Quantum measurements, EPR paradox and Bell's inequality

Sakurai 3.10
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Let us review what we have learned so far in the context of a spin 1/2 particle. Suppose that, at some instant of time, the particle is in a state

$$|\alpha\rangle = a|+\rangle + b|-\rangle \qquad (1)$$

We say that $|\alpha\rangle$ corresponds to a superposition of the states $|+\rangle$ and $|-\rangle$. It is not in $|+\rangle$, nor is it in $|-\rangle$. But it is also not in both, nor in neither. This is very strange and has no classical counterpart. Since there is no word for this in common language, we had to give it a new word: superposition.

However, if we measure the state of the particle, we find it either in $|+\rangle$ or in $|-\rangle$ with probabilities

$$P(+) = |a|^2 \qquad P(-) = |b|^2 \qquad (2)$$

From normalization it then follows that $|a|^2 + |b|^2 = 1$].
This is an experimental fact. So before the measurement the particle was not in any specific state, but after the measurement it was forced into one state or the other. This is one of the most peculiar things about QM. After the measurement we say the state collapsed to either $|+\rangle$ or $|-\rangle$

So the first thing I want to discuss is why this happens. The theory which describes this is called <u>decoherence theory</u> and the short answer is

"By a measurement we mean an interaction between a system and a large (classical) object. This interaction affects the state of the system, forcing it to collapse".

Suppose you want to measure the position of a dog that is free to walk around a dark arena. One way to do it is to shine light around and see the dog. But you see, to measure the position of the dog you must <u>interact</u> with it. Only in this case this interaction may be very small so the dog won't feel anything.

Another way to measure the position of the dog is to drive a car around. If you ever feel a bump you know where the dog was. In this case your measurement involve making the dog interact with something larger than it. Accordingly, note how your measurement affected the state of the dog (you killed it!).

Why can't we thus use a measurement apparatus smaller than the system? Because we, who are performing the experiment, are classical objects. To obtain information we thus need our system to interact with a classical object.

About Ignorance

Probabilities in the classical sense are always related to ignorance or lack of knowledge. When you throw a coin you usually assume a 50/50 chance of getting heads or tails. But that is only because you don't know enough details about the world. If you knew the position of every atom in the world, then in principle you could use a supercomputer to simulate Newton's law and predict whether your toss would yield Head or Tails. You see, probability is really a statement about lack of information

But superposition is different. Before the measurement no one knew which state the particle was. Not even God. The idea of superposition implies that there exits an intrinsic ignorance about the world; ie, that there is a limit of how much we can know about Nature. Think about it. This is very deep. It doesn't matter how much you improve your measurement apparatus, you will never be able to know everything deterministically.

This idea bothered a lot of people in the early days of QM. What they said was: "Maybe this idea of superposition is not correct. Maybe the system already knew which state it was before the measurement. You didn't know but the system did. Maybe there is some other hidden variable which we are not taking into account. But if we do, then we will know with certainty what state the particle will come out."

we usually call this a "hidden variable theory". If you think about it, it actually makes sense. QM, as we know it, makes predictions which agree with experiment. But this does not exclude the possibility that we actually missed something in our theory. In fact, for many decades this debate was closer to philosophy than physics since we couldn't really proove which possibility was the correct one.

But in 1964 a guy named John S. Bell changed this completely. Bell derived a simple inequality which could be tested experimentally to infer whether hidden variable theories existed or not. This was done years later and quantum mechanics won. Hidden variables are incompatible with QM. All you have learned about superposition is indeed correct.

