The central force problem

In these notes I will discuss one of the most important problems in classical mechanics: the motion of a particle under the influence of a potential $U(r)$, which depends only on its distance from the center. This is called the central force problem. The canonical examples are the gravitational and electrostatic forces, for which

$$U(r) = -\frac{\alpha}{r} \tag{1}$$

where α is a constant ($\alpha > 0$ in the gravitational problem, but may have any sign in the electrostatic case).

The Lagrangian I will consider is

$$\mathcal{L} = \frac{1}{2} m v^2 - U(r) \tag{2}$$

Now is a good time to remember how this Lagrangian may be derived from the two-body problem

$$\mathcal{L} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{3}$$

We have seen that this may be decomposed into the motion of the center of mass of the system, with position

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{4}$$

and the relative motion with coordinates

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{5}$$

In terms of these new variables Eq (3) is written as

$$\mathcal{L} = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} m \dot{\mathbf{r}}^2 - U(r) \tag{6}$$

where

$$M = m_1 + m_2$$
$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{7}$$

are respectively the total mass and the reduced mass.

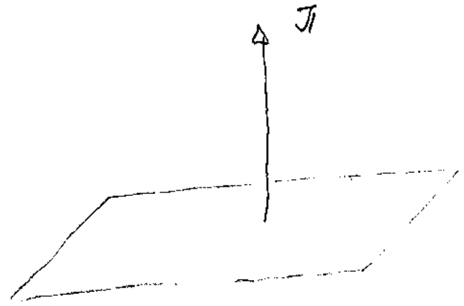
The Lagrangian (6) decomposes into two non-interacting parts, showing that the motion of the center of mass is independent of the relative motion.

The problem we will consider here [Eq (2)] is the second part of Eq (6). So the mass m in (2) is the reduced mass. Of course, quite oftenly, one body is much heavier than the other so the reduced mass ends up being approximately the mass of the lighter body.

The Lagrangian (2) is invariant under rotations, which you may see by noting that it depends only on scalars. Therefore, the angular momentum of the system is conserved

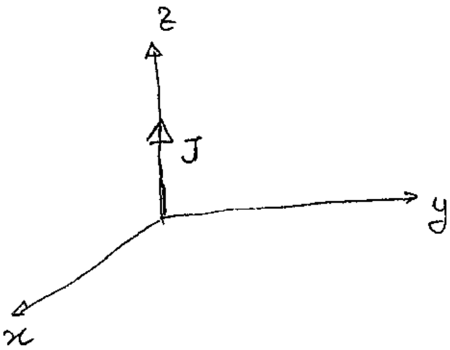
$$\mathbf{J} = m(\mathbf{r} \times \mathbf{v}) = \text{constant}. \tag{8}$$

This gives us 3 constants of the motion. By definition, $\mathbf{J}$ is perpendicular to both $\mathbf{r}$ and $\mathbf{v}$. Thus, since $\mathbf{J}$ is constant, the motion will occur in a plane perpendicular to $\mathbf{J}$



The direction of $\mathbf{J}$ is fixed by the initial conditions and from there on, the plane in which the motion occurs will be the same throughout.

It is convenient to work with a coordinate axis where $\mathbf{J}$ is parallel to $\hat{z}$



The motion will then occur in the xy plane.

We also use circular coordinates

$$x = r\cos\phi$$
$$y = r\sin\phi \qquad (18)$$

with which the Lagrangian (2) becomes

$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r) \qquad (19)$$

As can be seen, ϕ is a cyclic coordinate so its corresponding generalized momentum will be conserved

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} = \text{constant} \qquad (20)$$

This turns out to be the same as the magnitude of the angular momentum

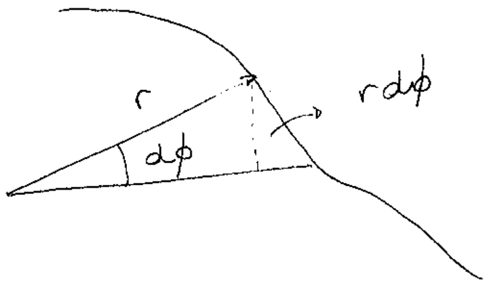
$$J = J_z = m(x\dot{y} - y\dot{x}) = m r^2 \dot{\phi} = p_\phi \qquad (21)$$

which makes sense: the fact that $\vec{J}$ = constant gives us 3 conserved quantities, But by fixing the direction of $\vec{J}$ we only use 2 of them.

We therefore have

$$\boxed{J = m r^2 \dot{\phi} = \text{constant}} \qquad (22)$$

This law has an important geometrical interpretation:



If we consider an infinitesimal chunk of the path, it has length $rd\phi$ and therefore the bounding area may be approximated by the area of a triangle:

$$\frac{1}{2} r \times rd\phi$$

We therefore see that J in (22) may be written as

$$J = 2m\dot{A}, \quad \dot{A} = \frac{1}{2} r^2 \dot{\phi} \tag{23}$$

A is the <u>sectorial area</u> of the path and hence $\dot{A}$ is the sectorial velocity.

