Motion in one dimension

We will now study in more detail the motion of a particle described by a single coordinate x. This is what is called 1D. But please note that x is not necessarily a cartesian coordinate. For instance, the pendulum is described by a single coordinate, even though the motion is actually in a plane.

I will consider the general Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U(x) \qquad (1)$$

If we use exotic generalized coordinates, m may be a function of x. We will not consider this case here.

The equations of motion in this case are simply Newton's law:

$$m\ddot{x} = F = -\frac{\partial U}{\partial x} \qquad (2)$$

This Eq. can only be solved exactly in a few special cases. When that is not possible, we must resort to numerical methods. For such simple systems, current methods are very efficient and a solution can be obtained almost instantaneously.

For 1D systems there is a more efficient way to analyze the problem. It consists in noting that the energy of the system is a conserved quantity:

$$\boxed{E = \frac{1}{2} m v^2 + U(x)} \tag{3}$$

The value of E is determined by the initial conditions, $x(0)$ and $v(0)$. It then remains constant throughout the rest of the motion.

Let us write (3) as

$$\frac{1}{2} m v^2 = E - U$$

or

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U)}$$

We now isolate the x-dependent part

$$\frac{dx}{\sqrt{\frac{2}{m}(E - U)}} = dt$$

and integrate from $t=0$ to t:

$$\boxed{\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}} = t} \tag{4}$$

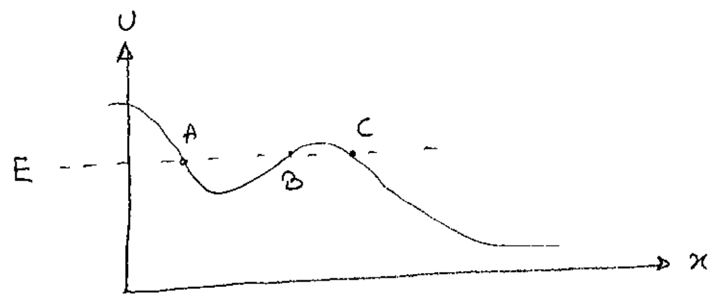
This is a formal solution of the problem. If we are able to compute the integral we will get t as a function of x. Then we would have to invert the solution to find $x(t)$.

In this framework, the initial conditions are specified by x_0 and E, instead of x_0 and v_0. That is actually convenient since in many cases knowing the energy is easier than knowing the velocity.

Now let us go back to Eq (3). Since $\frac{1}{2}mv^2$ is always a positive quantity, we must have

$$U(x) \leq E \tag{5}$$

It is therefore convenient to draw E as a horizontal line in the plot of $U(x)$



Due to (5), the motion will always occur below the line E. This means that for the particular choice in the figure, the motion will either occur in the interval $[A, B]$, or in the interval $[C, \infty)$. These are the only two possibilities.

The points A, B, C in the figure are special points which satisfy

$$U(x^i) = E \tag{6}$$

ie, at these points the velocity is zero. They are called <u>turning points</u>. At a turning point the particle stops and then start to move in the opposite direction.

When the motion is bounded by two turning points, it will occur in a finite region of space (as in [A,B]). Otherwise, the particle will fly off to infinity. So if a particle starts to the right of C, going left, it will keep on going up to C, then it stops and starts going right to infinity.

Let a finite motion be bounded by two turning points, x_1^i and x_2^i. Every finite motion will be oscillatory: the particle will oscillate back and forth between x_1^i and x_2^i.

We may use Eq (4) to obtain a formula for the period of oscillation. This is the time it takes to go from x_1^i to x_2^i and back. By symmetry, this is simply twice the time of the one-way trip:

$$\boxed{T(E) = 2\int_{x_1^i}^{x_2^i} \frac{dx}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}}} \tag{7}$$

When using this formula in practice, we must be careful to avoid double-valued regions. In practice we may need to break the integrals in several parts

Example: harmonic oscillator

The potential for the harmonic oscillator is

$$U(x) = \frac{1}{2} k x^2 \tag{8}$$

Thus, Newton's law reads

$$m\ddot{x} = -kx$$

or

$$\ddot{x} = -\omega^2 x \qquad \omega^2 = \frac{k}{m} \tag{9}$$

This equation is solved by $\cos(\omega t)$ and $\sin(\omega t)$. So we write the general solution as a linear combination

$$x(t) = a\cos(\omega t) + b\sin(\omega t) \tag{10}$$

The constants a and b are determined from the initial conditions.

Alternatively we may also write

$$x(t) = A\cos(\omega t - \phi) \tag{10'}$$

with new constants A and ϕ. This is equivalent to (10) because we may expand

$$x(t) = A\left[\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi\right]$$

Comparing with (10) we then find that

$$a = A\cos\phi \qquad b = A\sin\phi$$

The energy of the harmonic oscillator is

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \qquad (11)$$

Using (10') and noting that

$$v = \dot{x} = -A\omega \sin(\omega t - \phi)$$

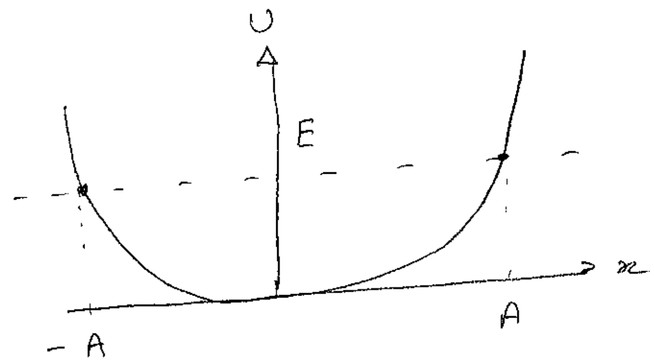
we get

$$E = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t - \phi) + \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t - \phi)$$

where I used $\omega^2 = k/m$. We therefore see that

$$E = \frac{1}{2} m\omega^2 A^2 = \frac{1}{2} k A^2 \qquad (12)$$

The constant A is the amplitude of the motion. It tells how far x goes.



So the turning points are exactly $\pm A$.

The period of oscillation of the harmonic oscillator is, according to (10) or (10'),

$$T = \frac{2\pi}{\omega} \qquad (13)$$

It is independent of energy, which is very peculiar of the harmonic oscillator.

