The tensor product

In these notes I want to give you a more solid understanding of how to work with systems containing more than one particle. We won't work out any specific examples. I just want to show you the notation and techniques that we use.

Suppose you have 2 systems which are independent of each other. System 1 has a Hilbert space $\mathcal{H}_1$, a basis $|e_i\rangle$ and a typical operator $\hat{A}$. Similarly system 2 has a Hilbert space $\mathcal{H}_2$, a basis $|f_j\rangle$ and a typical operator $\hat{B}$.

If we want to describe the two systems together then we say the total Hilbert space is a product space, for which we write

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{1}$$

The basis for this product space $\mathcal{H}$ must contain all possible states of systems 1 and 2. A possible choice is thus to combine the bases $|e_i\rangle$ and $|f_j\rangle$. We will write this in 3 different ways

$$\boxed{|e_i, f_j\rangle = |e_i\rangle|f_j\rangle = |e_i\rangle \otimes |f_j\rangle} \tag{2}$$

The first two ways are just abbreviations. The correct way is the third. We call $|e_i\rangle \otimes |f_j\rangle$ a tensor product. Mathematicians call it a Kronecker product. I like to call it the "kron" because it reminds me of a transformers movie. I read $|e_i\rangle \otimes |f_j\rangle$ as "e_i kron f_j".

The kron is defined as follows: let A, B, C, D be any object we want (bra, ket, operator, etc). Then

$$\boxed{(A \otimes B)(C \otimes D) = (AC) \otimes (BD)} \tag{3}$$

provided that the products AC and BD make sense. I call this "the one property, to rule them all!"

In terms of dimensions, we have

$$\boxed{\begin{array}{l} A \text{ is } n \times m \\ B \text{ is } p \times q \end{array} \Rightarrow A \times B \text{ is } np \times mq} \tag{4}$$

An operator $\hat{A}$ of system 1 is written in this language as

$$\hat{A}_1 = \hat{A} \otimes 1 \tag{5}$$

Similarly

$$\hat{B}_2 = 1 \otimes \hat{B} \tag{6}$$

We can now use Eq (3) to show that $\hat{A}_1$ and $\hat{B}_2$ commute:

$$\hat{A}_1 \hat{B}_2 = (\hat{A} \otimes 1)(1 \otimes \hat{B}) = (\hat{A} 1) \otimes (1 \hat{B}) = \hat{A} \otimes \hat{B}$$

$$\hat{B}_2 \hat{A}_1 = (1 \otimes \hat{B})(\hat{A} \otimes 1) = (1 \hat{A}) \otimes (\hat{B} 1) = \hat{A} \otimes \hat{B}$$

Thus

$$[\hat{A}_1, \hat{B}_2] = 0 \tag{7}$$

$\boxed{\text{Operators pertaining to different systems commute}}$

We can also use the kron to multiply matrices by vectors: recall that when we apply

$$\hat{A}_1 | e_i f_j \rangle$$

$\hat{A}_1$ only acts on the first part. The kron makes this very explicit:

$$\hat{A}_1 | e_i, f_j \rangle = (\hat{A} \otimes 1)(|e_i\rangle \otimes |f_j\rangle) = (\hat{A}|e_i\rangle) \otimes |f_j\rangle \tag{8}$$

Inner product

Suppose $|\alpha\rangle$ and $|\beta\rangle$ are kets of system 1 and $|a\rangle$ and $|b\rangle$ are kets of system 2. Then

$$(\langle\alpha| \otimes \langle a|)(|\beta\rangle \otimes |b\rangle) = \langle\alpha|\beta\rangle \otimes \langle a|b\rangle$$

Now let us look at the dimensions in Eq (4). Both $\langle\alpha|\beta\rangle$ and $\langle a|b\rangle$ are numbers; ie, they are 1×1. Thus the kronecke product will also be 1×1. We don't need the $\otimes$:

$$(\langle\alpha| \otimes \langle a|)(|\beta\rangle \otimes |b\rangle) = \langle\alpha|\beta\rangle\langle a|b\rangle \qquad (9).$$

We can do the same thing with the notation $|\alpha\rangle|a\rangle$.

$$(\langle\alpha|\langle a|)(|\beta\rangle|b\rangle) = \langle\alpha|\beta\rangle \langle a|b\rangle \qquad (10)$$

This notation is a bit simpler, but is also more confusing because you need to pay attention on who is acting on who.

