Momentum and energy

Generalized momentum

Let us write the Euler-Lagrange equations as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i} \qquad (1)$$

To each generalized coordinate q_i, we define the corresponding generalized momentum as

$$\boxed{p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}} \qquad (2)$$

Then Eq (1) becomes

$$\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i} \qquad (3)$$

If we use cartesian coordinates the generalized momentum will be the usual momentum:

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U \qquad (4)$$

$$\Rightarrow \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i \qquad (5)$$

Or, in vector notation,

$$\mathbf{p} = m\dot{\mathbf{r}} = m\mathbf{v} \qquad (5')$$

But if we use different types of coordinates we may obtain all sorts of crazy expressions. For instance, a particle in spherical coordinates has the Lagrangian

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) - U \tag{6}$$

Applying Eq (2) we then get

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \tag{7a}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} \tag{7b}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} \sin^2\theta \tag{7c}$$

So you see, the formulas are not very intuitive. We must always trust the definition (2).

By the way, note that the units of p_i depend on the units of q_i. Notwithstanding, the product $q_i p_i$ always has the same units

$$[q_i p_i] = \text{energy} \times \text{time} = \text{J}\cdot\text{s} \tag{8}$$

This combination of dimensions appear a <u>lot</u> in physics. The action has this energy and so does $\hbar$.

Now let us go back to Eq (3):

$$\frac{dp_i}{dt} = \frac{\partial \mathcal{L}}{\partial q_i}$$

We see that if $\mathcal{L}$ does not depend on q_i, then

$$\frac{dp_i}{dt} = 0 \qquad (9)$$

ie,

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \Rightarrow \quad p_i = \text{constant}} \qquad (10)$$

This is a conservation law. And it is very easy to use: just look at $\mathcal{L}$. If q_i does not appear in $\mathcal{L}$, then p_i is a constant.

When a coordinate q_i does not appear in $\mathcal{L}$, we call it a cyclic coordinate. So the generalized momentum associated to a cyclic coordinate is a conserved quantity.

For example, a particle under the effect of gravity has the Lagrangian

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \qquad (11)$$

Both x and y do not appear in this formula so p_x and p_y are conserved quantities. Their values are fixed by the initial conditions and then remain fixed through the rest of the motion. But z, on the other hand, is not a cyclic variable so p_z is not a constant: the particle accelerates downwards.

For cartesian coordinates we have $\frac{\partial \mathcal{L}}{\partial x_i} = F_i$ so

$$\frac{dp_i}{dt} = F_i$$

Thus, conservation of momentum implies that $F_i = 0$: when no force acts on the particle, its momentum is conserved. This is the same as (10) because $F_i = 0$ is equivalent to $\frac{\partial \mathcal{L}}{\partial x_i} = 0$.

But the difference is that (10) holds for generalized coordinates, whereas the Newtonian interpretation is valid only for cartesian coordinates.

Symmetries and conservation laws: Nöether's theorem

Symmetry is the most important concept in all of physics. And the more advanced is a theory, the more important is the role played by symmetry. I strongly recommend that you dedicate your time to this important subject

The meaning of symmetry is the following:

> "A symmetry is a transformation which does not alter the equations of motion"

When this happens we say "This system is invariant under this or that symmetry".

As an example, take a free particle. It is invariant under translations, because if you take it from here to there, it is still a free particle. A harmonic oscillator, on the other hand, is not invariant under translations: if you displace it you stretch the spring!

Let us now ask under what conditions may $\mathcal{L}$ change so as to keep the Euler-Lagrange equations invariant. Of course, if $\mathcal{L}$ does not change at all, than that is great. But we may also have more general changes which leave the Euler-Lagrange equations invariant. For instance, consider a change of the form

$$\mathcal{L} \to \mathcal{L}' = \alpha \mathcal{L} + \beta \qquad (12)$$

where α and β are constants.

A multiplicative or an additive constant change nothing in the motion. This shows, for instance, that the choice of the zero of U is irrelevant.

