Phonons and field theory (continued)

Summary of previous notes

In the previous set of notes we discussed how to diagonalize the Hamiltonian of a 1D harmonic chain

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + \omega^2 q_i^2\right) + \frac{g}{2}\sum_{i=1}^{N}\left(q_i - q_{i+1}\right)^2 \tag{1}$$

we learned that the whole task reduced to finding the eigenvalues and eigenvectors of a cyclic matrix

$$A = \begin{pmatrix} \omega^2+2g & -g & 0 & 0 & -g \\ -g & \omega^2+2g & -g & 0 & 0 \\ 0 & -g & \omega^2+2g & -g & 0 \\ 0 & 0 & -g & \omega^2+2g & -g \\ -g & 0 & 0 & -g & \omega^2+2g \end{pmatrix} \tag{2}$$

we learned that this matrix is diagonalized by a Fourier transform

$$A = U \Lambda U^{+} \tag{3}$$

where Λ is a diagonal matrix with entries

$$\lambda_k = \omega^2 + 2g(1-\cos k) \tag{4}$$

and

$$U_{ik} = \frac{1}{\sqrt{N}}\, e^{ikx_i} \qquad x_i = i \tag{5}$$

$$k = \frac{2\pi l}{N}, \quad l = 0, \pm 1, \pm 2, \dots, \pm N/2$$

As a consequence of $UU^{+}=1$ and $U^{+}U=1$, we get

$$\frac{1}{N}\sum_i e^{i(k-k')x_i} = \delta_{kk'}, \qquad \frac{1}{N}\sum_k e^{ik(x_i-x_j)} = \delta_{ij} \tag{6}$$

If we now define a new set of operators

$$Q_k = \sum_i v_{ik}\, q_i = \frac{1}{\sqrt{N}} \sum_i e^{ikx_i}\, q_i \qquad (7)$$

$$P_k = \sum_i v_{ik}^*\, p_i = \frac{1}{\sqrt{N}} \sum_i e^{-ikx_i}\, p_i \qquad (8)$$

then Eq (1) becomes, with $\Omega_k = \sqrt{\lambda_k}$,

$$H = \frac{1}{2} \sum_k \left(P_k^\dagger P_k + \Omega_k^2\, Q_k^\dagger Q_k \right) \qquad (9)$$

The operators Q_k and P_k still satisfy the canonical algebra

$$[Q_k, P_k] = i\, \delta_{kk'} \qquad (10)$$

But they are not Hermitian and $Q_k^\dagger = Q_{-k}$, $P_k^\dagger = P_{-k}$.

To put (9) in the standard form we then define creation and annihilation operators

$$a_k = \sqrt{\frac{\Omega_k}{2}} \left(Q_k + \frac{i P_k^\dagger}{\Omega_k} \right) \qquad\qquad Q_k = \frac{1}{\sqrt{2\Omega_k}} \left(a_k + a_{-k}^\dagger \right)$$

$$\text{or} \qquad\qquad (11)$$

$$a_k^\dagger = \sqrt{\frac{\Omega_k}{2}} \left(Q_k^\dagger - \frac{i P_k}{\Omega_k} \right) \qquad\qquad P_k = i\sqrt{\frac{\Omega_k}{2}} \left(a_k^\dagger - a_{-k} \right)$$