Let us use this as a sanity check to see how Eq (7) works. We define the turning points as $\pm x^*$, where

$$E = \frac{1}{2} k x^{*2}$$

Since the problem is symmetric, the integrals from $-x^*$ to 0 and from 0 to x^* give the same result. So we may multiply Eq (7) by 2 and write

$$T = 4 \int_0^{x^*} \frac{dx}{\sqrt{\frac{2}{m}\left[\frac{1}{2}kx^{*2} - \frac{1}{2}kx^2\right]}}$$

Since $k/m = \omega^2$, we get

$$T = \frac{4}{\omega} \int_0^{x^*} \frac{dx}{\sqrt{x^{*2} - x^2}}$$

$$= \frac{4}{\omega} \int_0^{x^*} \frac{dx/x^*}{\sqrt{1 - (x/x^*)^2}}$$

Now let $y = x/x^{\circ}$. Then

$$T = \frac{4}{\omega} \int_0^1 \frac{dy}{\sqrt{1-y^2}}$$

The result of this integral is

$$\int \frac{dy}{\sqrt{1-y^2}} = \arcsin(y) \qquad (14)$$

You can demonstrate this as follows:

$$\arcsin(\sin x) = x$$

so

$$\frac{d}{dx}(\arcsin(\sin x)) = 1$$

But

$$\frac{d}{dx} \arcsin(\sin x) = \arcsin'(\sin x) \cos x$$

Thus

$$\arcsin'(\sin x) = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}}$$

Replacing $\sin x$ with y we get

$$\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$$

This demonstrates (14)

We therefore get

$$T = \frac{4}{\omega} \arcsin(y) \Big|_0^1$$

$$= \frac{4}{\omega} \left[\frac{\pi}{2} - 0 \right]$$

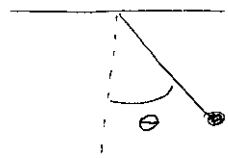
Thus

$$T = \frac{2\pi}{\omega}$$

as expected. Yay! Eq (7) works. Physics is consistent!

Example: pendulum

The lagrangian of a pendulum is



$$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos\theta . \qquad (15)$$

This falls under the category of Eq (1), provided we replace m with ml^2.

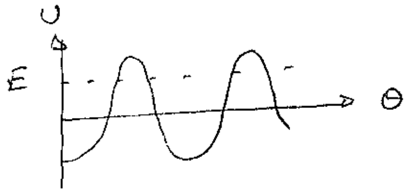
The energy is

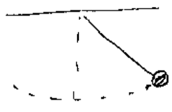
$$E = \frac{1}{2} m l^2 \dot{\theta}^2 - m g l \cos\theta \qquad (16)$$

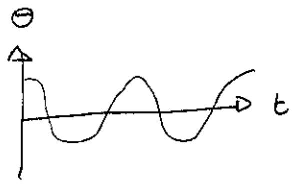
so the turning points will be $\pm\theta^*$, where

$$E = - m g l \cos\theta^* \qquad (17)$$

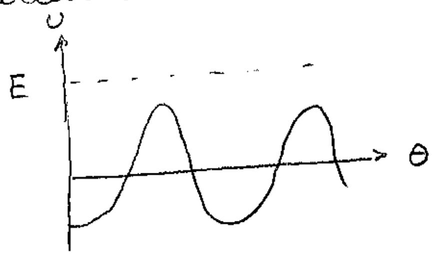
If $|E/mgl| < 1$ the pendulum goes up, stops, and then comes down again.

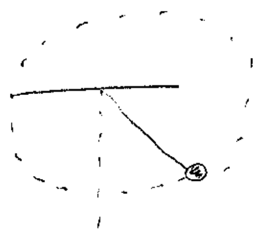






Otherwise, if $|E/mgl| > 1$, the energy is too big and the pendulum goes around in a circle.





In this case θ^* becomes imaginary, which means there are no turning points: the motion is unbounded.

We again use (11) over the interval $[0, \theta^*]$ so

$$T = 4 \int_0^{\theta^*} \frac{d\theta}{\sqrt{\frac{2}{ml^2}\left[-mgl\cos\theta^* + mgl\cos\theta\right]}}$$

Let

$$\omega^2 = g/l$$

Then we get

$$T = \frac{4}{\omega\sqrt{2}} \int_0^{\theta^*} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta^*}} \qquad (18)$$

This integral cannot be computed in closed form. So you have two choices. One is to compute it numerically. The other is to map it into a special function. That is not always possible, but when it is possible, it is usually worth the effort. In this case we can write T in terms of the elliptic integral of the first kind.

We start with

$$\cos\theta = \cos^2\theta/2 - \sin^2\theta/2 = 1 - 2\sin^2\theta/2$$

$$\Rightarrow \cos\theta - \cos\theta^* = 2(\sin^2\theta^*/2 - \sin^2\theta/2)$$

$$= 2\sin^2\theta^*/2\left[1 - \frac{\sin^2\theta/2}{\sin^2\theta^*/2}\right]$$

Eq (18) then becomes

$$T = \frac{2}{\omega} \int_0^{\theta^*} \frac{d\theta/(\sin\theta^*/2)}{\sqrt{1 - \frac{\sin^2\theta/2}{\sin^2\theta^*/2}}}$$

We now let

$$\sin y = \frac{\sin\theta/2}{\sin\theta^*/2}$$

The limits of integration change to:

$$\theta = 0 \quad \to \quad y = 0$$

$$\theta = \theta^* \quad \to \quad y = \pi/2$$

Moreover

$$\cos y \, dy = \frac{1}{2} \frac{\cos\theta/2 \, d\theta}{\sin\theta^*/2}$$

so

$$\frac{d\theta}{\sin\theta^*/2} = \frac{2\cos y}{\cos\theta/2} dy = \frac{2\cos y}{\sqrt{1-\sin^2\theta/2}} dy$$

Let

$$\lambda = \sin^2\theta^*/2 \qquad (19)$$

Then

$$\frac{d\theta}{\sin\theta^*/2} = \frac{2\cos y}{\sqrt{1-\lambda\sin^2 y}} dy$$

We then finally obtain

$$\boxed{T = \frac{4}{\omega}\int_0^{\pi/2} \frac{dy}{\sqrt{1-\lambda\sin^2 y}}} \qquad (20)$$