Measurements of compatible observables

Suppose you have 2 particles, 1 and 2. Let $\hat{A}_1$ and $\hat{B}_2$ be observables pertaining to each particle. Then we already learned that

$$[\hat{A}_1, \hat{B}_2] = 0 \tag{3}$$

ie, $\hat{A}_1$ and $\hat{B}_2$ are compatible. Let also

$$\begin{aligned} \hat{A}_1 |a_i\rangle &= a_i |a_i\rangle \\ \hat{B}_2 |b_j\rangle &= b_j |b_j\rangle \end{aligned} \tag{4}$$

Then in the product space we may use as basis the vectors

$$|a_i, b_j\rangle = |a_i\rangle \otimes |b_j\rangle \tag{5}$$

They satisfy

$$\begin{aligned} \hat{A}_1 |a_i, b_j\rangle &= a_i |a_i, b_j\rangle \\ \hat{B}_2 |a_i, b_j\rangle &= b_j |a_i, b_j\rangle \end{aligned} \tag{6}$$

Any vector $|v\rangle$ of the product space may be written as a linear combination of the vectors $|a_i, b_j\rangle$:

$$|v\rangle = \sum_{i,j} v_{ij} |a_i, b_j\rangle \tag{7}$$

We have also learned that if σ_{ij} may be written as
a product, $\sigma_{ij} = \alpha_i \beta_j$, then the two particles will not
be entangled because $|\sigma\rangle$ may be written as a product

$$|\sigma\rangle = \sum_{ij} \alpha_i \beta_j \, |a_i . b_j\rangle = \left[\sum_i \alpha_i |a_i\rangle \right] \otimes \left[\sum_j \beta_j |b_j\rangle \right] \qquad (8)$$

$$|\sigma\rangle = |\alpha\rangle \otimes |\beta\rangle .$$

Now suppose we measure $\hat{A}_1$. The outcomes will be one of its eigenvalues a_i. But with what probability? The answer is quite intuitive:

$$\boxed{P(a_i|\psi) = \sum_j |\langle a_i, b_j|\psi\rangle|^2} \qquad (9)$$

You simply sum over all possible states corresponding to the eigenvalue a_i. For $|\psi\rangle$ given by (7) we have

$$\langle a_i, b_j|\psi\rangle = \sum_{k,l} \psi_{kl} \langle a_i, b_j|a_k, b_l\rangle$$

$$= \psi_{ij}$$

so that

$$\boxed{P(a_i|\psi) = \sum_j |\psi_{ij}|^2} \qquad (10)$$

If the two states are not entangled, as in (8), then

$$P(a_i|\psi) = \sum_j |\alpha_i|^2 |\beta_j|^2$$

$$= |\alpha_i|^2 \underbrace{\sum_j |\beta_j|^2}_{1}$$

Thus, in this case we recover what we already knew

$$P(a_i|\psi) = |\alpha_i|^2 = |\langle a_i|\alpha\rangle|^2 \qquad (11)$$

e therefore conclude that if the state is a product, then he probabilities in a measurement of $\hat{A}_1$ are not affected by the state of particle 1. But if they are entangled, the state of 2 affects the outcome of particle 1. This is very strange because nowhere did we have to specify if the two particles were close to each other, or if they interact or not. This is what is really cool about entanglement: it is <u>non-local</u>.

The state after a measurement

If you have a single particle and you measure $\hat{A}$, you get a_i with prob. $|\langle a_i|\alpha\rangle|^2$ and, after the measurement the state collapses to $|a_i\rangle$. what if you have two particle and you measure $\hat{A}_1$. The prob. of getting a_i is given in (9), but what is the state after the measurement?. I want to convince you that the answer is

$$\boxed{|\psi, a_i\rangle = \frac{1}{\sqrt{P(a_i|\psi)}} \sum_j |a_i, b_j\rangle\langle a_i, b_j|\psi\rangle} \qquad (12)$$

To see why suppose first that there is no b_j. Then

$$P(a_i|\psi) = |\langle a_i|\psi\rangle|^2$$

so that

$$|\psi, a_i\rangle = \left[\frac{\langle a_i|\psi\rangle}{|\langle a_i|\psi\rangle|}\right]|a_i\rangle$$

The quantity inside [ ] is at most a complex phase so the state after the measurement is simply $|a_i\rangle$, as expected.

