Formal construction of second quantization

These notes are supposed to complement the previous set, which introduced 2^{nd} quantization informally. The main goal of these notes is to show you that there is no magic involved in 2^{nd} quantization. It is simply a natural extension of quantum mechanics for dealing with systems having a variable number of identical particles.

We start by labeling Hilbert spaces according to the number of particles in it. First we have a Hilbert space with no particles, which we call $\mathcal{H}_0$. It only has one state, the vacuum

$$|0\rangle \in \mathcal{H}_0 \tag{1}$$

Next we look at the Hilbert space of one particle, which we call $\mathcal{H}_1$. It is spanned by the single-particle kets $|\alpha\rangle, |\phi\rangle$, etc.:

$$|\alpha\rangle, |\phi\rangle, \ldots \in \mathcal{H}_1 \tag{2}$$

Then comes the space with 2 particles. This space is spanned by product states, like $|\alpha_1, \alpha_2\rangle$:

$$|\alpha_1, \alpha_2\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \in \mathcal{H}_2 \tag{3}$$

But now we impose that the particles are indistinguishable. As we have seen, this means that any state must either be symmetric or anti-symmetric, depending on whether we are dealing with Bosons or Fermions.

Take Bosons for instance. A general 2-particle state may be written as

$$|\psi\rangle = \sum_{\alpha_1,\alpha_2} c(\alpha_1,\alpha_2)\,|\alpha_1,\alpha_2\rangle \qquad (4)$$

But since $c(\alpha_1,\alpha_2) = c(\alpha_2,\alpha_1)$ we can group this as

$$|\psi\rangle = \frac{1}{2}\sum_{\alpha_1,\alpha_2} c(\alpha_1,\alpha_2)\left[|\alpha_1,\alpha_2\rangle + |\alpha_2,\alpha_1\rangle\right] \qquad (5)$$

we therefore see that, for Bosons, any $|\psi\rangle$ will always depend on symmetric combinations of $|\alpha_1,\alpha_2\rangle$ and $|\alpha_2,\alpha_1\rangle$. In other words, $|\psi\rangle$ will not live in the entire Hilbert space $\mathcal{H}_2$, but only on a subspace corresponding to symmetric states. The indistinguishability restricts the part of $\mathcal{H}_2$ that a $|\psi\rangle$ may inhabit.

We therefore define the subspaces $\mathcal{H}_2^{\varepsilon} \subset \mathcal{H}_2$, where recall,

$$\varepsilon = \begin{cases} +1 & \text{Bosons} \\ -1 & \text{Fermions} \end{cases} \qquad (6)$$

We call $\mathcal{H}_2^{+}$ the symmetric two-particle space and $\mathcal{H}_2^{-}$ the anti-symmetric two-particle space we also define a basis for $\mathcal{H}_2^{\xi}$ as

$$|\alpha_1, \alpha_2\rangle_\xi = A\left[|\alpha_1, \alpha_2\rangle + \xi\, |\alpha_2, \alpha_1\rangle \right] \tag{7}$$

where A is a normalization constant. In general $A = 1/\sqrt{2}$, but for Bosons we have the annoying possibility of $\alpha_1 = \alpha_2$, in which case we need to set $A = 1/2$. For Fermions we don't have this problem since $|\alpha_1, \alpha_1\rangle_- = 0$ by the Pauli principle. I will not worry about these normalization constants. All we need to know is that $|\alpha_1, \alpha_2\rangle_\xi$ is properly normalized and that

$$\begin{aligned} |\alpha_1, \alpha_2\rangle_+ &= |\alpha_2, \alpha_1\rangle_+ \\ |\alpha_1, \alpha_2\rangle_- &= -|\alpha_2, \alpha_1\rangle_- \end{aligned} \tag{8}$$

that is, they are symmetrized or anti-symmetrized states. the $|\alpha_1, \alpha_2\rangle_\xi$ form a basis for $\mathcal{H}_2^{\xi}$.

We will also need later a formula for the inner product between these states. For concreteness let us do the case of Fermions.

$$|\alpha_1, \alpha_2\rangle_- = \frac{1}{\sqrt{2}}\left[|\alpha_1, \alpha_2\rangle - |\alpha_2, \alpha_1\rangle \right]$$

$$|\beta_1, \beta_2\rangle_- = \frac{1}{\sqrt{2}}\left[|\beta_1, \beta_2\rangle - |\beta_2, \beta_1\rangle \right]$$