This integral is called the elliptic integral of the first kind and is efficiently implemented in any computer library:

$$K(\lambda) = \int_0^{\pi/2} \frac{dy}{\sqrt{1-\lambda\sin^2 y}} \qquad (21)$$

In Mathematica it is implemented as EllipticK[λ]. But please note that some libraries actually implement

$$K'(\lambda) = \int_0^{\pi/2} \frac{dy}{\sqrt{1-\lambda^2 \sin^2 y}}$$

with λ^2 instead of λ. So please be careful what library you use

The period is then

$$\boxed{T = \frac{4}{\omega} K(\sin^2\theta^*/2)} \tag{22}$$

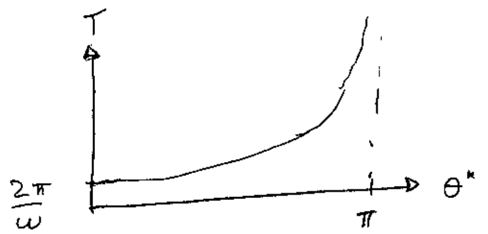
Recalling (17), we may also write

$$E = -mgl\,(1 - 2\sin^2\theta^*/2)$$

so that

$$\sin^2\theta^*/2 = \frac{1}{2}\left(1 + \frac{E}{mgl}\right) \tag{23}$$

Eq (22) looks like this:

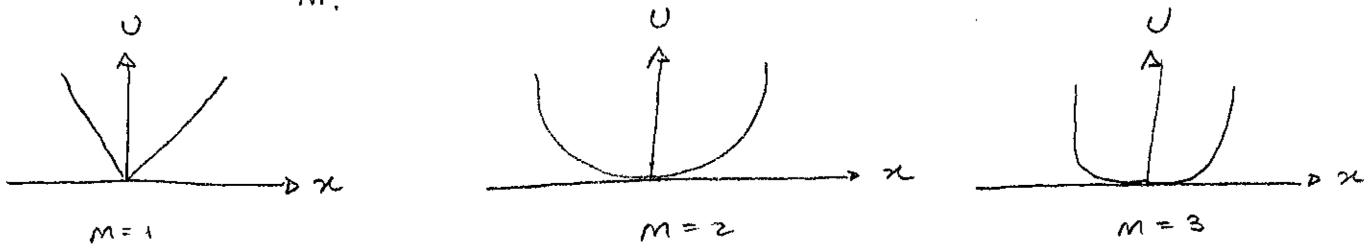


At $\theta^* \simeq 0$ it has the harmonic oscillator value $T = 2\pi/\omega$. This is the small angle approximation. At $\theta^* = \pi$ the period diverges. This corresponds to :  : it just stays up there.

It is also interesting to measure the relative error that is made when one assumes the period is harmonic

θ^* (°)	$T/(2\pi/\omega)$
15	0.43%
30	1.74%
45	4.0%

Example: $U = \dfrac{k}{m!} |x|^m$



In this case the turning points are $\pm x^*$, where

$$E = \frac{k}{m!} |x^*|^m \tag{24}$$

Again, we may consider only the interval $[0, x^*]$, where $|x| = x$. Then, (7) becomes

$$T(E) = 4 \int_0^{x^*} \frac{dx}{\sqrt{\frac{2}{m}\left[\frac{k}{m!} x^{*m} - \frac{k}{m!} x^m\right]}}$$

$$= \frac{4}{\sqrt{2}} \sqrt{m! \, \frac{m}{k}} \int_0^{x^*} \frac{dx}{\sqrt{(x^*)^m - x^m}}$$

Again, let $y = x/x^*$. Then

$$\frac{dx}{\sqrt{(x^*)^m - x^m}} = \frac{x^* \, dy}{(x^*)^{\frac{m}{2}} \sqrt{1 - y^m}}$$

$$\Rightarrow \quad T(E) = 2\sqrt{2} \sqrt{m! \, \frac{m}{k}} \, (x^*)^{1 - \frac{m}{2}} \int_0^1 \frac{dy}{\sqrt{1 - y^m}}$$

Let us write this in terms of E instead of x:

$$T = 2\sqrt{2m!\,\frac{m}{k}}\left(m!\,\frac{E}{k}\right)^{\frac{1-m/2}{m}}\int_0^1 \frac{dy}{\sqrt{1-y^m}}$$

$$= 2\sqrt{2m}\left(\frac{m!}{k}\right)^{1/m} E^{\frac{1}{m}-1/2}\int_0^1 \frac{dy}{\sqrt{1-y^m}}$$

The remaining integral is just a function of m. It turns out that it may be expressed in terms of Γ functions

$$\int_0^1 \frac{dy}{\sqrt{1-y^m}} = \sqrt{\pi}\,\frac{\Gamma\left(1+\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{m}\right)} \tag{25}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, dt \tag{26}$$

the Γ function is the generalization of the factorial

$$m! = \Gamma(m+1)$$

but it is not restricted to integers. We then finally obtain

$$\boxed{T(E) = \frac{2\sqrt{2\pi m}}{k^{1/m}}\, E^{\frac{1}{m}-\frac{1}{2}}\,(m!)^{1/m}\,\frac{\Gamma\left(1+\frac{1}{m}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{m}\right)}} \tag{27}$$

