Compatible vs. incompatible observables

Let $\hat{A}$ and $\hat{B}$ be two observables (ie, two Hermitian operators). The following nomenclature is commonly used:

$$[\hat{A}, \hat{B}] = 0 \Rightarrow \hat{A} \text{ and } \hat{B} \text{ are compatible observables}$$

$$[\hat{A}, \hat{B}] \neq 0 \Rightarrow \hat{A} \text{ and } \hat{B} \text{ are incompatible observables.}$$

The reason why we say this will be discussed in more detail soon. But if you want a quick answer think about the uncertainty principle

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right| \tag{1}$$

If they are compatible there is no restriction between the measurements of the observables. But if they are incompatible there is a restriction. The most famous example is $[\hat{x}, \hat{p}] = i\hbar$ which leads to

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \tag{2}$$

So $\hat{x}$ and $\hat{p}$ are incompatible observables. It is impossible to know both with arbitrary precision.

Compatible observables satisfy a very important property.

$\boxed{\text{If } [\hat{A}, \hat{B}] = 0 \text{ it is always possible to find a basis of vectors which are simultaneously eigenvectors of } \hat{A} \text{ and } \hat{B}\text{. In other words, } \hat{A} \text{ and } \hat{B} \text{ can be } \underline{\text{diagonalized}}\ \underline{\text{simultaneously}}}$

The key idea here is the expression "always possible". It means that you have to look for them. You have to search. They may not be easy to find, but in principle it is always possible to find them.

Let us begin by writting

$$\hat{A}\,|a_i\rangle = a_i\,|a_i\rangle \tag{3}$$

If $[\hat{A}, \hat{B}] = 0$ then we have

$$\boxed{\hat{A}\hat{B}\,|a_i\rangle = \hat{B}\hat{A}\,|a_i\rangle = a_i\,\hat{B}\,|a_i\rangle} \tag{4}$$

this is the important result. It shows that for each eigenvector $|a_i\rangle$, $\hat{B}\,|a_i\rangle$ is also an eigenvector with the same eigenvalue a_i.

Suppose first that the eigenvalues of $\hat{A}$ are non-degenerate. This means that to each a_i there corresponds just one eigenvector $|a_i\rangle$. But $\hat{B}\,|a_i\rangle$ must also be an eigenvector with eigenvalue a_i. thus, the only possibility is that $\hat{B}\,|a_i\rangle$ is a constant times $|a_i\rangle$. We call this constant b_i:

$$\hat{B}\,|a_i\rangle = b_i\,|a_i\rangle \tag{5}$$

But wait! this is simply the eigenvalue/ eigenvector Eq for B. We thus conclude that $|a_i\rangle$ is also an eigenvector of B with eigenvalue b_i

$$\begin{aligned} \hat{A}\,|a_i\rangle &= a_i\,|a_i\rangle \\ \hat{B}\,|a_i\rangle &= b_i\,|a_i\rangle \end{aligned} \tag{6}$$

If you want a more neutral notation you can define $|a_i\rangle = |i\rangle$. We then have

$$\hat{A}|i\rangle = a_i|i\rangle \tag{7}$$
$$\hat{B}|i\rangle = b_i|i\rangle$$

So when the spectrum of $\hat{A}$ is non-degenerate, the eigenvectors of $\hat{A}$ are <u>automatically</u> eigenvectors of $\hat{B}$.

<u>Example</u>: $\hat{A} = \hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \hat{B} = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} = 2 - 3\hat{\sigma}_x$

These two matrices commute. The eigenvectors of $\hat{A}$ are automatically eigenvectors of $\hat{B}$.

When the eigenvalues of $\hat{A}$ are degenerate, things get trickier. But this is the most important case in practice so let us deal with it in detail.

Instead of (3) we now write

$$\hat{A}\,|a_{i,j}\rangle = a_i\,|a_{i,j}\rangle \qquad (8)$$

The index j labels the degeneracy. Each a_i may have a different degeneracy. So we let g_i be the degeneracy of a_i, in which case $j = 1, \dots, g_i$ for each i.