There is also a more general way to change $\mathcal{L}$ which does not affect the motion: namely if

$$\boxed{\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{d}{dt} f(q_1, \dots, q_N, t)} \tag{13}$$

where f is an arbitrary function of q_i and t.

I will demonstrate this result in two ways. First, let us look at the actions corresponding to $\mathcal{L}$ and $\mathcal{L}'$:

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

and

$$S' = \int_{t_1}^{t_2} \mathcal{L}'\, dt = \int_{t_1}^{t_2} \mathcal{L}\, dt + \int_{t_1}^{t_2} \frac{df}{dt}\, dt$$

Thus we see that

$$S' = S + f(q_i(t_2), t_2) - f(q_i(t_1), t_1) \tag{14}$$

This means that the two actions differ only by a function of the end-points. But to obtain the Euler-Lagrange equations we compute δS with fixed end-points, so these terms will not contribute. Thus

$$\delta S = \delta S'$$

and consequently $\mathcal{L}$ and $\mathcal{L}'$ will both give the same equations of motion

Note that f can depend on q_i and t, but not on $\dot{q}_i$. This must be so because we fix $q_i(t_2)$ and $q_i(t_1)$, but not their derivatives.

If you are not convinced by this proof, we can also check it directly. For simplicity I will do this assuming a single variable q. Then

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}$$

Thus

$$\frac{\partial \mathcal{L}'}{\partial q} = \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial}{\partial q}\left(\frac{df}{dt}\right) = \frac{\partial \mathcal{L}}{\partial q} + \frac{\partial^2 f}{\partial q^2}\dot{q}^2 + \frac{\partial^2 f}{\partial q \partial t}$$

$$\frac{\partial \mathcal{L}'}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}}\left(\frac{df}{dt}\right) = \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial f}{\partial q}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial q}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) + \frac{\partial^2 f}{\partial q^2}\dot{q} + \frac{\partial^2 f}{\partial t^2}$$

Thus

$$\frac{\partial \mathcal{L}'}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}'}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

The equations of motion are the same; qed.

Ok! Now let's go back to our discussion about symmetries. There is a very powerful theorem due to Nöether which states that:

$$\boxed{\text{Nöether's theorem: to every symmetry there is an associated conserved quantity}}$$

We have already seen an example of this: we saw that when L does not depend on q_i, the corresponding momentum p_i is conserved. In this case the Lagrangian is invariant under the symmetry transformation $q_i \to q_i + \epsilon$. This symmetry implies a conserved quantity (p_i).

Depending on the symmetry, the corresponding conserved quantity may have a deep physical meaning. The four most important symmetries are

Symmetry	Conserved quantity
Time translations	Energy
Space translations	Momentum
Rotations	Angular momentum
Gauge invariance	Electric charge

In this course we will discuss all of these symmetries, one at a time.

Energy and the invariance under time translations

Consider a general Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t) \tag{15}$$

If $\mathcal{L}$ does not depend explicitly on time, we say the system is invariant under time translations. The reason is that, in this case, what you call $t=0$ is irrelevant, so you can translate the origin of time at will.

Let us compute the total time derivative of Eq (15):

$$\frac{d\mathcal{L}}{dt} = \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right\} + \frac{\partial \mathcal{L}}{\partial t} \tag{16}$$

Now we use the Euler-Lagrange equations to write

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$$

we get

$$\frac{d\mathcal{L}}{dt} = \sum_i \left\{ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i \right\} + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

I will write this as

$$\frac{d}{dt}\left(\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \right) = - \frac{\partial \mathcal{L}}{\partial t}$$

We now define a new quantity called the <u>energy</u> of the system as

$$E = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \tag{18}$$

or

$$\boxed{E = \sum_i p_i \dot{q}_i - \mathcal{L}} \tag{18'}$$

We then conclude that

$$\frac{dE}{dt} = - \frac{\partial \mathcal{L}}{\partial t} \tag{19}$$

If the Lagrangian is invariant under time translations then $\frac{\partial \mathcal{L}}{\partial t} = 0$ which implies that E = constant

$$\boxed{\frac{\partial \mathcal{L}}{\partial t} = 0 \quad \Rightarrow \quad E = \text{constant}} \tag{20}$$