As a sanity check, take $m=2$. Use

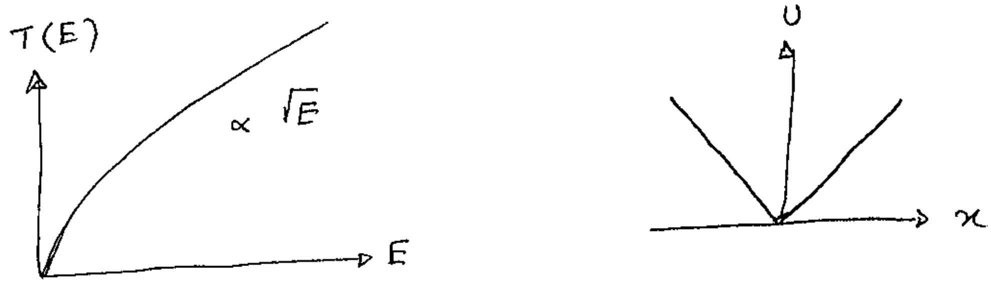
$$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(1) = 1$$

we then get the harmonic oscillator result:

$$T = 2\pi\sqrt{\frac{m}{k}} .$$

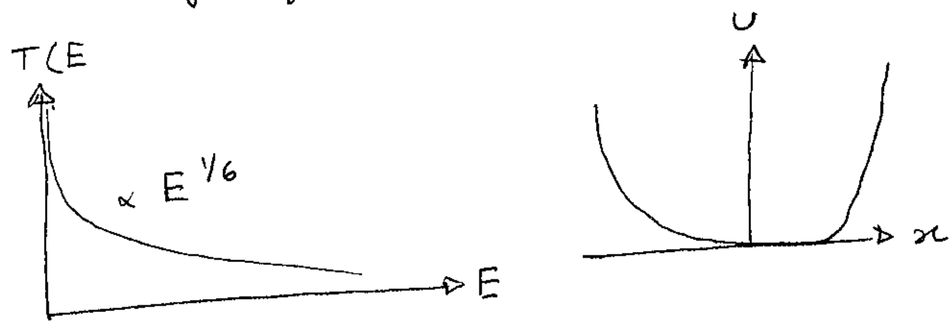
For other values of n we get many different behaviors.



$m=1$

For $m=1$ the period tends to zero when $E \to 0$.

For $n=3$, it tends to infinity



$m=3$

This is true for any $n \geq 3$. It is a consequence that, at $x \sim 0$, the potentials are flat, whereas for $m=1$ it is not

Numerical integration of the equations of motion

Fifty years ago, if someone asked you to study a certain mechanical problem, you could always say "Sorry, this problem cannot be solved exactly". Nowadays you have no excuse. Numerical solutions can be obtained in microseconds.

I even have the particular habbit of always solving numerically first. It takes no time at all and that way you know in advance what to expect before you attempt to solve the problem analytically.

The equations of motion are second order in time, which is numerically not very convenient. Numerical methods always use first order equations.

You can always convert a second order equation into 2 first order equations. Eq (4),

$$m\ddot{q} = F(q)$$

is the same as

$$\boxed{\begin{aligned} \dot{q} &= v \\ m\dot{v} &= F(q) \end{aligned}} \qquad (28)$$

Numerical methods convert this to a *difference equation* by discretizing time in steps of Δt. The naive approach consists in using the usual definition of the derivative

$$\dot{q}(t) \simeq \frac{q(t+\Delta t) - q(t)}{\Delta t} \qquad (29)$$

and similarly for v. Here Δt is a parameter which you have to choose.

This yields the <u>Euler method</u>

$$q(t+\Delta t) = q(t) + v(t)\,\Delta t$$
$$v(t+\Delta t) = v(t) + \frac{F(q(t))}{m}\,\Delta t \qquad (30)$$

The Euler method <u>sucks</u>. It is <u>terrible</u>. Please do not use it. We will see in a second an example of why it sucks so much.

A much better method is called the <u>velocity Verlet</u>

$$\boxed{\begin{aligned} q(t+\Delta t) &= q(t) + v(t)\Delta t + \frac{F(q(t))}{2m}\Delta t^2 \\ v(t+\Delta t) &= v(t) + \frac{\left[F(q(t)) + F(q(t+\Delta t))\right]}{2m}\Delta t \end{aligned}} \qquad (31)$$

This method must be applied sequentially. First you update $q(t+\Delta t)$. Then you use the old force $F(q(t))$ and the new force $F(q(t+\Delta t))$ (you need to keep both) to find the new velocity.

The velocity Verlet is the same as the leapfrog method, which you may have seen before. They are just written differently. The algorithm is quite good and I reccommend you always use it, unless you are going to do a very technical work. If you are, then please read the literature. There are many tricks that you can use which greatly increase the performance.

By the way, the velocity Verlet algorithm is symplectic.. This word means it preserves elements of areas $dq\,dp$. A symplectic algorithm is essential for <u>long-term stability</u>, which is what you need when you study, for instance, the motion of the planets.

Finally, note that Eq (31) is readily extended to the case where you have many particles. A very important example is <u>Molecular Dynamics</u>.

Example: harmonic oscillator

The equation for a harmonic oscillator is

$$\ddot{q} = -\omega^2 q$$

The frequency ω simply sets the time scale. So let us put $\omega = 1$. Then

$$\ddot{q} = -q \tag{32}$$

Or, in the form of two first order equations

$$\begin{aligned} \dot{q} &= v \\ \dot{v} &= -q \end{aligned} \tag{33}$$

The Euler recipe gives

$$\begin{aligned} q(t+\Delta t) &= q(t) + v(t)\Delta t \\ v(t+\Delta t) &= v(t) - q(t)\Delta t \end{aligned} \tag{34}$$

and the velocity Verlet method gives

$$\begin{aligned} q(t+\Delta t) &= q(t) + v(t)\Delta t - \frac{q(t)}{2}\Delta t^2 \\ v(t+\Delta t) &= v(t) - \left[\frac{q(t) + q(t+\Delta t)}{2}\right]\Delta t \end{aligned} \tag{35}$$

In either case, let us consider the energy

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k q^2$$

Setting all constants to 1, we get

$$E = \frac{1}{2}(v^2 + q^2) \tag{36}$$

Let us first compute the energy difference for the Euler method:

$$E(t+\Delta t) = \frac{1}{2}\left[v(t+\Delta t)^2 + q(t+\Delta t)^2\right]$$

$$= \frac{1}{2}\left[q(t)^2 + 2q(t)v(t)\Delta t + v(t)^2\Delta t^2\right] +$$

$$+ \frac{1}{2}\left[v(t)^2 - 2q(t)v(t)\Delta t + q(t)^2\Delta t^2\right]$$

Thus

(Euler): $$E(t+\Delta t) = E(t)(1+\Delta t^2) \qquad (37)$$

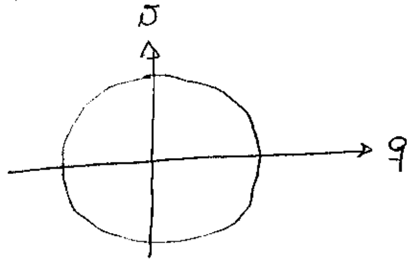
the energy in the Euler method always increases. It is the energy in the previous step plus a little extra. This is terrible for long-term stability.