The simplest example is

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad (9a)$$

We then have

$$a_1 = 1 \qquad |a_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (9b)$$

$$a_2 = 2 \qquad |a_{2,1}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad |a_{2,2}\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (9c)$$

The key point is that the eigenvectors of degenerate eigenvalues are not unique. In fact, my choice in (9c) was just for simplicity. Any linear combination of eigenvectors is also an eigenvector

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2c_1 \\ 2c_2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix}$$

So for any c_1, c_2,

$$c_1 |a_{2,1}\rangle + c_2 |a_{2,2}\rangle$$

is also an eigenvector.

We say that the Hilbert space where $\hat{A}$ acts factors into subspaces corresponding to each eigenvalue a_i

The dimension of each subspace is g_i

Equation (4) continues to be valid

$$A\left(\hat{B}\,|a_{i,j}\rangle\right) = a_i\left(\hat{B}\,|a_{i,j}\rangle\right) \tag{10}$$

So that if $|a_{i,j}\rangle$ is an eigenvector with eigenvalue a_i, then so is $\hat{B}\,|a_{i,j}\rangle$.

But now we can no longer say that $\hat{B}\,|a_{i,j}\rangle \propto |a_{i,j}\rangle$, because any linear combination of the $|a_{i,j}\rangle$ will also be an eigenvector of $\hat{A}$ with eigenvalue a_i. So all we can say is that $\hat{B}\,|a_{i,j}\rangle$ will be some linear combination of the $|a_{i,j}\rangle$

$$\hat{B}\,|a_{i,j}\rangle = \sum_{j=1}^{g_i} b_{ij}\,|a_{ij}\rangle \tag{11}$$

This means that the matrix $\hat{B}$ only has matrix elements connecting states with the same a_i. For instance

$$\langle a_{1,j}\,|\,\hat{B}\,|\,a_{7,k}\rangle = 0$$

or any j, k. This is a consequence of $[\hat{A},\hat{B}]=0$:

$$0 = \langle a_{i,j}|\,[\hat{A},\hat{B}]\,|\,a_{k,e}\rangle = \langle a_{i,j}|\hat{A}\hat{B}|a_{k,e}\rangle - \langle a_{ij}|\hat{B}\hat{A}|a_{ke}\rangle$$

$$= a_i\langle a_{i,j}|\hat{B}|a_{k,e}\rangle - a_k\langle a_{ij}|\hat{B}|a_{ke}\rangle$$

that is

$$(a_i - a_k)\langle a_{i,j}|\hat{B}|a_{k,l}\rangle = 0 \tag{12}$$

So if $a_i \neq a_k$ it must mean that $\langle a_{i,j}|\hat{B}|a_{k,l}\rangle = 0$.

We thus conclude that in the basis $|a_{i,j}\rangle$ the matrix $\hat{B}$ factors into blocks

$$\hat{B} = \begin{bmatrix} \boxed{B_1} & & & \\ & \boxed{B_2} & & \\ & & \boxed{B_3} & \\ & & & \ddots \end{bmatrix} \tag{13}$$

Each block correspond to a given a_i and has dimension g_i. the block do not communicate with each other. However, within each block, $\hat{B}$ will not necessarily be diagonal. For instance, given $\hat{A}$ in (9a), suppose

$$\hat{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix} \tag{14}$$

This matrix commutes with $\hat{A}$, but the lower 2×2 block is not diagonal.

Thus we conclude that when there is degeneracy the eigenvectors of $\hat{A}$ are <u>not</u> <u>necessarily</u> eigenvectors of $\hat{B}$

But not all hope is lost because the eigenvectors of $\hat{A}$ with degenerate eigenvalues are not unique. We can find linear combinations of them which <u>are</u> eigenvectors of $\hat{B}$. This is <u>always possible</u> because $\hat{B}$ factors into blocks.