Everything we just said is also valid for more than 2 particles. For instance, if we have 4 spin 1/2 particles a typical element of the basis will be

$$|++-+\rangle = |+\rangle \otimes |+\rangle \otimes |-\rangle \otimes |+\rangle \tag{11}$$

the operators $\hat{\sigma}_x$ for each particle are

$$\begin{aligned}
\hat{\sigma}_1^x &= \hat{\sigma}_x \otimes 1 \otimes 1 \otimes 1 \\
\hat{\sigma}_2^x &= 1 \otimes \hat{\sigma}_x \otimes 1 \otimes 1 \\
\hat{\sigma}_3^x &= 1 \otimes 1 \otimes \hat{\sigma}_x \otimes 1 \\
\hat{\sigma}_4^x &= 1 \otimes 1 \otimes 1 \otimes \hat{\sigma}_x
\end{aligned} \tag{12}$$

We can now do our calculations using any notation we like. Recall that

$$\hat{\sigma}_x |\pm\rangle = |\mp\rangle \tag{13}$$

So, for instance

$$\hat{\sigma}_2^x |++-+\rangle = |+--+\rangle \tag{14}$$

Or, if we want to use kron:

$$\begin{aligned}
\hat{\sigma}_2^x |++-+\rangle &= (1 \otimes \hat{\sigma}_x \otimes 1 \otimes 1)(|+\rangle \otimes |+\rangle \otimes |-\rangle \otimes |+\rangle) \\
&= |+\rangle \otimes (\hat{\sigma}_x |+\rangle) \otimes |-\rangle \otimes |+\rangle \\
&= |+\rangle \otimes |-\rangle \otimes |-\rangle \otimes |+\rangle \\
&= |+--+\rangle .
\end{aligned} \tag{15}$$

As you can see, the kron is not very practical! Indeed, in most calculations we will use the notation in Eq (14). However, the kron is useful for two reasons: first, it helps clarify the meaning of certain operations we are doing. And secondly, the kron will give us an automated way of constructing matrices for many particle systems.

<u>Example</u>: Simplify $\sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^z$

I will use the following properties: $(\sigma)^2 = 1$ and $\sigma^x \sigma^y = i\sigma^z$.

<u>Method 1</u>: Operators pertaining to different particles commute so

$$\sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^z = \sigma_1^x \underbrace{\sigma_2^z \sigma_2^z}_{1} \sigma_1^y = \sigma_1^x \sigma_1^y = i\sigma_1^z$$

<u>Method 2</u>:

$$\sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^z = (\sigma_x \otimes 1)(1 \otimes \sigma_z)(\sigma_y \otimes 1)(1 \otimes \sigma_z)$$

$$= (\underbrace{\sigma_x 1 \sigma_y 1}_{i\sigma_z}) \otimes (\underbrace{1 \sigma_z 1 \sigma_z}_{1})$$

$$= i\,\sigma_z \otimes 1$$

$$= i\,\sigma_1^z$$

Matrix elements of operators

The basis for the product space is $|e_i f_j\rangle$. It has two indices. It is customary to order them as follows. Suppose $i = 1, \dots, m$ and $j = 1, \dots, p$. Then we order them as

$$|e_1 f_1\rangle, \dots, |e_1, f_p\rangle, |e_2, f_1\rangle \dots |e_2, f_p\rangle \dots \tag{16}$$

Always go through the second index first.

For a general operator acting on $\mathcal{H}$, its matrix elements are

$$G_{ij,kl} = \langle e_i f_j | \hat{G} | e_k f_l\rangle \tag{17}$$

Suppose we now want the matrix elements of an operator $\hat{A}_1$ acting only on particle 1. Then, since $\hat{A}_1$ only acts on the first index

$$\langle e_i f_j | \hat{A}_1 | e_k, f_l\rangle = \langle e_i | \hat{A} | e_k\rangle \langle f_j | f_l\rangle = A_{ik}\,\delta_{jl} \tag{18}$$

So the matrix $\hat{A}_1$ is diagonal with respect to the second set of indices. We can do the same thing using the kron

$$\begin{aligned}\langle e_i f_j | \hat{A}_1 | e_k f_l\rangle &= (\langle e_i| \otimes \langle f_j|)(\hat{A} \otimes 1)(|e_k\rangle \otimes |f_l\rangle) \\ &= \langle e_i | \hat{A} | e_k\rangle \otimes \langle f_j | f_l\rangle \\ &= A_{ik} \otimes \delta_{jl} \\ &= A_{ik}\,\delta_{jl}\end{aligned}$$

since the kron of two numbers is also a number.