The exact solution of the harmonic oscillator in reduced units, Eq (32), is

$$q(t) = A\cos(t+\phi) \qquad (38)$$

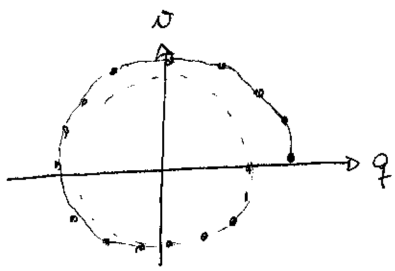
$$v(t) = -A\sin(t+\phi)$$

this describes a circle, whose radius is given by equation (36)



E determines radius

In the Euler method the solution spirals out



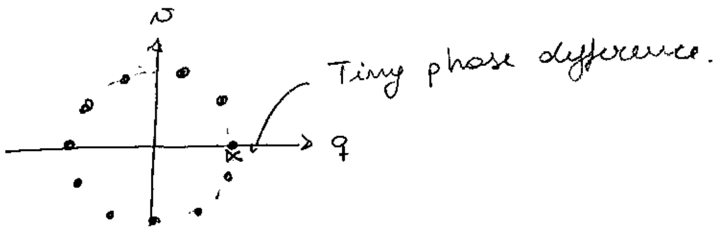
If you want to minimize this catastrophe, you need to use really small Δt. But if what you are interested in the solution after many periods, using a small Δt becomes computationally very expensive

Now let's do the same for the velocity Verlet. Using (35) to compute the energy (36), we get

$$E(t+\Delta t) = E(t) + \frac{q(t)\,v(t)}{4}\Delta t^3 + \mathcal{O}(\Delta t)^4 \tag{39}$$

We see two things. First, the error is now of the order Δt^3, instead of Δt^2. But secondly, and most importantly, the error may be either positive or negative, and therefore does not accumulate as much in the long run.

With the velocity Verlet the motion stays on a circle



there is, notwithstanding, a tiny phase difference, so that after one period we do not get back exactly to where we started.

For better plots, see the Mathematica notebook entitled "Numerical integration".

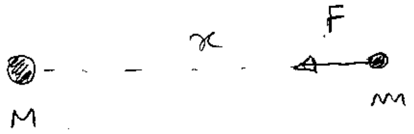
Example : gravitational force

The gravitational force between two bodies of mass M and m is given by the famous formula

$$F = -\frac{G M m}{x^2} \tag{40}$$

where x is the distance between them and G is the gravitational constant

$$G = 6.67408 \times 10^{-11} \, \frac{m^3}{kg\, s^2} \tag{41}$$



It is a surprising fact, and one which we will demonstrate later, that Eq (40) also holds when you are in the surface of the earth. In this case it is as if the entire mass M of the earth were concentrated on its center. this is a peculiar fact of a "inverse square force". In this case $x = R$ is the radius of the earth. Using

$$\begin{aligned} M &= 5.972 \times 10^{24}\ kg \\ R &= 6371\ km \\ &= 6.371 \times 10^{6}\ m \end{aligned} \tag{42}$$

we get

$$\frac{GM}{R^2} = g = 9.819 \ m/s^2$$

this only holds at the surface of the earth. otherwise, we must use (40).

Of course, the earth is not exactly spherical and its density is not really homogeneous. So you shouldn't try to estimate g with too many decimal places.

The potential energy associated to (40) is obtained by inverting

$$F = -\frac{dU}{dx}$$

or

$$U = -\int F dx + \text{const}$$

using (40) we get

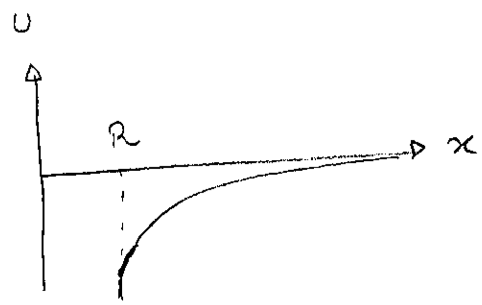
$$U = GMm \int \frac{dx}{x^2} + \text{const}$$

$$= GMm\left(-\frac{1}{x}\right) + \text{const}$$

It is customary to choose const = 0, which implies that U tends to zero at infinity. We then get

$$\boxed{U = -\frac{GMm}{x}} \tag{43}$$

This is the gravitational potential



For our purposes, we consider it only for $x > R$. When $x < R$, we get inside the earth and Eq (43) is no longer valid (we will see how it is modified later).

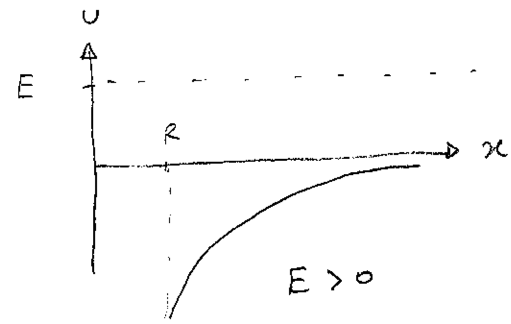
The total energy of the system is a conserved quantity. It reads:

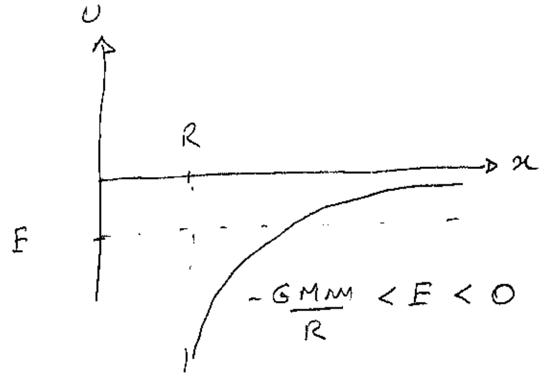
$$E = \frac{1}{2} m v^2 - \frac{GMm}{x} \tag{44}$$

Let us consider the situation where a particle is launched vertically from the surface of the earth with velocity v_0. The energy will then be

$$E = \frac{1}{2} m v_0^2 - \frac{GMm}{R} \tag{45}$$

We must now consider two situations:





When $E > 0$ the motion will be infinite: the particle is launched and simply goes off to space. The velocity when it is very far away is obtained from (44) by setting $x \to \infty$:

$$v_{\text{really far away}} = \sqrt{\frac{2E}{m}} \tag{46}$$

It tends to zero when $E \to 0$.