For instance, in our example the submatrix $\hat{B}$ within the subspace of $a_2 = 2$ is

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = 1 - 2\hat{\sigma}_x$$

This matrix has eigenvalues and eigenvectors

$$-1 \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|a_{2,1}\rangle + |a_{2,2}\rangle}{\sqrt{2}}$$

$$3 \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|a_{2,1}\rangle - |a_{2,2}\rangle}{\sqrt{2}}$$

Thus, instead of using $|a_1\rangle$, $|a_{2,1}\rangle$ and $|a_{2,2}\rangle$ in Eq (9) we could use the eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{15}$$

These eigenvectors are simultaneously eigenvectors of $\hat{A}$ and $\hat{B}$. You see, it is <u>always possible</u> to diagonalize $\hat{A}$ and $\hat{B}$ simultaneously when $[\hat{A}, \hat{B}] = 0$.

It is customary to label the eigenvectors as $|a_i b_j\rangle$ so we write

$$\boxed{\begin{aligned} \hat{A}|a_i b_j\rangle &= a_i |a_i b_j\rangle \\ \hat{B}|a_i b_j\rangle &= b_j |a_i b_j\rangle \end{aligned}} \tag{16}$$

For our example we have

$$|1,3\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

because

$$\hat{A}|1,3\rangle = 1\,|1,3\rangle \qquad \hat{B}|1,3\rangle = 3\,|1,3\rangle$$

Moreover

$$|2,-1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad |2,3\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The first index is the eigenvalue of $\hat{A}$ and the second is the eigenvalue of $\hat{B}$.

The example of $\hat{A}$ in Eq (9) was sort of easy because $\hat{A}$ was already diagonal. But the procedure for more general cases is the same. First we find the $|a_{i,j}\rangle$ and then, for each a_i, we construct the submatrix B_i in the subspace spanned by the $|a_{i,j}\rangle$. We then diagonalize this matrix to find the $|a_i b_j\rangle$.

<u>Example</u>: $\hat{A} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad \hat{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

The eigenvalues and eigenvectors of $\hat{A}$ are (please check)

$a_1 = 0 \qquad |a_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$|a_{2,1}\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$a_2 = 1$

$|a_{2,2}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Since a_1 is not degenerate and $[\hat{A}, \hat{B}] = 0$ we expect that $|a_1\rangle$ will also be an eigenvector of $\hat{B}$. Indeed

$$\hat{B}|a_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = 2|a_1\rangle.$$

So one eigenvalue of $\hat{B}$ is $b_1 = 2$.

On the other hand, since a_2 is degenerate, $|a_{2,1}\rangle$ and $|a_{2,2}\rangle$ will not necessarily be eigenvectors of $\hat{B}$ (maybe we are lucky and they turn out to be; but in general they will not). Indeed

$$\hat{B}|a_{2,1}\rangle = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

we can write this as

$$\hat{B}|a_{2,1}\rangle = 2|a_{2,1}\rangle - \sqrt{2}\,|a_{2,2}\rangle$$

Similarly

$$\hat{B}|a_{2,2}\rangle = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

which we may write as

$$\hat{B}|a_{2,2}\rangle = 2|a_{2,2}\rangle - \sqrt{2}\,|a_{2,1}\rangle$$

Note how $\hat{B}$ is fully contained within the block $\{|a_{2,1}\rangle, |a_{2,2}\rangle\}$ there is no connection with $|a_1\rangle$. This is exactly what we expected.

We must now find linear combinations of $|a_{z,1}\rangle$ and $|a_{z,2}\rangle$ which diagonalize $\hat{B}$. To do this we write the sub matrix $\hat{B}_2$

$$\hat{B}_2 = \begin{bmatrix} \langle a_{z,1} | \hat{B} | a_{z,1} \rangle & \langle a_{z,1} | \hat{B} | a_{z,2} \rangle \\ \langle a_{z,2} | \hat{B} | a_{z1} \rangle & \langle a_{z2} | \hat{B} | a_{z2} \rangle \end{bmatrix}$$

$$\therefore \quad \hat{B}_2 = \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} = 2 - \sqrt{2}\,\hat{\sigma}_x$$

Sanity check: $\hat{B}_2$ is Hermitian since $\hat{B}$ is Hermitian.