The conservation of angular momentum therefore implies that the velocity with which the area defined by r is swept is constant. In other words, in equal times the radius vector will sweep equal areas. This is <u>Kepler's second law</u>.

The energy of the system is also a conserved quantity. From the Lagrangian (19) we find

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) + U(r) \tag{24}$$

Since J is constant we may substitute

$$\dot{\phi} = \frac{J}{mr^2} \tag{25}$$

then

$$\frac{1}{2} m r^2 \dot{\phi}^2 = \frac{1}{2} m r^2 \frac{J^2}{m^2 r^4} = \frac{1}{2} \frac{J^2}{mr^2}$$

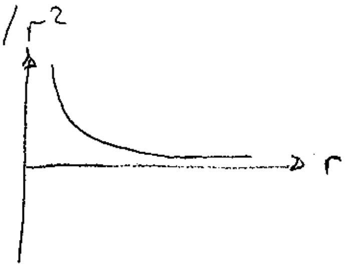
The energy may, in this way, be written as a function of r only:

$$\boxed{E = \frac{1}{2} m \dot{r}^2 + \frac{J^2}{2mr^2} + U(r)} \tag{26}$$

We see from this equation that the motion resembles that of a one-dimensional particle under the influence of an effective potential

$$\boxed{U_{eff}(r) = \frac{J^2}{2mr^2} + U(r)} \tag{27}$$

The term $J^2/2mr^2$ is called the <u>centrifugal energy</u>. It looks like



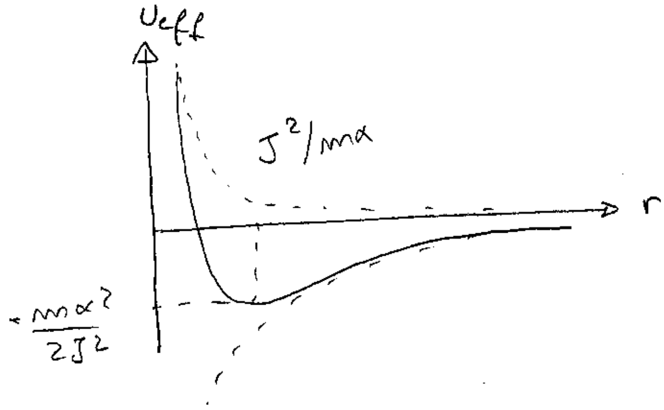
Thus, it has the tendency to expel the particle from the origin. Note how it is determined by the angular momentum.

Let us consider the gravitational potential in Eq (5).

Then

$$U_{eff} = \frac{J^2}{2mr^2} - \frac{\alpha}{r} \qquad (28)$$

This effective potential has the following form



The gravitational interaction pulls the particle to the center while the centrifugal force pulls it away. This competition leads to a minimum, whose value is

$$U_{eff, min} = -\frac{m\alpha^2}{2J^2} \qquad (28')$$

and occurs at

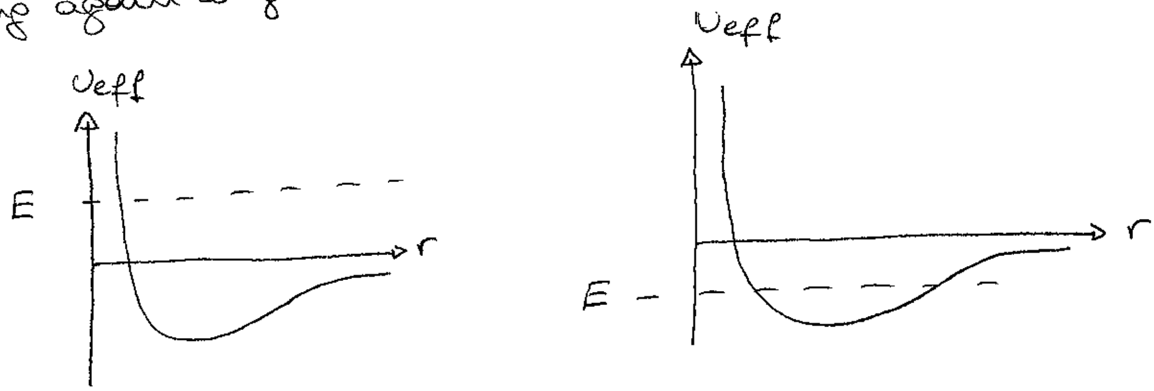
$$r_{min} = J^2/m\alpha \qquad (28'')$$

The type of motion is determined by analysing the energy, just as was done in the 1D case. The turning points are those where $\dot{r} = 0$ or

$$E = U_{eff}(r^*) \tag{29}$$

At the turning points the radial velocity $\dot{r}$ is zero. This does not mean, however, that the particle will stop since we also have the angular motion, which never stops.