On the other hand, when $E < 0$, there will be a return point: the particle will go up, stop, and then fall down again. The return point is obtained from (44) by setting $v = 0$:

$$x^* = -\frac{GMm}{E} \tag{47}$$

This is positive because $E < 0$.

The case $E = 0$ is very special. It corresponds to the minimum energy required to escape the earth's gravitational field. In this case it reaches $x = \infty$ exactly with zero velocity (see Eq (46)).

The initial velocity which gives $E = 0$ is called the <u>escape</u> <u>velocity</u>. From (45)

$$\boxed{v_e = \sqrt{\frac{2GM}{R}}} \tag{48}$$

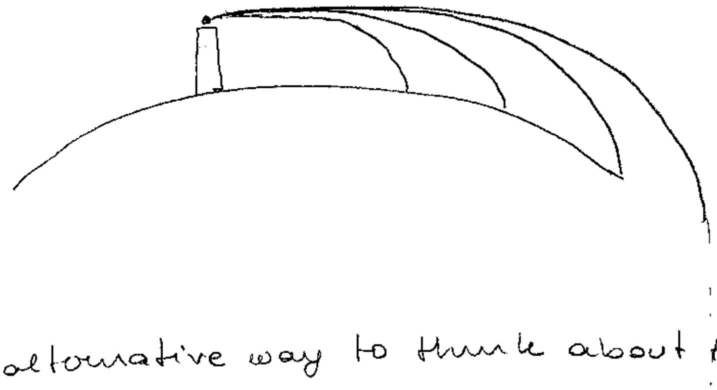
The escape velocity is independent of the mass m of the body. For the earth, (41) and (42) give

$$v_e = 40\,000 \text{ km/h} \simeq 11.2 \text{ km/s} \tag{49}$$

For the sun it is 617.5 km/s and for Pluto it is 1.2 km/s.

The escape velocity assumes a single initial impulse, followed by a movement without any external forces. Of course, that is not how rockets work: they accelerate continuously. But still, the escape velocity gives a measure of how fast you must go to be able to leave the earth (we are also ignoring air friction, which would further modify the results).

We can also form a picture based on orbits.



← It escaped.

This is just an alternative way to think about the problem

The earth is actually rotating and that affects the escape velocity. At the equatorial line the velocity of rotation is 465 m/s. If you are launched to the east the escape velocity is 10.375 km/s. But to the west it is 11.665 km/s.

Since the surface velocity of rotation decreases as you move away from the equatorial line, most launch pads are close to the equator to take advantage of the rotation of the earth. Examples include Cape Canaveral in Florida, US, or Alcântara, Maranhão, Brazil.

It is a very surprising coincidence that formula (48) for the escape velocity coincides with a formula developed by Schwarzschild in general relativity to measure the radius of a black hole.

A hole is "black" when not even light can escape from it. So consider a certain star of mass M and let us use Eq (48) to ask "what should its radius R be so that not even light would be able to escape from it?". The rationale is that the smaller is R, with M fixed, the stronger will the gravitational force be.

Setting $v_e = c$ in (48) and isolating R we get the <u>Schwarzschild radius</u>

$$\boxed{R_S = \frac{2GM}{c^2}} \tag{50}$$

Any object whose radius is smaller than its corresponding Schwarzschild radius is defined as a black hole. R_S also defines the <u>event horizon</u> of the black hole. If you pass this horizon you cannot escape, even if you move with speed c.

The Schwarzschild radius of the earth is ~ 9 mm (LOL). So to make earth a black hole you would have to squeeze all its mass (us included) into a 9mm sphere. The Schwarzschild radius of the sun is 3km

Back to reality (oh, there goes gravity). Let us again write (44) as a formal solution

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\left(E + \frac{GMm}{x}\right)} \qquad (51)$$

where I am assuming $v > 0$ because the particle is going up. Integrating we get

$$t = \int_R^x \frac{dx'}{\sqrt{\frac{2}{m}\left(E + \frac{GMm}{x'}\right)}} \qquad (52)$$

Suppose $E < 0$. then the motion will stop at x^* given by (47) :

$$E = -\frac{GMm}{x^*}$$

Thus

$$t = \frac{1}{\sqrt{2GM}} \int_R^x \frac{dx'}{\sqrt{\frac{1}{x'} - \frac{1}{x^*}}} = \sqrt{\frac{x^*}{2GM}} \int_R^x \frac{dx'}{\sqrt{\left(\frac{x^*}{x'}\right) - 1}}$$

Now let

$$x' = x^* \cos^2\theta$$

$$x' = R \quad \Rightarrow \quad \theta = \arccos\left(\sqrt{R/x^*}\right) := \theta_0$$

$$x' = x \quad \Rightarrow \quad \theta = \arccos\left(\sqrt{x/x^*}\right) := \theta$$

$$dx' = -x^* \, 2 \sin\theta \cos\theta \, d\theta$$

Then

$$t = \sqrt{\frac{x^*}{2GM}} \int_{\theta_0}^{\theta} \frac{-2x^* \sin\theta \cos\theta \, d\theta}{\sqrt{\frac{1}{\cos^2\theta} - 1}}$$

$$= -2x^* \sqrt{\frac{x^*}{2GM}} \int_{\theta_0}^{\theta} \frac{\sin\theta \cos\theta \, d\theta}{\frac{\sin\theta}{\cos\theta}}$$

$$= -2x^* \sqrt{\frac{x^*}{2GM}} \int_{\theta_0}^{\theta} \cos^2\theta \, d\theta$$