then

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}$$

$$[a_k, a_{k'}] = 0 \qquad (12)$$

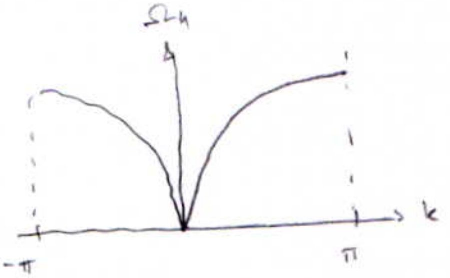
and the Hamiltonian becomes

$$\boxed{H = \sum_k \Omega_k \left(a_k^\dagger a_k + 1/2 \right)} \qquad (13)$$

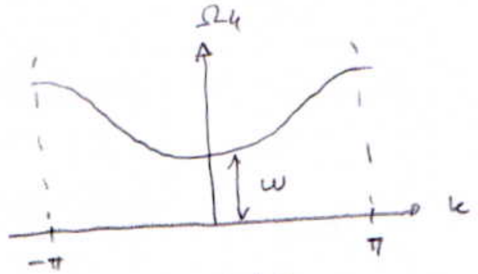
with

$$\Omega_k = \sqrt{\omega^2 + 2g(1-\cos k)} \tag{14}$$

From all this we emerged with the following interpretation: the operator $a_k^\dagger$ creates a quasi-particle, which we call a phonon, that has momentum k and energy Ω_k. The relation between Ω_k and k is called the dispersion relation. It looks like this



$\omega = 0$



$\omega \neq 0$

For small k we may expand (14) in a Taylor series to get

$$\Omega_k \simeq \sqrt{\omega^2 + g k^2} \tag{15}$$

which is just like the relativistic dispersion relation

$$E = \sqrt{m^2c^4 + p^2c^2} \tag{16}$$

(recall that $E = \hbar\Omega$ and $p = \hbar k$). Thus, the pinning frequency ω plays the role of a "mass" term, whereas the spring constant g represents the speed of sound.

Phonons are not particles. They are excitations of the vibrational modes of a chain. However, for all stands and purposes, they do behave like particles, with definite momentum and energy

Finally, we discussed how the Hamiltonian (13) is already diagonal in the Fock basis

$$|m\rangle = |m_{k_1}\rangle \otimes |m_{k_2}\rangle \otimes \cdots \tag{17}$$

where $k_1, k_2, \ldots,$ are all the allowed values of k. We have

$$H\,|m\rangle = E(m)\,|m\rangle \tag{18}$$

where

$$E(m) = \sum_k \Omega_k (m_k + 1/2) \quad , \quad m_k = 0, 1, 2, \ldots \tag{19}$$

With have therefore N quantum numbers m_k, each of which can be in any value $0, 1, 2, \ldots$. Each m_k represents the number of phonons in mode k.

Phonon - phonon interactions

The Hamiltonian (13) of a harmonic chain represents non-interacting phonons. the reason is that the Fock states are eigen-states of H, so if we create one phonon in the chain, it simply stays there

Phonon interactions appear if we consider anharmonic terms. For instance

$$H = \frac{1}{2} \sum_i (p_i^2 + \omega^2 q_i^2) + \frac{g}{2} \sum_i (q_i - q_{i+1})^2 + \frac{\mu}{3} \sum_i (q_i - q_{i+1})^3 + \frac{\lambda}{4} \sum_i (q_i - q_{i+1})^4 \tag{20}$$

we will not study these anharmonic terms in detail at this point, but there are some messages that I wanted to [illegible]. First, note that the quadratic part (the first line) is already diagonal (it is given by (13)) As for the other terms, we use (7) and (11) to write

$$q_i = \frac{1}{\sqrt{N}} \sum_k e^{-ikx_i} Q_k = \frac{1}{N} \sum_k \frac{e^{-ikx_i}}{\sqrt{2\Omega_k}} (a_k + a_{-k}^+) \tag{21}$$

In the last term we can change $k \to -k$ in the sum, to obtain the more symmetric form

$$\boxed{q_i = \frac{1}{N} \sum_k \frac{1}{\sqrt{2\Omega_k}} (a_k e^{-ikx_i} + a_k^+ e^{ikx_i})} \tag{22}$$

Both (21) and (22) can be useful.