Since $\langle a_i, b_j|\psi\rangle = \psi_{ij}$ we may also write (12) as

$$\boxed{|\psi, a_i\rangle = \frac{1}{\sqrt{P(a_i|\psi)}} \sum_j \psi_{ij}\, |a_i, b_j\rangle} \qquad (12')$$

If the state is a product then $\vartheta_{ij} = \alpha_i \beta_j$ and $P(a_i|\vartheta) = |\alpha_i|^2$.

then

$$|\vartheta, a_i\rangle = \frac{1}{|\alpha_i|} \sum_j \alpha_i \beta_j |a_i, b_j\rangle$$

$$= \frac{\alpha_i}{|\alpha_i|} |a_i\rangle \otimes \left[\sum_j \beta_j |b_j\rangle \right]$$

The quantity $\alpha_i/|\alpha_i|$ is just a complex phase which may be neglected. Moreover, the state in [ ] is just $|\beta\rangle$, the original state of particle 2. Thus

$$|\vartheta, a_i\rangle = |a_i\rangle \otimes |\beta\rangle \tag{13}$$

This is very intuitive. The state after the measurement corresponds to a collapse of particle 1, while nothing happens to particle 2.

In conclusion, if the two particles are not entangled, measuring one does not affect the other in any way. But if they are entangled, it does.

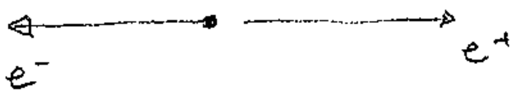
The EPR paradox

Now I want to discuss a paradox proposed by Einstein, Podolski and Rosen which favours the existence of hidden variables.

Consider the decay of the π meson into an electron and a positron (the anti-particle of the electron):

$$\pi \rightarrow e^- + e^+ \tag{14}$$

If the π is initially at rest, then after the decay the two particles will fly off in opposite directions due to the conservation of momentum

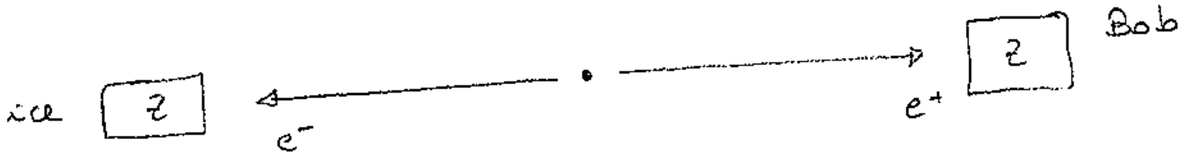


However, we also know that radioactive decays conserve angular momentum (spin). The spin of the π meson is 0 and the spin of the electron and the positron are both 1/2. Thus, the pair $e^- + e^+$ must be in a state whose total spin is zero. We know which state this is: its the singlet! Hence, the spin part of the (e^-, e^+) pair must be in the state

$$|0\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \tag{15}$$

This state is <u>entangled</u>. The radioactive decay produces the pair (e^-, e^+) in an entangled state!

Now suppose that at each end there is a physicist with a etector measuring the spin of each particle



Alice measures the spin of the e^- and Bob measures the spin of the e^+.

Suppose Alice goes first (she is a tiny bit closer than Bob). What is the prob. she will get + ? According to Eq (9)

$$P(\text{Alice} \uparrow \mid \text{Singlet}) = \underbrace{|\langle ++|0\rangle|^2}_{0} + \underbrace{|\langle +-|0\rangle|^2}_{1/2} = \frac{1}{2}$$

So she has a 50/50 chance of getting $\uparrow$ or $\downarrow$. And what is the state after the measurement? According to Eq (12)

$$|0;\text{Alice}\uparrow\rangle = \frac{1}{\sqrt{1/2}}\Big[\,|++\rangle \overbrace{\langle ++|0\rangle}^{0} + |+-\rangle \overbrace{\langle +-|0\rangle}^{1/\sqrt{2}}\,\Big]$$

$$= \sqrt{2}\,\frac{1}{\sqrt{2}}\,|+-\rangle$$

$$= |+-\rangle$$

So after Alice's measurement the state has collapsed to $|+-\rangle$. If, subsequentially, Bob measures the spin of the positron, he will <u>with certainty</u> find $|-\rangle$.