Similarly

$$\langle e_i f_j | \hat{B}_2 | e_k f_\ell \rangle = \langle e_i | e_k \rangle \langle f_j | \hat{B}_2 | f_\ell \rangle = \delta_{ik} B_{j\ell} \qquad (19)$$

Writting matrices using kron

From the previous examples it may seem that there is no real advantage in using the kron notation. The big advantage is in obtaining the matrices, such as $\hat{A}_1$ and $\hat{B}_2$, in the composite basis. If we order the basis $|e_i f_j\rangle$ by running through the f_j first then the e_i, then it is possible to write

$$\hat{A} \otimes \hat{B} = \begin{bmatrix} a_{11}\hat{B} & \dots & a_{1m}\hat{B} \\ \vdots & & \\ a_{m1}\hat{B} & \dots & a_{mm}\hat{B} \end{bmatrix} \tag{20}$$

where $\hat{B}$ is to be interpreted as a block of the big matrix. This is confusing, I know. So let me clarify with examples

Example 1:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 0\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ 0\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 1\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

It is very clear that

$$\boxed{\hat{A} \otimes \hat{B} \neq \hat{B} \otimes \hat{A}} \tag{21}$$

Example 2 :

$$\hat{\sigma}_1^x = \hat{\sigma}_x \otimes 1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

This looks like a $\hat{\sigma}^x$ with $1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\hat{\sigma}_2^x = 1 \otimes \hat{\sigma}^x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and this looks like a identity with $1 \to \hat{\sigma}^x$.

Example 3:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & 0 & b \\ c & 0 & 0 & d & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 \\ 0 & 0 & c & 0 & 0 & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{bmatrix}$$

If $\hat{A}$ is $m \times n$ and $\hat{B}$ is $p \times q$ then

$$\hat{A} \otimes \hat{B} \text{ is } mp \otimes nq \tag{22}$$

Example 4:

$$|\alpha\rangle \otimes |\beta\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \otimes \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_1 \beta_2 \\ \alpha_2 \beta_1 \\ \alpha_2 \beta_2 \end{bmatrix}$$

In this case $|\alpha\rangle$ is 2×1 and $|\beta\rangle$ is 2×1 so $|\alpha\rangle \otimes |\beta\rangle$ is 4×1.

For 2 spin 1/2 particles we have

$$|++\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |+-\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|-+\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |--\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{23}$$

Example 5:

$$\langle\alpha| \otimes \langle\beta| = [\alpha_1^* \quad \alpha_2^*] \otimes [\beta_1^* \quad \beta_2^*]$$

$$= [\alpha_1^* [\beta_1^* \quad \beta_2^*] \quad \alpha_2^* [\beta_1^* \quad \beta_2^*]]$$

$$= [\alpha_1^* \beta_1^* \quad \alpha_1^* \beta_2^* \quad \alpha_2^* \beta_1^* \quad \alpha_2^* \beta_2^*]$$

Now $\langle\alpha|$ is 1×2 and $\langle\beta|$ is 1×2 so $\langle\alpha| \otimes \langle\beta|$ is 1×4.

We can also mix all types of objects we want. Of course, if they are useful in practice is another story. So we can have things like

$$\langle\alpha| \otimes |\beta\rangle$$

$$|\alpha\rangle \otimes \langle\beta|$$

$$1 \otimes |\alpha\rangle$$

$$1 \otimes \langle\alpha|$$

$$\langle\alpha| \otimes 1$$

etc.

Entanglement ("Emaranhamento")

Suppose system 1 is in a state

$$|\alpha_1\rangle = \sum_i \alpha_i |e_i\rangle \tag{24}$$

and system 2 is in a state

$$|\beta_2\rangle = \sum_j \beta_j |f_j\rangle \tag{25}$$

If we want to consider the two particles together then their joint state is

$$|\psi\rangle = |\alpha_1, \beta_2\rangle = |\alpha_1\rangle \otimes |\beta_2\rangle = \left[\sum_i \alpha_i |e_i\rangle\right] \otimes \left[\sum_j \beta_j |f_j\rangle\right]$$
$$= \sum_i \alpha_i \beta_j \; |e_i, f_j\rangle \tag{26}$$

Note, however, that not all states of $\mathcal{H}$ are of this form. For instance, if we have two spin 1/2 particles, the <u>singlet</u> state

$$|0\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \tag{27}$$

cannot be written as a tensor product of two single-particle states.