Let's now work out the cubic term

$$(q_i - q_{i+1}) = \frac{1}{N} \sum_k e^{-ikx_i}(1 - e^{-ik})(a_k + a^\dagger_{-k})$$

$$(q_i - q_{i+1})^3 = \frac{1}{N} \sum_{kk'k''} e^{-i(k+k'+k'')x_i} (1 - e^{-ik})(1 - e^{-ik'})(1 - e^{-ik''}) (a_k + a^\dagger_{-k})(a_{k'} + a^\dagger_{-k'})(a_{k''} + a^\dagger_{-k''}) \tag{23}$$

we now carry out the sum over i and use (6). then

$$\frac{1}{N} \sum_i e^{-i(k+k'+k'')x_i} = \delta(k + k' + k'' = 0) \tag{24}$$

where $\delta(a = b)$ is just the Kronecker δ written a bit differently

Eq (23) thus becomes

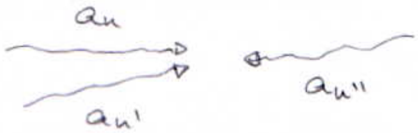
$$\sum_{i=1}^{N} (q_i - q_{i+1})^3 = \sum_{k,k'k''} \delta(k + k' + k'' = 0)(1 - e^{-ik})(1 - e^{-ik'})(1 - e^{-ik''}) (a_k + a^\dagger_{-k})(a_{k'} + a^\dagger_{-k'})(a_{k''} + a^\dagger_{-k''}) \tag{25}$$

this is still very complicated, but we can now start to have an idea of what kinds of processes are generated by a cubic term. What is really important is that these processes conserve momentum

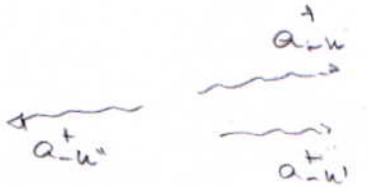
$$k + k' + k'' = 0 \tag{26}$$

this is a consequence of the translation invariance of our model

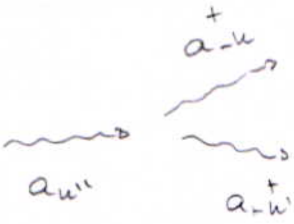
For instance, there will be a term in (25) which is of the form $a_k a_{k'} a_{k''}$. This is a triple annihilation



Two phonons collide with a third one and they are all annihilated, with momentum being conserved. Since H is Hermitian, we must also have triple creation processes $a^{\dagger}_{-k} a^{\dagger}_{-k'} a^{\dagger}_{-k''}$



And, of course, we have mixed processes, like $a^{\dagger}_{-k} a^{\dagger}_{-k'} a_{k''}$. This represents something like



Phonon $a_{k''}$ is annihilated and two other phonons are created, with momenta $k + k' = k''$.

Recall that H is the guy that drives Schrödinger's equation. Thus, if we start in an arbitrary state, as time goes on, we will start to see all these scattering events, leading to a extremely complicated dynamics.

So here are the messages I want you to take home

1) Any Hamiltonian which is <u>quadratic</u> in a and $a^\dagger$, is easy to deal with: you only need to diagonalize a matrix.

$$\boxed{\text{Quadratic = free particles}}$$

2) Any term which is cubic or higher is usually really really complicated. Normally handled with perturbation theory.

$$\boxed{\text{cubic, quartic = interactions / scattering processes}}$$

3) Momentum is always conserved when your lattice is translationally invariant (homogeneous and periodic boundary conditions).

Thermal properties

Let's go back to the start: I give you a 1D chain with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{N} \left(p_i^2 + \omega^2 q_i^2 \right) + \frac{g}{2} \sum_{i=1}^{N} \left(q_i - q_{i+1} \right)^2 \qquad (27)$$

and ask you to compute the thermal properties of this model. At first this seems like a very hard task: each oscillator lives in an infinite dimensional Hilbert space and there are N of them. How in the world will we be able to compute $e^{-\beta H}$?

