let us compute the kinetic energy.
[If you want, you can skip some work using (13)].

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\left[\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi\right]$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\left[\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi\right]$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2\sin^2\theta + r^2\left\{(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)^2 + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)^2\right\}$$

$$+ 2r\dot{r}\sin\theta\left\{\cos\phi(\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) + \sin\phi(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\right\}$$

$$= \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\phi}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta$$

Now we add $\dot{z}^2$:

$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{z}^2 = \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta$$

Thus

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta \qquad (16)$$

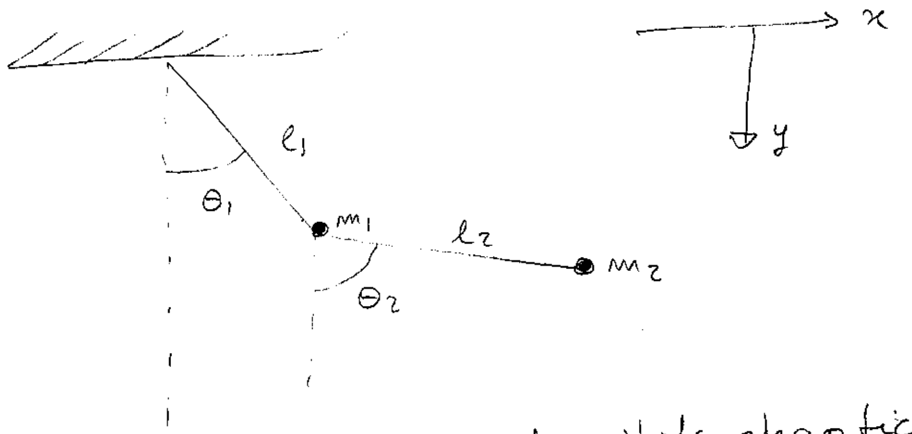
and the Lagrangian in spherical coordinates becomes

$$\boxed{\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - U(r,\theta,\phi)} \qquad (17)$$

Again, no restrictions have been imposed yet. If it happen that the motion should occur at the surface of a sphere, then $\dot{r} = 0$. If $\dot{\theta} = 0$ then the motion occurs on a fixed circle of lattitude, whereas $\dot{\phi} = 0$ means a fixed meridian.

Example: double pendulum

Now here is a problem that would be a mess to study using Newtonian mechanics: a double pendulum



This is an important problem because it exibits chaotic behavior. In principle there are 4 variables: x_1, y_1, x_2, y_2. But the masses are attached to the strings so we may completely describe the problem using the two angles, θ_1 and θ_2 (as defined in the figure).

The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + m_1 g y_1 + m_2 g y_2$$

the positions of the first mass are related to θ_1 by

$$x_1 = l_1 \sin\theta_1$$
$$y_1 = l_1 \cos\theta_1$$

For the mass m_2 we need to be a bit more careful:

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

Thus

$$\dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1 \qquad \Rightarrow \qquad \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\theta}_1^2$$
$$\dot{y}_1 = -l_1 \dot{\theta}_1 \sin\theta_1$$

$$\dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$$
$$\dot{y}_2 = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \dot{\theta}_2 \sin\theta_2$$

$$\Downarrow$$

$$\dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$
$$= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

In terms of the generalized coordinates θ_1 and θ_2 we then have

$$\boxed{\begin{aligned} \mathcal{L} = {} & \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ & + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 . \end{aligned}}$$

This is the Lagrangian of a double pendulum. The first two terms describe the kinetic energies of the two particles and the last two terms describe the interaction of the masses with the gravitational field.

The real complication comes from the term in the middle, which couples both pendulums. Note that it depends on $\theta_1 - \theta_2$, so it is zero when the two masses are aligned.