This is a manifestation of Nöether's theorem. Invariance under time-translation (the symmetry) implies that energy is conserved (the conservation law)

Let us now compute the energy for a few systems. For a particle in cartesian coordinates

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U$$

so Eq (18') gives

$$E = p_x \dot{x} + p_y \dot{y} + p_z \dot{z} - \mathcal{L}$$

$$= m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U$$

Thus we arrive at the intuitive result

$$E = T + U = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U \qquad (21)$$

The total energy is simply the kinetic energy plus the potential energy.

In fact this is true in most cases, but there are exceptions, I therefore recommend that you always follow the definition.

In spherical coordinates

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) - U$$

so

$$E = (m\dot{r})\dot{r} + (mr^2\dot{\theta})\dot{\theta} + (mr^2\dot{\phi}\sin^2\theta)\dot{\phi} - \mathcal{L}$$

we again obtain

$$E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) + U \qquad (22)$$

Relativistic free particle

Nothing propagates faster than the velocity of light c. This is not included in classical mechanics. The theory which includes this is called <u>special relativity</u>. The Lagrangian of a free particle in special relativity is

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} \tag{23}$$

where

$$v^2 = v_x^2 + v_y^2 + v_z^2 \tag{24}$$

Let us first check what happens when $v << c$. Using the Taylor series expansion

$$\sqrt{1 - x^2} \simeq 1 - \frac{1}{2}x^2$$

we get

$$\mathcal{L} \simeq -mc^2\left[1 - \frac{1}{2}\frac{v^2}{c^2}\right]$$

or

$$\mathcal{L} = -mc^2 + \frac{1}{2}mv^2 \tag{25}$$

The first term is a constant and therefore is irrelevant. The second term is the kinetic energy we are used to. So when $v << c$ we recover the results of classical mechanics. That makes sense.

Now let us compute the generalized momenta. We write

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{(v_x^2 + v_y^2 + v_z^2)}{c^2}}$$

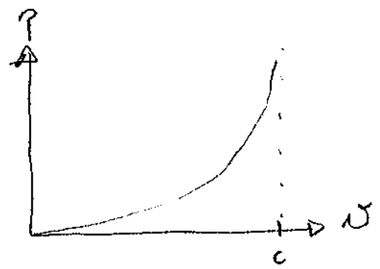
Then

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} = -mc^2 \frac{1}{2} \frac{(-2v_i/c^2)}{\sqrt{1-v^2/c^2}}$$

Thus

$$\boxed{p_i = \frac{mv_i}{\sqrt{1-v^2/c^2}}} \tag{26}$$

So the momentum in relativity will not be mv_i. It will be close to this when $v << c$, but not so otherwise.



When $v \sim c$ the momentum becomes huge.

Next let us compute the energy. From (18') we have

$$E = \sum_i p_i v_i - \mathcal{L}$$

$$= \sum_i \frac{m v_i^2}{\sqrt{1 - v^2/c^2}} + mc^2 \sqrt{1 - v^2/c^2}$$

$$= \frac{m v^2 + mc^2 (1 - v^2/c^2)}{\sqrt{1 - v^2/c^2}}$$

Thus

$$\boxed{E = \frac{mc^2}{\sqrt{1 - v^2/c^2}}} \qquad (27)$$

At zero speed the energy tends to a finite value

$$E = mc^2 \qquad (28)$$

which is a formula you may have seen before.

For $v << c$ we expand

$$E \simeq mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2} \right)$$

or

$$E = mc^2 + \frac{1}{2} m v^2 \qquad (29)$$

So, except for the constant mc^2, we again recover the classical result.