Well, we first diagonalize this Hamiltonian, by writing it as

$$H = \sum_k \Omega_k \left(a_k^\dagger a_k + 1/2 \right) \qquad (28)$$

this procedure may require some effort. But now that it's in diagonal form, we simply have N independent harmonic oscillators, so computing thermal properties is trivial. For instance

$$Z = \operatorname{tr} e^{-\beta H} = \operatorname{tr} e^{-\beta \sum_k \Omega_k \left(a_k^\dagger a_k + 1/2 \right)} \qquad (29)$$

But the $a_k^\dagger a_k$ commute with each other so

$$\begin{aligned} Z &= \operatorname{tr} \left[\prod_k e^{-\beta \Omega_k \left(a_k^\dagger a_k + 1/2 \right)} \right] \\ &= \prod_k \operatorname{tr}_k \left(e^{-\beta \Omega_k \left(a_k^\dagger a_k + 1/2 \right)} \right) \end{aligned} \qquad (30)$$

We now have N independent traces, each identical to what we already found before in (26).

Thus

$$Z = \prod_k \frac{e^{-\beta\Omega_k/2}}{(1-e^{-\beta\Omega_k})} = \prod_k Z_k \qquad (31)$$

Similarly, the Gibbs state becomes simply

$$\boxed{\rho = \prod_k \rho_k = \prod_k (1-e^{-\beta\Omega_k})\, e^{-\beta\Omega_k a_k^\dagger a_k}} \qquad (32)$$

For instance, we know that for one harmonic oscillator

$$\Omega(\langle a^\dagger a\rangle + 1/2) = \Omega(\bar{n} + 1/2) = \frac{\Omega}{2}\coth\left(\frac{\Omega}{2T}\right) \qquad (33)$$

where

$$\bar{n} = \frac{1}{e^{\beta\Omega}-1}$$

Thus, since (87) is just a sum of independent oscillators, we get

$$\boxed{U = \langle H\rangle = \sum_k \Omega_k(\bar{n}_k + 1/2) = \sum_k \frac{\Omega_k}{2}\coth\left(\frac{\Omega_k}{2T}\right)} \qquad (34)$$

A similar reasoning holds for all other thermodynamic quantities. For instance

$$\langle a_k^\dagger a_k\rangle = \bar{n}_k = \frac{1}{e^{\beta\Omega_k}-1} \qquad (35)$$

Let's now compute some more interesting quantities. For instance using Eq (22) we can compute

$$\langle q_i \rangle = \frac{1}{N} \sum_{k} \frac{1}{\sqrt{2\Omega_k}} \left(\langle a_k \rangle e^{-ikx_i} + \langle a_k^\dagger \rangle e^{ikx_i} \right)$$

But $\langle a_k \rangle = \langle a_k^\dagger \rangle = 0$ in equilibrium so

$$\langle q_i \rangle = 0 \tag{36}$$

This was expected as in equilibrium the oscillators should be equally likely to oscillate to the left or to the right.

Instead, let us now compute a correlation function, like $\langle q_i q_{i+r} \rangle$. We have, from (22)

$$\langle q_i q_{i+r} \rangle = \frac{1}{N^2} \sum_{kk'} \frac{1}{\sqrt{2\Omega_k}} \frac{1}{\sqrt{2\Omega_{k'}}} \langle (a_k e^{-ikx_i} + a_k^\dagger e^{ikx_i}) \cdot (a_{k'} e^{-ik'x_{i+r}} + a_{k'}^\dagger e^{ik'x_{i+r}}) \rangle$$

But now we use the fact that, in equilibrium

$$\begin{aligned} \langle a_k^\dagger a_{k'} \rangle &= \bar{n}_k \delta_{kk'} \\ \langle a_k a_{k'}^\dagger \rangle &= (\bar{n}_k + 1) \delta_{kk'} \\ \langle a_k a_{k'} \rangle &= \langle a_k^\dagger a_{k'}^\dagger \rangle = 0 \end{aligned} \tag{37}$$