Conclusion: not all states of the composite system can be written as a product of states of individual particles. When the state cannot be written as a product we say the two systems are <u>entangled</u>. Entanglement is the source of remarkable properties which are unique of quantum systems. We will view a few of them later.

A general state may be written as

$$|\upsilon\rangle = \sum_{i,j} \upsilon_{ij} \, |e_i f_j\rangle \tag{28}$$

comparing with (26) we see that a state will <u>not</u> be entangled only if

$$\upsilon_{ij} = \alpha_i \beta_j \tag{29}$$

ie, the coefficients factor as a product.

Description in terms of wave-functions

Suppose a certain system is in a general state $|\sigma\rangle$ like that in Eq (28). Let $|x_1\rangle$ and $|x_2\rangle$ denote the position kets of systems 1 and 2. Then the wavefunction corresponding to $|\sigma\rangle$ is

$$\Psi(x_1, x_2) = \langle x_1 x_2 | \sigma \rangle = \left[\langle x_1 | \otimes \langle x_2|\right] |\sigma\rangle \tag{30}$$

Now let

$$\begin{aligned} \langle x_1 | e_i \rangle &= \psi_i(x_1) \\ \langle x_2 | f_j \rangle &= \phi_j(x_2) \end{aligned} \tag{31}$$

Then

$$\Psi(x_1, x_2) = \sum_{i,j} \sigma_{ij} \; \psi_i(x_1)\, \phi_j(x_2) \tag{32}$$

In terms of wave functions it is more obvious to see the role of entanglement. Suppose the system is in a state

$$\Psi(x_1, x_2) = \psi_i(x_1)\, \phi_j(x_2) \tag{33}$$

This state is a product. The two particles are not entangled. The probabilities are

$$|\Psi(x_1, x_2)|^2 = |\psi_i(x_1)|^2 \, |\phi_j(x_2)|^2 \tag{34}$$

the probabilities are just products. This is what we intuitively expect for two independent particles. But now consider a state of the form

$$\Psi(x_1, x_2) = \frac{\psi_i(x_1)\,\phi_j(x_2) + \psi_k(x_1)\,\phi_l(x_2)}{\sqrt{2}} \qquad (35)$$

where the factor of 2 is just to ensure that Ψ is normalized. the probability is now

$$|\Psi(x_1, x_2)|^2 = \frac{1}{2}\Big\{ |\psi_i(x_1)|^2\,|\phi_j(x_1)|^2 + |\psi_k(x_1)|^2\,|\phi_l(x_2)|^2 + 2\,\mathrm{Re}\left[\psi_i(x_1)\,\psi_k^*(x_1)\,\phi_j(x_2)\,\phi_l^*(x_2)\right]\Big\} \qquad (36)$$

The first two terms make sense: they are simply the probabilities for each system. But the third term is new. It leads to interference. You see: entanglement affects the probabilities! This is a genuinely quantum mechanical effect. Note that the key point is that in QM we add probability amplitudes $\langle\alpha|\beta\rangle$, not probabilities $|\langle\alpha|\beta\rangle|^2$.

Comment about Hamiltonians, eigenvalues and eigenvectors

Suppose the two systems do not interact. Let $\hat{H}_1$ be the Hamiltonian of system 1. We write its eigenvalue / eigenvector Eq as

$$\hat{H}_1 |m_1\rangle = E^1_{m_1} |m_1\rangle \tag{37}$$

Sorry about the notation $E^1_{m_1}$: it is E^1 because it is the energy of system 1 and m_1 is the quantum number of system 1. Similarly, for system 2 we have

$$\hat{H}_2 |m_2\rangle = E^2_{m_2} |m_2\rangle \tag{38}$$

If we want to consider both systems together we may use as basis

$$|m_1, m_2\rangle = |m_1\rangle \otimes |m_2\rangle \tag{39}$$

these states simultaneously diagonalize $\hat{H}_1$ and $\hat{H}_2$. The total energy is

$$\hat{H} = \hat{H}_1 + \hat{H}_2 \tag{40}$$

and its eigenvectors are precisely the $|m_1, m_2\rangle$.