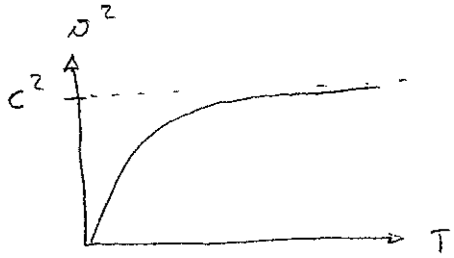
We may define the kinetic energy as

$$T = E - mc^2 = mc^2\left[\frac{1}{\sqrt{1-v^2/c^2}} - 1\right] \tag{30}$$

If you want to invert this you get

$$v^2 = c^2 T \frac{(T + 2mc^2)}{(T + mc^2)^2} \tag{31}$$

this function looks like this



You can control T by accelerating a particle through a potential difference. This result then shows that as you increase the energy, the velocity saturates and never surpasses c.

Here is some real data for electrons

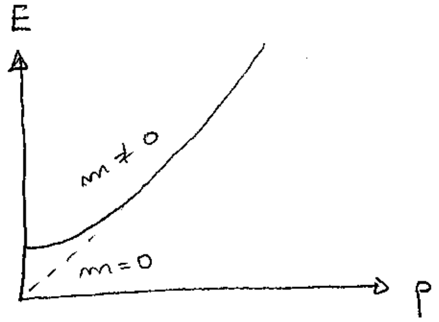
T (MeV)	v^2 ($\times 10^{16}$ m^2/s^2)
0.5	6.8
1.0	7.5
1.5	8.3
4.5	8.8
15	9.0

Note how it saturates around c^2 ($c \simeq 3\times10^8$ m/s)

It is useful to relate energy with momentum. We have:

$$\boxed{E = \sqrt{m^2 c^4 + p^2 c^2}} \tag{32}$$

Please check that this actually works. This function looks like this.



We say that close to $p = 0$ there is a gap between $E = 0$ and $E = mc^2$. This gap goes to zero when $m \to 0$.

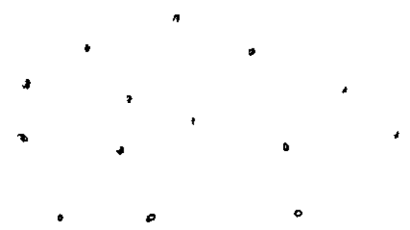
In fact, even though we started with a Lagrangian for a massive particle, we see that Eq (32) predicts the existence of particles with $m = 0$. For these particles

$$E = pc \tag{33}$$

these are the photons of light.

Momentum and the invariance under space translations

Now let us consider a system of N particles isolated from the rest of the world but interacting among themselves. This could for instance be a galaxy, where each particle represents a star.



This system shows invariance under space translations, because if you take all particles and translate by the same amount, the physics does not change. This is a consequence of the fact that the interaction potential depends only on the relative distance between the particles. So if you translate all particles by the same amount, their relative distances do not change.

Let us now state this mathematically. Let $\mathbf{r}_a$ be the position vector of the a-th atom where $a = 1, \dots, N$. The Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2} \sum_a m_a v_a^2 - U(\mathbf{r}_1, \dots, \mathbf{r}_N) \tag{34}$$

Usually, the potential depends on the relative distance between the particles. Something like this.

$$U = \frac{1}{2} \sum_{a, b \neq a} V_{ab}(\mathbf{r}_a - \mathbf{r}_b) \tag{35}$$

where we put the 1/2 so that we don't double count any of the terms.

For the gravitational force

$$V_{ab}(\mathbf{r}_a - \mathbf{r}_b) = -\frac{G\, m_a m_b}{|\mathbf{r}_a - \mathbf{r}_b|} \qquad (36)$$

But I want to keep the discussion general. I want to impose that a given Lagrangian $\mathcal{L}(\mathbf{r}_a, \mathbf{v}_a)$ is invariant under the simultaneous translation of all particles. In symbols this reads

$$\mathbf{r}_a \rightarrow \mathbf{r}_a' = \mathbf{r}_a + \boldsymbol{\epsilon} \qquad (37)$$

where ϵ is a constant vector. The Lagrangian will correspondingly change to

$$\mathcal{L}' = \mathcal{L}(\mathbf{r}_a + \boldsymbol{\epsilon}, \mathbf{v}_a)$$

We only need to consider an infinitesimal translation because we can always compose a finite translation into a sequence of infinitesimal ones. In fact, this is a trick used very often: always study infinitesimal transformations.