We then get

$$\langle q_i\, q_{i+r} \rangle = \frac{1}{N^2} \sum_{kk'} \frac{1}{2\sqrt{\Omega_k \Omega_{k'}}} \Big\{ \langle a_k a_{k'}^\dagger \rangle\, e^{-ikx_i}\, e^{ik'x_{i+r}} + \langle a_k^\dagger a_{k'} \rangle\, e^{ikx_i}\, e^{-ik'x_{i+r}} \Big\}$$

$$= \frac{1}{N^2} \sum_{k} \frac{1}{2\Omega_k} \Big\{ (\bar{n}_k + 1)\, e^{ikx_r} + \bar{n}_k\, e^{-ikx_r} \Big\}$$

We see that the result is independent of i, which is again a consequence of translation invariance. Using the fact that both Ω_k and $\bar{n}_k$ are even in k, we can also write

$$\langle q_i\, q_{i+r} \rangle = \frac{1}{N^2} \sum_{k} \frac{1}{2\Omega_k} (2\bar{n}_k + 1)\, e^{ikx_r}$$

$$= \frac{1}{N^2} \sum_{k} \frac{1}{2\Omega_k} (2\bar{n}_k + 1)\, e^{-ikx_r}$$

Taking the average of the two results, we get

$$\boxed{\langle q_i\, q_{i+r} \rangle = \frac{1}{N^2} \sum_{k} \frac{(\bar{n}_k + 1/2)}{\Omega_k} \cos(kr)} \qquad (38)$$

Note that even at $T=0$ we still get some correlation because at $T=0$ we get $\bar{n}_k = 0$, but the "1" is still there

$$\langle q_i\, q_{i+r} \rangle = \frac{1}{N^2} \sum_{k} \frac{1}{2\Omega_k} \cos(kr) \qquad (T=0) \qquad (39)$$

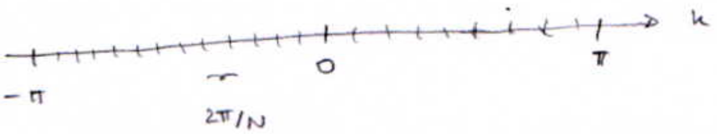
Converting sums to integrals

We see in both (34) and (38) that thermodynamic quantities are expressed in terms of momentum sums. Carrying out these sums is in general not easy. But if we are only interested in the thermodynamic limit ($N \to \infty$), then we can convert them to integrals.

Recall that

$$k = \frac{2\pi \ell}{N}, \quad \ell = 0, \pm 1, \pm 2, \dots \pm \frac{N}{2}$$

Thus we have something like



For N large the spacings become increasingly smaller.

To convert a k-sum to a k-integral, you should always identify the convenient 1: in our case

$$\Delta k = \frac{2\pi}{N}$$

so

$$\frac{N}{2\pi} \Delta k = 1 \tag{40}$$

This is our convenient 1.

Given an arbitrary function $f(k)$, we can now write

$$\sum_k f(k) = \frac{N}{2\pi} \sum_k \Delta k \, f(k)$$

the resulting sum is now in the form of a Riemann sum, so when N is sufficiently large we may approximate

$$\boxed{\sum_k f(k) = \frac{N}{2\pi} \sum_k \Delta k \, f(k) = \frac{N}{2\pi} \int_{-\pi}^{\pi} dk \, f(k)} \qquad (41)$$

If you always follow this recipe, you will never run into trouble. I also always suggest that you try to work with discrete k for as much as you can, and only convert to an integral at the very end.

Eq (34) may now be written as

$$U = \frac{N}{2\pi} \int_{-\pi}^{\pi} dk \, \Omega_k (\bar{n}_k + 1/2) \qquad (42)$$

and we see that, as expected, the energy is extensive. Similarly Eq (38) becomes

$$\langle q_i q_{i+r} \rangle = \frac{1}{2\pi N} \int_{-\pi}^{\pi} dk \, \frac{(\bar{n}_k + 1/2)}{\Omega_k} \cos(kr) \qquad (43)$$

The specific heat of phonons

Specific heat measurements are extremely important in condensed matter, as they are relatively simple to perform, but still provide valuable information. The specific heat of a solid will contain contributions from the several degrees of freedom, such as the vibrations, the conduction electrons, the magnetic moments and so on.