To solve the remaining integral we write

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\Rightarrow \int \cos^2\theta \, d\theta = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta$$

whence

$$\boxed{t = -x^* \sqrt{\frac{x^*}{2GM}} \left\{ \theta + \frac{1}{2}\sin 2\theta - \left(\theta_0 + \frac{1}{2}\sin 2\theta_0\right) \right\}} \qquad (53)$$

$$x = x^* \cos^2\theta$$

$$R = x^* \cos^2\theta_0$$

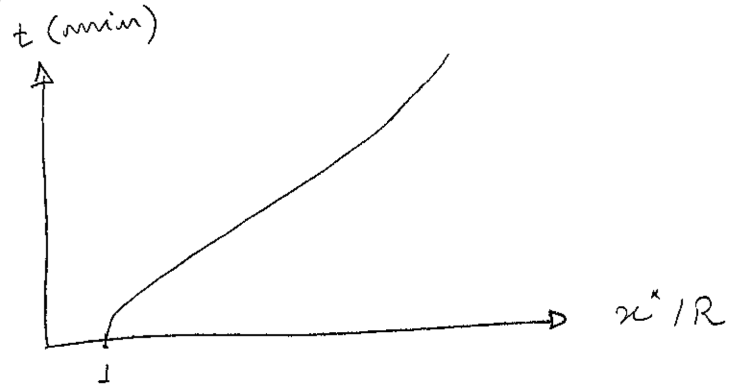
This is the implicit solution which gives $t(\theta(x))$. We may then invert it numerically to find $x(t)$.
Please note that this holds for the upward motion, with

$$R \leq x \leq x^*$$

In particular, we may compute the time t^* which takes to get to the return point. We obtain this by setting $x = x^*$, which gives $\theta = 0$. Thus

$$\boxed{\begin{aligned} t^* &= x^* \sqrt{\frac{x^*}{2GM}} \left(\theta_0 + \frac{1}{2} \sin 2\theta_0 \right) \\ \theta_0 (x^*) &= \arccos \left(\sqrt{R/x^*} \right) \end{aligned}} \qquad (54)$$

This gives the time to reach the return point as a function of the return point.



The value of x^* is entirely determined by the initial velocity:

$$x^* = -\frac{GMm}{E}$$

$$E = \frac{1}{2} m v_0^2 - \frac{GMm}{R}$$

$$\Rightarrow \quad x^* = \frac{-GMm}{\frac{1}{2} m v_0^2 - \frac{GMm}{R}}$$

or

$$x^* = \frac{1}{\frac{1}{R} - \frac{1}{2} \frac{v_0^2}{GM}}$$

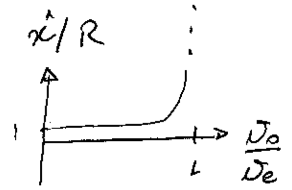
It is clearer to write

$$x^* = \frac{R}{1 - \frac{v_0^2 R}{2GM}}$$

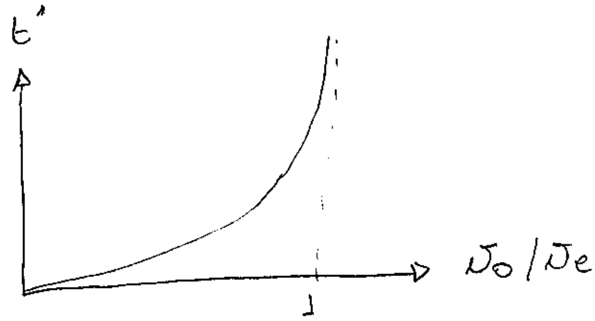
Recall that the escape velocity [Eq (48)] is $v_e^2 = \frac{2GM}{R}$.

We then get

$$\boxed{x^* = \frac{R}{1 - (v_0/v_e)^2}} \qquad (55)$$



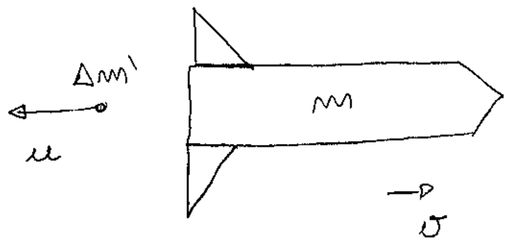
Come on! Isn't this a neat formula?! The return point diverges when $v_0 \to v_e$. This makes sense since it then never returns.



v_0/v_e	$x^* - R$ (km)	t (min)
0.2	265	4.0
0.5	2123	13.98
0.9	27160	173.31
0.99	313780	5305.2

Rocket Motion (Vrum! Vrum!)

Rockets move forward by expelling mass backwards.



We can analyze the effects of expelling mass using momentum conservation. Trying to use Newton's law with a time dependent mass $m(t)$ is not recommended since it may cause a lot of confusion.

Consider initially the rocket with mass m and velocity v. It's momentum will be

$$P_i = m v \tag{56a}$$

Suppose now that at an instant later a certain mass $\Delta m'$ is ejected backwards with a velocity u. We assume u is known. It depends on the design of the rocket and the type of fuel being used.

So the velocity of the rocket will increase to $v + \Delta v$. So the final momentum will be

$$P_f = (m - \Delta m')(v + \Delta v) + \Delta m'(v - u) \tag{56b}$$

Please take a second to appreciate the last term: u is the velocity the mass is ejected. But since the rocket was moving to the right with velocity v, someone on the ground will see the mass being expelled with velocity $v - u$.

We are considering only infinitesimal processes, so let us retain in p_f only the terms of linear order

$$p_f = m v + m \Delta v - \Delta m' v + \Delta m' v - \Delta m' u + \mathcal{O}(\Delta m' \Delta v)$$

$$= m v + m \Delta v - \Delta m' u$$

The change in momentum is therefore

$$\boxed{\Delta p = m \Delta v - \Delta m' u} \qquad (57)$$

$\Delta m' > 0$ was the mass ejected by the rocket. So the change in mass of the rocket was

$$\Delta m = - \Delta m' < 0$$

Thus

$$\boxed{\Delta p = m \Delta v + u \Delta m} \qquad (57)$$

According to Newton's law

$$F_{ext} = \frac{\Delta p}{\Delta t} \tag{58}$$

where F_{ext} encompasses all external forces, like gravity. In our case Δp is not simply $m \Delta v$: there is also the new term $u \Delta m$ in (57'). Thus we conclude