We thus reach a very strange conclusion. Due to entanglement, a measurement by Alice instantaneously affects the outcome of the experiment performed by Bob. If Alice gets ↑, Bob gets ↓ and if Alice gets ↓, Bob will get ↑. If Alice does not measure anything, Bob may get ↑ or ↓ with a 50/5[0] chance.

The reason why this is so strange is because Alice and Bob may be light years apart! So there seems to be information propagating faster than light. Einstein went crazy over this. The pillar of the theory of Relativity is that nothing propagates faster than the speed of light.

So the only explanation can be a Hidden variable: the two particles already knew in which state they were. We, dumb humans, didn't. But the particles did.

However, note that no information is propagated. If we perform the experiment several times we may obtain something like

Alice	Bob
+	−
−	+
−	+
+	−
+	−
+	−
−	+
+	−
−	+
−	+
+	−

[Th]e results of Alice and Bob are perfectly anti-correlated. [H]owever, Bob doesn't know that until he travels toward [A]lice to compare their data sheets. As far as Bob can tell, his results are just a pile of random numbers.

Thus, no information is propagated. So in this sense, this does not violate the principle of relativity.

Bell's Theorem

In 1964 John S. Bell proposed a simple test to check if Hidden variables exist or not. According to his test, hidden variables are incompatible with QM. This test was later realized experimentally by many authors and, indeed, QM wins: there are no hidden variables

Bell's theorem consists in studying the following experiment. Instead of measuring the z component of the spin, suppose Alice measures the spin a a certain directio $\vec{a}$ and Bob measures the spin in another direction $\vec{b}$. Thus, what we are doing is measuring the observables

$$\vec{a}\cdot\vec{\sigma_1} = a_x \hat{\sigma}_1{}^x + a_y \hat{\sigma}_1{}^y + a_z \hat{\sigma}_1{}^z \tag{16a}$$

and

$$\vec{b}\cdot\vec{\sigma_2} = b_x \hat{\sigma}_2{}^x + b_y \hat{\sigma}_2{}^y + b_z \hat{\sigma}_2{}^z \tag{16b}$$

Bell proposed that we look at the following quantity

$$\boxed{Q(\vec{a},\vec{b}) = \langle 0 | (\vec{a}\cdot\vec{\sigma_1})(\vec{b}\cdot\vec{\sigma_2}) | 0 \rangle} \tag{17}$$

that is, the expectation value of $(\vec{a}\cdot\vec{\sigma_1})(\vec{b}\cdot\vec{\sigma_2})$ calculated in the singlet state. It can be shown that

$$\boxed{Q(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b}} \tag{18}$$

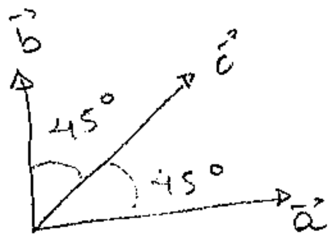
However, Bell showed (see Appendix) that *if* *hidden variables exist*, then

$$\boxed{|Q(\vec{a},\vec{b}) - Q(\vec{a},\vec{c})| \leq 1 + Q(\vec{b},\vec{c})} \qquad (19)$$

this is the famous Bell's inequality. Here $\vec{c}$ is just some other arbitrary direction.

If Hidden variables exist, Eq (19) must hold. Now we will choose a simple situation to show that (19) does *not* hold. This means that hidden variables are incompatible with QM.

choose



Then from (18)

$$Q(\vec{a},\vec{b}) = 0$$

$$Q(\vec{a},\vec{c}) = Q(\vec{b},\vec{c}) = -\cos(45^\circ) \approx -0.707.$$

Applying (19) we get

$$|0 - (-0.707))| \leq 1 + (-0.707)$$

$$0.707 \leq 0.293 \qquad !!! \text{ CRAZY.}$$

Bell's inequality has been violated. This is in fact observed experimentally, which means that there are no hidden variables. QM is correct. It may seem weird sometimes but it is nonetheless correct.