Expanding $\mathcal{L}'$ in a Taylor series we get

$$\mathcal{L}' = \mathcal{L} + \sum_a \frac{\partial \mathcal{L}}{\partial \mathbf{r}_a} \cdot \boldsymbol{\epsilon} \qquad (38)$$

We now impose that $\mathcal{L}$ and $\mathcal{L}'$ give the same physics for any vector ϵ. This implies that

$$\boxed{\sum_a \frac{\partial \mathcal{L}}{\partial \mathbb{r}_a} = 0} \tag{39}$$

But, from Newton's law

$$\frac{\partial \mathcal{L}}{\partial \mathbb{r}_a} = \mathbb{F}_a$$

so we conclude that

$$\boxed{\sum_a \mathbb{F}_a = 0} \tag{40}$$

This is <u>Newton's</u> <u>third</u> <u>law</u>. For instance, if we have only two particles we get

$$\mathbb{F}_1 + \mathbb{F}_2 = 0 \tag{41}$$

To every action there is an equal and opposite reaction.

Continuing, we also know from Newton's second law that

$$\mathbb{F}_a = \frac{d}{dt}(m_a \mathbb{v}_a)$$

Thus, (40) also implies that

$$\sum_a \frac{d}{dt}(m_a \mathbb{v}_a) = 0$$

We define the total momentum of the system as

$$\boxed{\mathbf{P} = \sum_a m_a \mathbf{v}_a} \qquad (42)$$

We then conclude that

$$\boxed{\frac{d\mathbf{P}}{dt} = 0} \qquad (43)$$

or

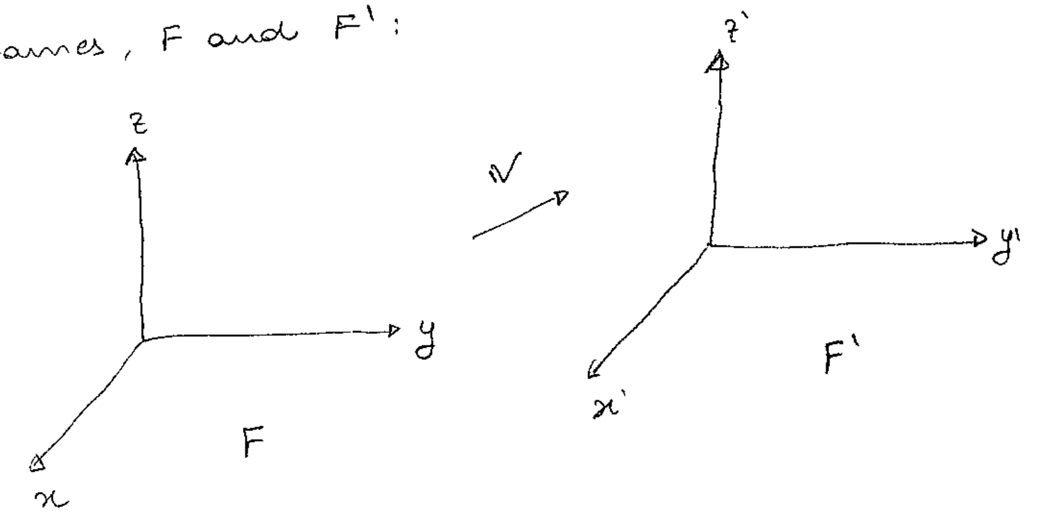
$$\boxed{\text{Invariance under space translations} \Rightarrow \mathbf{P} = \text{constant}}$$

this is another manifestation of Nöether's theorem.