he eigenvectors of $\hat{B}_2$ are the eigenvectors of $\hat{\sigma}_x$; ie

$$2 - \sqrt{2} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|a_{z,1}\rangle + |a_{z,2}\rangle}{\sqrt{2}}$$

$$2 + \sqrt{2} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|a_{z1}\rangle - |a_{z,2}\rangle}{\sqrt{2}}$$

hus we conclude that the eigenvectors of $\hat{B}$ are

$b_1 = 2$

$b_2 = 2 - \sqrt{2}$

$b_3 = 2 + \sqrt{3}$

Thus we finally conclude that the 3 eigenvectors which simultaneously diagonalize $\hat{A}$ and $\hat{B}$ are

$$\Rightarrow \quad |a_1, b_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \begin{matrix} a_1 = 0 \\ b_1 = 2 \end{matrix}$$

$$|a_2 b_2\rangle = \frac{|a_{21}\rangle + |a_{22}\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad |a_2 b_2\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \qquad \begin{matrix} a_2 = 1 \\ b_2 = 2 - \sqrt{2} \end{matrix}$$

$$|a_2 b_3\rangle = \frac{|a_{21}\rangle - |a_{22}\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad |a_2 b_3\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \qquad \begin{matrix} a_2 = 1 \\ b_3 = 2 - \sqrt{2} \end{matrix}$$

These 3 vectors form a orthonormal basis and simultaneously diagonalize $\hat{A}$ and $\hat{B}$.

Maximal set of commuting observables

If we have an observable $\hat{A}$ and we find that its eigenvalues are degenerate, then there is an ambiguity in the choice of the eigenvectors. But we can find another operator $\hat{B}$ such that $[\hat{A}, \hat{B}] = 0$. We then diagonalize both simultaneously. This "clarifies" the eigenvectors of $\hat{A}$ in the sense that it removes the ambiguity.

However, it is possible that this degeneracy continues, either in $\hat{A}$ or in $\hat{B}$ or in both. In this case we need to find another observable $\hat{C}$ which commutes with $\hat{A}$ and $\hat{B}$:

$$[\hat{A}, \hat{B}] = [\hat{A}, \hat{C}] = [\hat{B}, \hat{C}] = 0$$

We then diagonalize the 3 operators simultaneously. If this removes all ambiguities we are done. Otherwise we need to find yet another operator $\hat{D}$ which commutes with everyone, and then repeat the process.

The final set of operators $\hat{A}, \hat{B}, \hat{C}, \ldots$ which all commute with each other and which, when diagonalized simultaneously, removes all possible ambiguities in the eigenvectors, is called a maximal set of commuting operators. When you have found this set you cannot add any other operators to the bucket, unless they be simple combinations of the $\hat{A}, \hat{B}, \hat{C}, \ldots$

The meaning of "compatible"

Suppose we have 2 operators which do not commute. Then they cannot be diagonalized simultaneously, so we write

$$\hat{A}\,|a_i\rangle = a_i\,|a_i\rangle \tag{17a}$$

$$\hat{B}\,|b_j\rangle = b_j\,|b_j\rangle \tag{17b}$$

Suppose now that the system in in a certain state $|\alpha\rangle$ and we measure $\hat{B}$. The probability of observing b_i is

$$P(\alpha \to b_j) = |\langle b_j|\alpha\rangle|^2 \tag{18}$$

Now suppose instead of measuring $\hat{B}$ we first measure $\hat{A}$ and then $\hat{B}$. When we measure $\hat{A}$ we get

$$P(\alpha \to a_i) = |\langle a_i|\alpha\rangle|^2$$

Afterwards, the system will be in $|a_i\rangle$ (I'm assuming no degeneracy for now; I will teach you what to do when there is degeneracy later). Thus, when we measure $\hat{B}$ right afterwards we get

$$P(\alpha \to a_i \to b_j) = |\langle b_j|a_i\rangle|^2\,|\langle a_i|\alpha\rangle|^2$$