Appendix: derivation of Bell's inequality

We now derive Eq (19). The spin of the electron at Alice's position may take on the values $+1$ or -1. This will depend on the direction $\vec{a}$ at which Alice sets her apparatus and on the hidden variable, which we denote by λ. We denote this as

$$A(\vec{a}, \lambda) = \pm 1 \qquad (A.1)$$

Similarly, Bob's spin is

$$B(\vec{b}, \lambda) = \pm 1 \qquad (A.2)$$

The only thing we know is that, if both apparatuses are aligned in the same direction, then when Alice measures $+$, Bob measures $-$ and vice-versa. Thus

$$A(\vec{a}, \lambda) = - B(\vec{a}, \lambda). \qquad (A.3)$$

The quantity $Q(\vec{a}, \vec{b})$ in Eq (17) may be written in the language of hidden variables as

$$Q(\vec{a}, \vec{b}) = \int \rho(\lambda)\, A(\vec{a}, \lambda)\, B(\vec{b}, \lambda)\, d\lambda \qquad (A.4)$$

which corresponds to a "classical average" over the hidden variables with distribution $\rho(\lambda)$, where

$$\int \rho(\lambda)\, d\lambda = 1 \qquad (A.5)$$

using (A.3) in (A.4) we may write

$$Q(\vec{a},\vec{b}) = -\int \rho(\lambda)\, A(\vec{a},\lambda)\, A(\vec{b},\lambda)\, d\lambda \qquad (A.6)$$

substituting $\vec{b}$ for $\vec{c}$ we also have

$$Q(\vec{a},\vec{c}) = -\int \rho(\lambda)\, A(\vec{a},\lambda)\, A(\vec{c},\lambda)\, d\lambda \qquad (A.7)$$

Subtracting the two

$$Q(\vec{a},\vec{b}) - Q(\vec{a},\vec{c}) = -\int \rho(\lambda)\, A(\vec{a},\lambda)\left[A(\vec{b},\lambda) - A(\vec{c},\lambda)\right] d\lambda \qquad (A.8$$

But since $A = \pm 1$ we have that $A^2 = 1$. Thus we may write

$$A(\vec{b},\lambda) - A(\vec{c},\lambda) = A(\vec{b},\lambda)\left[1 - A(\vec{b},\lambda)\, A(\vec{c},\lambda)\right]$$

Thus

$$Q(\vec{a},\vec{b}) - Q(\vec{a},\vec{c}) = -\int d\rho\; A(\vec{a},\lambda)\, A(\vec{b},\lambda)\left[1 - A(\vec{b},\lambda)\, A(\vec{c},\lambda)\right] d\lambda$$

We are almost done. Since the A's are always ± 1, we have that

$$1 - A(\vec{b},\lambda)\,A(\vec{c},\lambda) = \begin{cases} 0 \\ \text{or} \\ 2 \end{cases}$$

and since $\rho(\lambda) \geqslant 0$ we conclude that

$$\rho(\lambda)\,[1 - A(\vec{b},\lambda)\,A(\vec{c},\lambda)] \geqslant 0$$

Moreover, we have that

$$-1 \leqslant A(\vec{a},\lambda)\,A(\vec{b},\lambda) \leqslant 1$$

from which it follows that

$$|Q(\vec{a},\vec{b}) - Q(\vec{a},\vec{c})| \leqslant \int \rho(\lambda)\,[1 - A(\vec{b},\lambda)\,A(\vec{c},\lambda)]\,d\rho$$

or

$$\boxed{|Q(\vec{a},\vec{b}) - Q(\vec{a},\vec{c})| \leqslant 1 + Q(\vec{b},\vec{c})}$$

which is Eq (19). q.e.d.