In a typical insulator, the main contribution comes from the phonons. A typical specific heat curve would look like this

For high temperatures it tends to the value of $3k_B$, which is known as the law of Dulong and Petit. But as you cool down to cryogenic temperatures, c goes down all the way to zero when $T \to 0$. This is a quantum effect and cannot be explained by classical physics at all.

The first hint came from Einstein, who realized the similarity between this curve and the specific heat of a single quantum harmonic oscillator. Recall that for one oscillator

$$U = \omega (\bar{n} + 1/2)$$
$$\bar{n} = \frac{1}{e^{\beta\omega} - 1} \qquad (44)$$

Thus

$$C = \frac{\partial U}{\partial T} = \frac{(\beta\omega)^2 e^{\beta\omega}}{(e^{\beta\omega}-1)^2} \tag{45}$$

This curve looks a lot like the one in the figure above. In particular, when $T \to \infty$ ($\beta \to 0$) we get

$$C \simeq \frac{(\beta\omega)^2 \, e^{\beta\omega}}{(1+\beta\omega-1)^2} \simeq 1 \tag{46}$$

Thus, Einstein argued that the Dulong and Petit value of $3k_B$ (for us $k_B = 1$) was due to the fact that each atom in the crystal has 3 degrees of freedom, associated with the vibrations in the x, y and z directions.

There is a problem with Einstein's result, however. If we look at c for very low temperatures, it is found experimentally that

$$c \sim A T^3 \tag{47}$$

where A is a constant. This, however, is not what (45) predicts. Instead, it predicts an exponential decay

$$c \simeq e^{-\beta\omega} \qquad T \to 0 \tag{48}$$

The reason for this discrepancy is that Einstein's model assumes the atoms all oscillate with the same frequency. And we already know this is not true: the atoms undergo collective oscillations related to the normal modes, each with a different frequency Ω_k. Debye was the first to realize this and use the concept of phonons to predict the T^3 law.

Before we talk about Debye's model, let's quickly discuss how phonons appear in realistic materials. the dispersion relations of phonons in real systems can be insanely complicated. First, momentum is now a vector

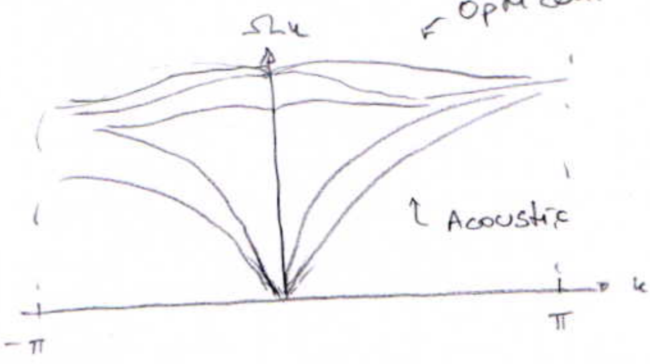
$$\mathbf{k} = (k_x, k_y, k_z) \tag{49}$$

each k_i, quantized as before, with independent quantum numbers

$$k_i = \frac{2\pi l_i}{N_i}, \qquad l_i = 0, \pm 1, \ldots, \pm \frac{N_i}{2} \tag{50}$$

where N_i is the number of atoms in each direction.

But the though part is that in real systems the dispersion relations will have many branches. Something like



Oops: I forgot to say: in real systems there is no pinning (that's a theoretician's thing).