Galileo transformations

Let us now discuss in more detail the concept of frames of reference. In practice we always work with inertial frames. An inertial frame is defined as being such that a free particle moves with constant velocity.

This means that two frames of reference move with respect to each other with a constant velocity. Consider two frames, F and F':



Let $\mathbb{r}$ and $\mathbb{r}'$ be the coordinates of a particle in F and F' respectively. They are related as follows:

$$\boxed{\mathbb{r} = \mathbb{r}' + \mathbb{v}\, t} \tag{44}$$

where we have chosen our clock in such a way that the two sistems coincide at $t=0$.

Eq. (44) is called a Galileo transformation. It is a postulate of classical mechanics that the laws of physics should be invariant under Galileo transformations.

This turns out not to be true in special relativity. You can see this quite clearly by computing the relative velocities from (44):

$$\boxed{v = v' + V} \tag{45}$$

Tune v' and V appropriately and you can easily tune v to become larger than c.

So (44) and (45) are postulates of classical mechanics but only hold at low speeds.

Now let me show you something cool. We have already discussed how the Lagrangian of a free particle can only depend on the absolute value of v. Let us write this as $\mathcal{L} = \mathcal{L}(v^2)$ [we could have written $\mathcal{L} = \mathcal{L}(v^9)$ or whatever. But it is convenient to use v^2.]

The Lagrangian in the frames F and F' are related by

$$\mathcal{L}(v^2) = \mathcal{L}(v'^2 + 2v' \cdot V + V^2)$$

Suppose V is infinitesimal and expand

$$\mathcal{L}(v^2) \simeq \mathcal{L}(v'^2) + \frac{\partial \mathcal{L}}{\partial v^2}(2v' \cdot V) \tag{46}$$

We now go back to Eq (13): for the physics to be the same, the two Lagrangians may differ by at most the total time derivative of a function of r and t.

the only way this will be true for (46) is if

$$\frac{\partial \mathcal{L}}{\partial v'^2} = \text{constant}$$

because then

$$\mathcal{L}(v^2) \simeq \mathcal{L}(v'^2) + \frac{d}{dt}(\text{const} \times 2\mathbf{r}\cdot\mathbf{V})$$

Thus we conclude that if a system is to be invariant under Galileo transformations, we must have

$$\mathcal{L} \propto v^2$$

We cannot say what the constant of proportionality is. That is an experimental information. But I find it quite interesting to note that the fact that the kinetic energy is $\frac{1}{2}mv^2$ is a consequence of Galileo invariance!

Let us look now at

$$\mathcal{L} = \frac{1}{2}mv^2$$

Using (45) we get

$$\mathcal{L} = \frac{1}{2}m\left(v'^2 + 2v'\cdot\mathbf{V} + V^2\right)$$

which we may write as

$$\mathcal{L} = \frac{1}{2}mv'^2 + \frac{d}{dt}\left[m\mathbf{r}'\cdot\mathbf{V} + \frac{1}{2}mV^2t\right]$$

which shows that $\mathcal{L}$ and $\mathcal{L}'$ differ only by the time derivative of a function of positions and time

Now let us go back to the definition of the total momentum in (42):

$$\mathbb{P} = \sum_a m_a v_a$$

If we use (45) to move to the frame F' we get

$$\mathbb{P} = \sum_a m_a v_a' + \sum_a m_a \mathbb{V}$$

Define the total mass as

$$M = \sum_a m_a \tag{47}$$

Then we conclude that

$$\boxed{\mathbb{P} = \mathbb{P}' + M\mathbb{V}} \tag{48}$$

This is the law of transformation of momentum from one frame to another.

We can always choose $\mathbb{V}$ such that in the frame F' the system as a whole is at rest. To do this we set $\mathbb{P}' = 0$ to obtain

$$\boxed{\mathbb{V} = \frac{\mathbb{P}}{M} = \frac{\sum_a m_a v_a}{\sum_a m_a}} \tag{49}$$

This is the velocity which the system moves as whole, as measured from the frame F.