$$\hat{H}|m_1, m_2\rangle = \hat{H}_1 |m_1 m_2\rangle + \hat{H}_2 |m_1 m_2\rangle \tag{41}$$

$$= E^1_{m_1} |m_1 m_2\rangle + E^2_{m_2} |m_1 m_2\rangle$$

thus

$$\hat{H}\,|m_1, m_2\rangle = \left(E^1_{m_1} + E^2_{m_2}\right)|m_1\, m_2\rangle \tag{42}$$

the whole discussion may also be done in terms of wavefunctions:

$$\hat{H}_1\, \psi_{m_1}(x_1) = E^1_{m_1}\, \psi_{m_1}(x_1) \tag{43}$$

$$\hat{H}_2\, \phi_{m_2}(x_2) = E^2_{m_2}\, \phi_{m_2}(x_2)$$

the basis for the product space is then

$$\Psi_{m_1, m_2}(x_1, x_2) = \psi_{m_1}(x_1)\, \phi_{m_2}(x_2) \tag{44}$$

and

$$\hat{H}\, \Psi_{m_1, m_2} = \left(E^1_{m_1} + E^2_{m_2}\right) \Psi_{m_1, m_2} \tag{45}$$

Conclusion: when two systems do not interact, the total energy is the sum of the energies and the total eigenvector / eigenfunction is the product of the individual eigenvector / eigenfunction

when we are talking about kets, then by product we mean the tensor product. For wavefunctions the product is actually the product.

Now suppose we allow the two systems to interact. then the total Hamiltonian becomes

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12} \tag{46}$$

where $\hat{H}_{12}$ is the interaction term. The eigenvectors in this case will, in general, no longer be the $|m_1, m_2\rangle$.

One way to understand this is as follows. Let $\hat{A}_\alpha$ be a series of operators of system 1. For instance $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ for a spin 1/2 particle. Then the Hamiltonian $\hat{H}_1$ may be written as

$$\hat{H}_1 = \sum_\alpha c_\alpha \left(\hat{A}_\alpha \otimes 1\right) \tag{47}$$

for some coefficients c_α. Similarly, if $\hat{B}_\beta$ are a series of operators for system 2 then

$$\hat{H}_2 = \sum_\beta d_\beta \left(1 \otimes \hat{B}_\beta\right) \tag{48}$$

However, $\hat{H}_{12}$ will be written as a combination of both $\hat{A}_\alpha$ and $\hat{B}_\beta$

$$\hat{H}_{12} = \sum_{\alpha,\beta} w_{\alpha,\beta} \; \hat{A}_\alpha \otimes \hat{B}_\beta \tag{49}$$

The vectors $|m_1, m_2\rangle$ which diagonalize $\hat{H}_1$ and $\hat{H}_2$ will not diagonalize H_{12} in general (there are exceptions; one is the Ising model).

Spinors

Here is a nice theorem: to completely describe a quantum particle you need 4 quantum numbers. For instance it may be $|x\rangle, |y\rangle, |z\rangle, |s\rangle$ where s is the spin, $s = \pm 1$. If $|v\rangle$ is a general state, its component in the basis

$$|x, y, z, s\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle \otimes |s\rangle \tag{50}$$

is

$$\psi_s(x, y, z) = \langle x, y, z, s | v\rangle \tag{51}$$

Note how we have "two" wave functions

$$\begin{aligned} \psi_+(x,y,z) &= \langle x, y, z, + | v\rangle \\ \psi_-(x,y,z) &= \langle x, y, z, - | v\rangle \end{aligned} \tag{52}$$

this is what we call a spinor. We may even organize them in a column vector

$$\begin{bmatrix} \psi_+(x,y,z) \\ \psi_-(x,y,z) \end{bmatrix} \tag{53}$$

For many problems in life, the spin part is independent of the spatial part. this is why it makes sense to talk about Stern-Gerlach experiments and etc. In these cases

$$|\psi\rangle = |\psi_{space}\rangle \otimes |\psi_{spin}\rangle \tag{54}$$

so that the corresponding wave function is

$$\Psi_s(x, y, z) = \psi(x, y, z)\,\chi_s \tag{55}$$

where
$$\chi_s = \langle s | \psi_{spin}\rangle \tag{56}$$

In this case we may study the two problems separately. The χ_s are simply the components of a general spin 1/2 vector

$$|\psi_{spin}\rangle = \begin{bmatrix} \chi_+ \\ \chi_- \end{bmatrix} \tag{57}$$