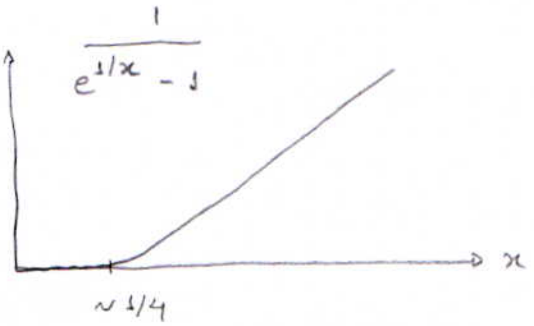
Each branch describes a possible type of vibration and they fall into two categories:

Acoustic branches: $\Omega_k \propto |k|$ for small k

Optical branches: $\Omega_k \propto \text{const}$ for small k

Moreover, any 3D material always has 3 acoustic branches, but a varying number of optical branches.

Lucky for us, as far as the specific heat is concerned, all that matters are the acoustic branches, specially at low temperatures. The reason is simple to understand. The plot of a phonon occupation looks like this



If $T/\Omega \lesssim 1/4$, that mode is practically unoccupied and will thus not contribute to the specific heat. Hence, for a mode to contribute, we must have

$$\frac{T}{\Omega_k} \gg 1 \quad \text{or} \quad \Omega_k \ll T \tag{51}$$

The optical modes oscillate at high frequency (see figure) and thus contribute negligibly. In fact, even in the acoustic modes, the only significant contribution will come from the part near k small.

This is good. We can now have a look at what to expect at low temperatures. We have 3 acoustic modes and we can approximate their dispersion relations as

$$\Omega_{\mathbf{k}} \simeq c\,|\mathbf{k}| \tag{52}$$

for some speed of sound c. The energy will then be given by
(44)

$$U = 3 \sum_{\mathbf{k}} \frac{\Omega_{\mathbf{k}}}{e^{\beta \Omega_{\mathbf{k}}} - 1} \tag{53}$$

where I omit the zero-point fluctuations for simplicity. We now convert the sum to an integral by introducing the "convenient 1"

$$\frac{V}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z = 1 \tag{54}$$

where $V = N_x N_y N_z$ is the volume. We then get

$$U = \frac{3V}{(2\pi)^3} \int d^3\mathbf{k} \, \frac{\Omega_{\mathbf{k}}}{e^{\beta \Omega_{\mathbf{k}}} - 1} \tag{55}$$

The limits of integration are in principle ± π, but the integrand is only significant for low $\mathbf{k}$, so we can extend the limits of integration to infinity.

We now move to spherical coordinates. Ω_k depends only on $k = |\mathbf{k}|$, so we can carry out the integration over the angles, leading to

$$U = \frac{3V}{(2\pi)^3} 4\pi \int_0^\infty dk\, k^2 \frac{(ck)}{e^{\beta ck} - 1}$$

And we are almost done. To finish we change integration variables to

$$x = \beta c k$$

Then we get

$$U = \frac{3V}{(2\pi)^3} 4\pi\, c\, \frac{1}{(\beta c)^4} \int_0^\infty dx\, \frac{x^3}{e^x - 1} \qquad (56)$$

I love this trick! Stop and realize what we did. We had a really complicated integral, but we changed variable and put all relevant physical parameters outside! Now the remaining integral is just a number

$$\int_0^\infty dx\, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \quad \text{(what the hell!)}$$

But none of these constants matter. All that matters is how U depends on T:

$$U = \text{const} \times T^4$$

Consequently,

$$\boxed{c \propto T^3} \qquad (57)$$

We have therefore succeeded to show that the existence of quantized collective vibrations of a crystal explains an experimental observation.

This result had a remarkable historical importance for the early development of quantum theory and quantum statistical mechanics. The reason is that specific heat is a macroscopic quantity. It was therefore quite remarkable that a clear quantum signature could be observed on such a bulky quantity.