$$F_{ext} = m \frac{\Delta v}{\Delta t} + u \frac{\Delta m}{\Delta t} \tag{59}$$

Finally, taking the limit $\Delta t \to 0$ we get

$$\boxed{m(t) \frac{dv}{dt} = F_{ext} - u \frac{dm}{dt}} \tag{60}$$

this is the generalization of Newton's law to a system where mass is ejected with velocity u. Note that $\dot{m} < 0$ so the second term in (60) is positive: it accelerates the rocket. this is called the thrust:

$$\text{Thrust} = - u \frac{dm}{dt} \tag{61}$$

Now suppose there are no external forces (a rocket in free space). Then

$$m\frac{dv}{dt} = -u\frac{dm}{dt}$$

$$dv = -u\frac{dm}{m}$$

Integrating, and assuming u is constant, we get

$$v = v_0 - u\log\left(\frac{m}{m_0}\right)$$

or

$$\boxed{v = v_0 + u\log\left(\frac{m_0}{m}\right)} \quad (62)$$

This is called the Tsiolkovsky-Moore Eq, or the ideal rocket equation. It gives the velocity as a function of the mass m. We may have to specify how m changes with time. For instance

$$m(t) = m_0 - \alpha t \quad (63)$$

means you burn mass at a constant rate

$$\frac{dm}{dt} = -\alpha$$

Or you may use some other protocol you prefer.

Of course (63) should never take you to $m=0$. Then there would be no rocket left! You only burn the fuel. Not the fuselage.

Next consider the motion under a constant gravitational force: Eq (60) becomes

$$m\frac{dv}{dt} = -mg - u\frac{dm}{dt}$$

$$dv = -g\,dt - u\frac{dm}{m}$$

Integrating we get

$$v(t) = v_0 - gt + u\log\left(\frac{m_0}{m(t)}\right) \qquad (64)$$

Now let us do a more detailed bookkeeping on the rocket's mass. Initially, the total mass m_0 consists of fuselage and fuel. Let

m_r = mass of fuselage

m_f = mass of fuel.

$m_0 = m_f + m_r$

Suppose a constant burning rate:

$$\frac{dm}{dt} = -\alpha = \text{constant} \qquad (65)$$

For the Saturn V rocket, $\alpha = 14$ tom/s

The time it takes to burn all the fuel is obtained by integrating (65):

$$\int_{m_o}^{m_r} dm = -\alpha t_b$$

where t_b is called the burn-out time. We then get

$$t_b = \frac{m_f}{\alpha} \qquad (66)$$

which is an intuitive result.

For the Saturn V, $m_f = 2100$ ton so

$$t_b = \frac{2100}{14} \simeq 150 \text{ s}.$$

The S-V was launched 13 times between 1966 and 1973. It continues to be the tallest, heaviest and most powerful rocket ever used.

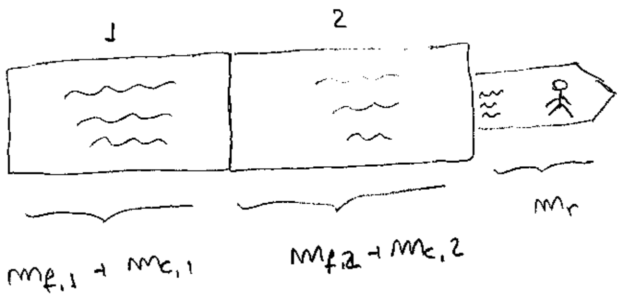
Assuming $v_o = 0$, the speed after burnout will be, according to (64)

$$v_b = -g t_b + u \log\left(\frac{m_r + m_f}{m_r}\right)$$

Multi-stage rockets

Just burning fuel is not enough to get us to space. In practice, rockets also throw away part of the fuselage during take-off. This not only makes them lighter, but also gives an additional boost in momentum.

Here is the idea



The part where the astronauts go is tiny and has mass m_r. Then there is a container with a bunch of fuel. The fuel weight $m_{f,1}$ and the container weighs $m_{c,1}$. Similarly, there is another stage with $m_{f,2}$ of fuel in a container with mass $m_{c,2}$. This drawing represents a 3-stage rocket.

When the rocket takes off it initially has mass

$$m_0 = m_r + m_{f,2} + m_{c,2} + m_{f,1} + m_{c,1} \tag{67}$$

Then it burns $m_{f,1}$. After burnout its mass will be

$$m_1 = m_r + m_{f,2} + m_{c,2} + m_{c,1} \tag{68}$$

and its velocity will be, according to (64)

$$v_1 = -g\,t_1 + u \log\left(\frac{m_0}{m_1}\right) \tag{69}$$

where t_1 is the burnout time

$$t_1 = \frac{m_{f,1}}{\alpha_1} \tag{70}$$

with α_1 being the burning rate of the first stage.

At this point we throw away the first container (mass $m_{c,1}$). This is not like throwing fuel away because fuel goes with a velocity u, whereas the container is thrown from rest. But what it means is that the total mass of the rocket becomes smaller:

$$m_1' = m_1 - m_{c,1} = m_r + m_{f,2} + m_{c,2} \tag{71}$$

We now proceed to burn stage two. After burnout the mass will be

$$m_2' = m_r + m_{c,2} \tag{72}$$

Thus, the velocity after burnout will be

$$v_2 = v_1 - g t_2 + u \log\left(\frac{m_1'}{m_2'}\right) \tag{73}$$

where

$$t_2 = \frac{m_{f,2}}{\alpha_2} \tag{74}$$

Substituting for v_1 in (69) we get

$$v_2 = -g(t_1 + t_2) + u \log\left(\frac{m_0}{m_1}\right) + u \log\left(\frac{m_1'}{m_2'}\right)$$

$$v_2 = -g(t_1 + t_2) + u \log\left(\frac{m_0 \, m_1'}{m_1 \, m_2'}\right) \qquad (75)$$

If we had not dumped $m_{c,1}$ we would've had $m_1' = m_1$. In this case the result would be that of a single burning stage of $m_{f,1}$ and $m_{f,2}$.

The product $\frac{m_0 \, m_1'}{m_1 \, m_2'}$ can be made much larger than m_0/m_2 since the containers (which include motors and other stuff) are really heavy

Staging makes rockets much more complex and much harder to build. Consequently it is also one of the primary sources of failure during launch (separation, ignition, etc). Notwithstanding, staging leads to such large savings that literally every rocket that was ever used to go to orbit has used some type of staging.