Now suppose we ask; what is the prob. of getting b_j irrespectively of the value we obtained for $\hat{A}$? This can be computed by summing over all values of a_i

$$P'(\alpha \to b_j) = \sum_i P(\alpha \to a_i \to b_j)$$

$$= \sum_i \langle b_j | a_i \rangle \langle a_i | b_j \rangle \langle a_i | \alpha \rangle \langle \alpha | a_i \rangle \qquad (19)$$

we can now compare this with (18). In (18) we didn't measure $\hat{A}$ and in (19) we measured it but didn't care about the values it took, so we then averaged over all possible values of a_i. Are the two similar. In classical mechanics they would certainly be. But here they are not! To see this recall that the $|a_i\rangle$ form a basis so we may write (18) as

$$P(\alpha \to b_j) = \langle b_j | \alpha \rangle \langle \alpha | b_j \rangle$$

$$= \sum_{i,k} \langle b_j | a_i \rangle \langle a_i | \alpha \rangle \langle \alpha | a_k \rangle \langle a_k | b_j \rangle \qquad (20)$$

This is not the same as (19). Here there is a double sum. Thus, measuring $\hat{A}$ affects the outcomes of $\hat{B}$. They are incompatible. This is the essence of quantum mechanics.

Now I'm going to show that when $[\hat{A}, \hat{B}]=0$ it doesn't matter if we measure $\hat{A}$ or not; it will not affect the measurements of $\hat{B}$.

This is easy to see in the case where the spectrum of $\hat{A}$ is non-degenerate. In this case $|b_j\rangle = |a_j\rangle$ so Eq (19) becomes

$$P'(\alpha \to b_j) = \sum_i \delta_{ji} \, |\langle a_i|\alpha\rangle|^2 = |\langle a_j|\alpha\rangle|^2 \tag{21}$$

which is the same as (18). Thus, the probabilities are the same whether you measure $\hat{A}$ or not.

In the case of degeneracy it is a little bit harder. Now

$$\begin{aligned} \hat{A}\,|a_i\, b_j\rangle &= a_i\,|a_i b_j\rangle \\ \hat{B}\,|a_i\, b_j\rangle &= b_j\,|a_i\, b_j\rangle \end{aligned} \tag{22}$$

Suppose we now measure $\hat{A}$. What is the probability of finding a_i? And what is the state after the measurements? For some reason these questions are usually avoided in most books. The probability of getting a_i is

$$p_i = P(\alpha \to a_i) = \sum_{j=1}^{g_i} |\langle a_i\, b_j|\alpha\rangle|^2 \tag{23}$$

You simply sum over all possibilities.

As far the final state after the measurement, the answer is actually a bit trickier:

$$|\alpha\rangle \longrightarrow \frac{1}{\sqrt{P_i}} \sum_{j=1}^{g_i} |a_i b_j\rangle\langle a_i b_j|\alpha\rangle \qquad (24)$$

Compare this to the non-degenerate case

$$|\alpha\rangle \rightarrow |a_i\rangle\langle a_i|\alpha\rangle$$

They are very similar. The big difference is the term $1/\sqrt{P_i}$ which is used for normalization purposes. Thus, when we measure $\hat{B}$ after $\hat{A}$ we must get one of the states in (24); ie

$$P(a_i \rightarrow b_j) = \frac{|\langle a_i b_j|\alpha\rangle|^2}{P_i}$$

Hence,

$$P(\alpha \rightarrow a_i \rightarrow b_j) = \frac{|\langle a_i b_j|\alpha\rangle|^2}{P_i} P_i = |\langle a_i b_j|\alpha\rangle|^2$$

Summing over i:

$$P'(\alpha \rightarrow b_j) = \sum_i |\langle a_i b_j|\alpha\rangle|^2$$

which is the same we would have obtained if we only had measured $\hat{B}$. Similarly to (23), in this case we would have obtained

$$P(\alpha \rightarrow b_j) = \sum_i |\langle a_i b_j|\alpha\rangle|^2$$

Thus, compatible really means <u>measuring</u> <u>one</u> <u>does</u> <u>not</u> <u>interfere</u> <u>with</u> <u>the</u> <u>other</u>.