Supposing again a gravitational potential, we now see two possibilities



If (29) has only one solution r^*_{min}, the motion will be infinite. This is what happens with comets, which approach our solar system, hang around for a while, and then leave.

Conversely, if (29) has 2 roots, r_{min} and r_{max}, then the motion will be bounded.

Formal solution

Proceeding like we did in the 1D case, we may now obtain a formal solution starting from the energy in (26):

$$E = \frac{1}{2} m \dot{r}^2 + U_{eff}(r) \qquad (30)$$

So

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left[E - U_{eff}(r)\right]}} \qquad (31)$$

Integrating this equation gives $t(r)$:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left[E - U_{eff}(r)\right]}} + \text{const} \qquad (32)$$

Similarly, we may use (22) to find $\phi(r)$. We first write

$$d\phi = \frac{J}{mr^2}\, dt \qquad (33)$$

and then use (31) to eliminate dt:

$$d\phi = \frac{J}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}\left[E - U_{eff}(r)\right]}} \qquad (34)$$

Integrating we find

$$\boxed{\phi = \int \frac{J}{mr^2} \frac{dr}{\sqrt{\frac{2}{m}\left[E - U_{eff}(r)\right]}} + \text{const}} \qquad (35)$$

This is the formal solution for the <u>path</u> of a particle. It is useful to note that, since $\dot{\phi} = J/mr^2 > 0$, the angle ϕ always increases with time. That is, the angular motion is monotonic. Usually we choose the constant to be zero.

The paths of the gravitational problem

Now let us apply (35) to the case $U = -\alpha/r$;

$$\phi = \frac{J}{m} \int \frac{dr/r^2}{\sqrt{\frac{2}{m}\left[E - \frac{J^2}{2mr^2} + \frac{\alpha}{r}\right]}} \tag{36}$$

Before we compute this integral, let us clean it up a bit.

Define

$$p = \frac{J^2}{m\alpha} \tag{37}$$

this quantity has units of length so we may define a dimensionless quantity

$$x = \frac{r}{p} \tag{38}$$

Then

$$\frac{J^2}{2mr^2} = \frac{p\alpha}{2r^2} = \frac{\alpha}{2px^2}$$

and

$$E - \frac{J^2}{2mr^2} + \frac{\alpha}{r} = E - \frac{\alpha}{p}\frac{1}{2x^2} + \frac{\alpha}{px}$$

$$= \frac{\alpha}{p}\left[\frac{Ep}{\alpha} - \frac{1}{2x^2} + \frac{1}{x}\right]$$

Let

$$2 = \frac{Ep}{\alpha} = \frac{J^2 E}{m\alpha^2} \tag{39}$$

Note that 2 may be either positive or negative.

with these changes Eq (36) becomes

$$\phi = \frac{J}{mp} \int \frac{dx/x^2}{\sqrt{\frac{2\alpha}{mp}\left(2 - \frac{1}{2x^2} + \frac{1}{x}\right)}}$$

the constant in front will be

$$\frac{J}{mp}\sqrt{\frac{mp}{2\alpha}} = \sqrt{\frac{J^2}{2\alpha mp}} = \frac{1}{\sqrt{2}}$$

thus

$$\phi = \frac{1}{\sqrt{2}} \int \frac{dx/x^2}{\sqrt{2 - \frac{1}{2x^2} + \frac{1}{x}}} \qquad (40)$$

Now that we have a clean integral, the first step to compute it is to complete squares:

$$2 - \frac{1}{2x^2} + \frac{1}{x} = 2 - \frac{1}{2}\left[\frac{1}{x^2} - \frac{2}{x}\right]$$

$$= 2 - \frac{1}{2}\left[\left(\frac{1}{x} - 1\right)^2 - 1\right]$$

$$= 2 + \frac{1}{2} - \frac{1}{2}\left(\frac{1}{x} - 1\right)^2$$

Now let

$$y = \frac{1}{x} - 1 \quad \text{or} \quad x = \frac{1}{y+1}$$

$$dx = -\frac{1}{(y+1)^2}\,dy$$

then

$$\phi = \frac{1}{\sqrt{2}} \int \frac{(-dy)}{(1+y)^2} (1+y)^2 \frac{1}{\sqrt{\frac{1}{2}+2-y^2/2}}$$

$$= \frac{-1}{\sqrt{2}} \frac{1}{\sqrt{\frac{1}{2}+2}} \int \frac{dy}{\sqrt{1-\left(\frac{y^2}{1+2\eta}\right)}}$$