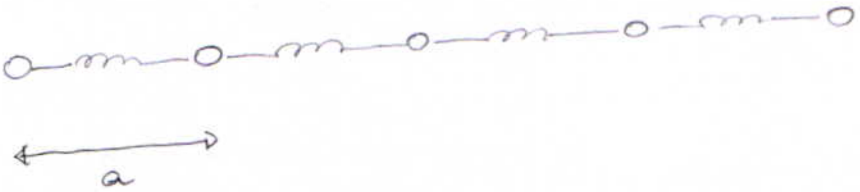
Continuum limit and field theory

As our last topic in the phonon realm, I want to discuss what happens when we take the continuum limit of the phonon model. This will serve for us as a gentle introduction to field theory.

We consider once again the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + w^2 q_i^2\right) + \frac{g}{2}\sum_{i}\left(q_i - q_{i+1}\right)^2 \qquad (58)$$

But now let's introduce explicitly the equilibrium separation between the oscillators, a



What I mean by continuum limit is to assume that the lattice spacing a itself is very tiny compared to the vibrations we are interested in. That is, we want to see what happens from a distance

↑ We are looking at the chain from far away.

We will therefore define a <u>field</u>

$$\phi(x_i) = \frac{q_i}{\sqrt{a}} \qquad (59)$$

where the factor of $1/\sqrt{a}$ is just for convenience (as we will see below). The idea is that in this continuum limit $\phi(x_n)$ can become $\phi(x)$, a operator that depends continuously on the position x.

The potential energy can now be written as

$$\frac{g}{2}\sum_{i=1}^{N}(q_i - q_{i+1})^2 = \frac{ga}{2}\sum_{i=1}^{N}\left[\phi(x_i) - \phi(x_{i+1})\right]^2$$

I want to make this look like a derivative, so we write

$$\frac{g}{2}\sum_{i=1}^{N}(q_i - q_{i+1})^2 = \frac{ga^3}{2}\sum_{i=1}^{N}\left[\frac{\phi(x_i) - \phi(x_{i+1})}{a}\right]^2$$

This now looks like a derivative because a is like Δx

$$\frac{\phi(x_{i+1}) - \phi(x_i)}{a} \simeq \partial_x \phi$$

Thus the kinetic energy becomes

$$\frac{g}{2}\sum_{i=1}^{N}(q_i - q_{i+1})^2 = \frac{ga^2}{2}\sum_{i} a\,(\partial_x \phi)^2$$

I left an "a" inside the sum because this now has the shape of a Riemann sum

$$\frac{g}{2}\sum_{i}(q_i - q_{i+1})^2 \simeq \frac{ga^2}{2}\int_0^L dx\ (\partial_x \phi)^2 \tag{60}$$

this is the continuum limit. To finish we just do the same for the first term in (58).

$$\frac{1}{2}\sum_{i=1}^{N} \omega^2 q_i^2 = \frac{1}{2}\sum_{i=1}^{N} \omega^2 a\, \phi(x_i)^2$$

$$= \frac{\omega^2}{2}\int dx\ \phi(x)^2$$

For the momentum we similarly define

$$\Pi(x_i) = \frac{p_i}{\sqrt{a}} \tag{61}$$

Then

$$\sum_{i}\frac{p_i^2}{2} = \frac{1}{2}\sum_{i} a\, \Pi(x_i)^2$$

$$= \frac{1}{2}\int dx\ \Pi(x)^2$$

thus we finally arrive at the continuum Hamiltonian

$$\boxed{H = \int_0^L dx \left\{ \frac{\Pi(x)^2 + \omega^2 \phi(x)^2}{2} + \frac{\lambda^2}{2}(\partial_x \phi)^2 \right\}} \tag{62}$$

where $\lambda = ga^2$

We have now passed from a picture where we had discrete oscillators at different positions, to a picture where the vibrations are described by a continuous field $\phi(x)$, which represents the elastic field of the material, describing the deformations in each position. So you see, this is essentially a change from an atomistic description of the vibrations of a solid, to a macroscopic description where we focus on the deformations of an elastic medium.

Thus, field theory is capable of capturing the low energy of long wavelength behavior. And that is why it is so important in condensed matter. It functions as an effective description of more complex behavior, which focuses on low energy excitations.