Eq (49) shows that the relation of $\mathbb{V}$ and $\mathbb{P}$ for a system of particles is the same as that of a single particle with mass M. This is the <u>additivity of mass</u>. It is true in classical mechanics but <u>not</u> true in relativity.

Let us define a quantity $\mathbb{R}$ such that $\mathbb{V} = \dot{\mathbb{R}}$. Then

$$\boxed{\mathbb{R} = \frac{\sum_a m_a \mathbb{r}_a}{\sum_a m_a}} \qquad (50)$$

This is the position of the <u>center of mass</u>. The conservation of momentum (43) implies that the center of mass moves with constant velocity $\mathbb{V}$.

The two-body problem

Let us practice what we learned by considering a very important problem in physics. Namely that of two particles interacting by a potential which depends only on their distance. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{51}$$

where

$$\dot{\mathbf{r}}_1^2 = \dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 = \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2$$

The form of U is inviting us to use as generalized coordinate

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{52}$$

which is the distance between the two particles.

But we also need another set of generalized coordinates. It is convenient to choose these as the center of mass coordinate

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{53}$$

Let us invert (45) and (46):

$$\mathbf{R} - \frac{m_1 \mathbf{r}}{m_1 + m_2} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} - \frac{m_1 \mathbf{r}_1 - m_1 \mathbf{r}_2}{m_1 + m_2} = \frac{(m_2 + m_1)\, \mathbf{r}_2}{m_1 + m_2} = \mathbf{r}_2$$

Thus

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1 \mathbf{r}}{m_1 + m_2}$$

Similarly

$$\mathbb{R} + \frac{m_2 \mathbb{r}}{m_1 + m_2} = \frac{m_1 \mathbb{r}_1 + m_2 \mathbb{r}_2}{m_1 + m_2} + \frac{m_2 \mathbb{r}_1 - m_2 \mathbb{r}_2}{m_1 + m_2} = \mathbb{r}_1$$

thus, the transformations are

$$\boxed{\begin{aligned} \mathbb{r} &= \mathbb{r}_1 - \mathbb{r}_2 & \qquad \mathbb{r}_1 &= \mathbb{R} + \frac{m_2 \mathbb{r}}{m_1 + m_2} \\ \mathbb{R} &= \frac{m_1 \mathbb{r}_1 + m_2 \mathbb{r}_2}{m_1 + m_2} & \qquad \mathbb{r}_2 &= \mathbb{R} - \frac{m_1 \mathbb{r}}{m_1 + m_2} \end{aligned}} \qquad (54)$$

the kinetic energy now becomes

$$\dot{\mathbb{r}}_1^2 = \dot{\mathbb{R}}^2 + \left(\frac{m_2}{m_1 + m_2}\right)^2 \dot{\mathbb{r}}^2 + \frac{2 m_2}{m_1 + m_2} \dot{\mathbb{R}} \cdot \dot{\mathbb{r}}$$

$$\dot{\mathbb{r}}_2^2 = \dot{\mathbb{R}}^2 + \left(\frac{m_1}{m_1 + m_2}\right)^2 \dot{\mathbb{r}}^2 - \frac{2 m_1}{m_1 + m_2} \dot{\mathbb{R}} \cdot \dot{\mathbb{r}}$$

Thus

$$\frac{1}{2} m_1 \dot{\mathbb{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbb{r}}_2^2 = \frac{1}{2} (m_1 + m_2) \dot{\mathbb{R}}^2 + \frac{1}{2} \left(\frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2} \right) \dot{\mathbb{r}}^2$$

The cross term cancels out

We see two "masses" naturally appearing. The total mass

$$M = m_1 + m_2 \tag{55}$$

and the <u>reduced mass</u>

$$\mu = \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1 + m_2)^2} = m_1 m_2 \frac{(m_1 + m_2)}{(m_1 + m_2)^2}$$

or

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{56}$$

In terms of our generalized coordinates the Lagrangian becomes

$$\boxed{\mathcal{L} = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(r)} \tag{57}$$

Note how the problem separates in two: the equations of motion for $\mathbf{R}$ will be independent of $\mathbf{r}$. In fact, we see that $\mathbf{R}$ is a cyclic coordinate because it does not appear explicitly on $\mathcal{L}$. As a consequence, the corresponding momentum will be a conserved quantity:

$$\mathbf{P} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{R}}} = M \dot{\mathbf{R}} = m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 \tag{58}$$

This is nothing but the total momentum

The two-body problem therefore reduces to two independent problems. One is the translation of a system with mass $M = m_1 + m_2$ through space and the other is the motion of a "particle" of mass μ in a central field $U(r)$.

If $m_1 \gg m_2$ as, for instance, in the problem of the earth and the sun, then

$$\mathbb{R} = \frac{m_1 \mathbb{r}_1 + m_2 \mathbb{r}_2}{m_1 + m_2} \approx \mathbb{r}_1$$

the center of mass will be close to $\mathbb{r}_1$. Similarly $M \approx m_1$.

But, as for the reduced mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \approx \frac{m_1 m_2}{m_1} = m_2$$

so the earth orbits the sun with a reduced mass that is practically it's own.

Mathematical appendix A: $\frac{d}{dt} \neq \frac{\partial}{\partial t}$

Allow me to clarify something:

$$\boxed{\frac{d}{dt} \text{ is not the same as } \frac{\partial}{\partial t}}$$

The meaning of the partial derivative is simple:

$$\boxed{\frac{\partial}{\partial x} = \text{differentiate with respect to } x \text{ while keeping everything else fixed}}$$

But the total derivative may have a more subtle meaning. Take for example a simple Lagrangian of the form $\mathcal{L}(q, \dot{q}, t)$. Here I wrote an explicit dependence on t. But even if it doesn't depend explicitly on t, there is always an implicit dependence since $q(t)$ and $\dot{q}(t)$ depend on t

Thus, in such cases, $\frac{\partial \mathcal{L}}{\partial t}$ will mean "differentiate with respect to t, while keeping all else fixed". In particular $\frac{\partial \mathcal{L}}{\partial t} = 0$ when $\mathcal{L}$ does not depend explicitly on t.

On the other hand, $\frac{d\mathcal{L}}{dt}$ is defined as

$$\boxed{\frac{d\mathcal{L}}{dt} = \lim_{\Delta t \to 0} \frac{\mathcal{L}(q(t+\Delta t), \dot{q}(t+\Delta t), t+\Delta t) - \mathcal{L}(q(t), \dot{q}(t), t)}{\Delta t}} \tag{A.1}$$

It is the total variation of $\mathcal{L}$ with time.

To compute this formula we may use a Taylor series expansion

$$q(t+\Delta t) \simeq q(t) + \dot{q}(t)\Delta t$$
$$\dot{q}(t+\Delta t) \simeq \dot{q}(t) + \ddot{q}(t)\Delta t$$

then we expand the Lagrangian as

$$\mathcal{L}(q(t+\Delta t), \dot{q}(t+\Delta t), t+\Delta t) = \mathcal{L}(q+\dot{q}\Delta t, \dot{q}+\ddot{q}\Delta t, t+\Delta t)$$
$$\simeq \mathcal{L}(q,\dot{q},t) + \frac{\partial \mathcal{L}}{\partial q}\dot{q}\Delta t + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q}\Delta t + \frac{\partial \mathcal{L}}{\partial t}\Delta t$$

whence, we conclude that

$$\boxed{\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial q}\dot{q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\ddot{q} + \frac{\partial \mathcal{L}}{\partial t}} \qquad (A.2)$$

The generalization to the case when there is more than one generalized coordinate is quite straightforward:

$$\boxed{\frac{d\mathcal{L}}{dt} = \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i \right\} + \frac{\partial \mathcal{L}}{\partial t}} \qquad (A.3)$$