This integral may be computed because

$$\frac{d}{dz} \arccos(z) = \frac{-1}{\sqrt{1-z^2}}$$

Setting $z = \frac{y}{\sqrt{1+2\eta}}$ we then get

$$\phi = \arccos\left(\frac{y}{\sqrt{1+2\eta}}\right)$$

Let

$$\boxed{\epsilon = \sqrt{1+2\eta} = \sqrt{1+\frac{2J^2E}{m\alpha^2}}} \qquad (40)$$

we then get

$$\epsilon \cos\phi = y = \frac{1}{x} - 1 = \frac{p}{r} - 1$$

which finally gives

$$\boxed{\frac{p}{r} = 1 + \epsilon\cos\phi} \qquad (41)$$

This is the formula describing the trajectory of the particle. It depends on $p = J^2/m\alpha$, which is determined by the initial angular momentum and is called the latus rectum, and on ϵ, which is called the eccentricity and depends on the energy.

Eq (41) specifies how r depends on ϕ. We may then use it to find the coordinates

$$x = r\cos\phi \qquad (42)$$
$$y = r\sin\phi$$

Eq (41) describes a conic section; ie, it describes the intersection of a cone with a plane. This may lead to 3 types of curves, an ellipse, a parabola and a hyperbole, depending on the value of ϵ. The classification, as we will investigate in detail, is:

$\epsilon = 0$	$E = U_{eff,min} = -\frac{m\alpha^2}{2J^2}$	circle
$0 < \epsilon < 1$	$E < 0$	ellipse
$\epsilon = 1$	$E = 0$	parabola
$\epsilon > 1$	$E > 0$	hyperbole.

Finite motion: ellipse

when $E<0$ the motion is finite because there are two turning points. They are determined from the Eq

$$\frac{J^2}{2mr^2} - \frac{\alpha}{r} = E$$

or

$$\frac{1}{2x^2} - \frac{1}{x} = \frac{Ep}{\alpha} = \lambda$$

Multiplying by x^2:

$$\lambda x^2 + x - 1/2 = 0$$

so

$$x = -\frac{1}{2\lambda} \pm \frac{1}{2\lambda}\sqrt{1+2\lambda}$$

$$= -\frac{1}{2\lambda}\left[1 \mp \epsilon\right]$$

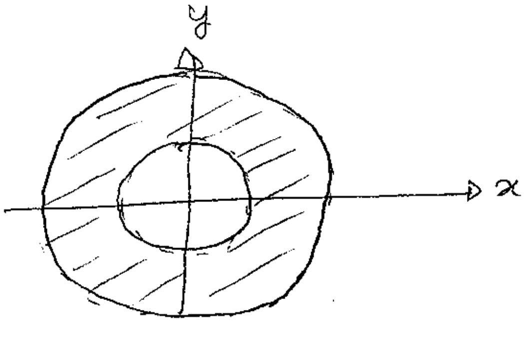
Since $E<0$ we arrive at

$$x = \frac{\alpha}{2p|E|}\left(1 \pm \epsilon\right)$$

The turning points are therefore

$$\boxed{r_{min} = \frac{\alpha}{2|E|}\left(1-\epsilon\right) \qquad r_{max} = \frac{\alpha}{2|E|}\left(1+\epsilon\right)} \tag{43}$$

The motion must therefore occur within an annulus bounded by these two radius vectors



Next let us consider the path in cartesian coordinates.
Using (41) we have

$$\frac{p}{r} = 1 + \epsilon \cos\phi = 1 + \frac{\epsilon x}{r}$$

Thus
$$(p - \epsilon x) = r = \sqrt{x^2 + y^2}$$

$$(p - \epsilon x)^2 = x^2 + y^2$$

$$p^2 - 2p\epsilon x + \epsilon^2 x^2 = x^2 + y^2$$

$\Rightarrow$
$$x^2(1-\epsilon^2) + 2p\epsilon x + y^2 = p^2$$

$$x^2 + \frac{2p\epsilon}{1-\epsilon^2} x + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{1-\epsilon^2}$$

Now we complete squares on x: let

$$x_0 = \frac{p\epsilon}{1-\epsilon^2} \tag{44}$$

Then we get

$$(x+x_0)^2 + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{1-\epsilon^2} + x_0^2$$

But

$$\frac{p^2}{1-\epsilon^2} + x_0^2 = \frac{p^2}{1-\epsilon^2} + \frac{p^2\epsilon^2}{(1-\epsilon^2)^2}$$

$$= \frac{p^2}{1-\epsilon^2}\left[\frac{1-\epsilon^2+\epsilon^2}{1-\epsilon^2}\right]$$

$$= \frac{p^2}{(1-\epsilon^2)^2}$$

Thus

$$(x+x_0)^2 + \frac{y^2}{1-\epsilon^2} = \frac{p^2}{(1-\epsilon^2)^2}$$

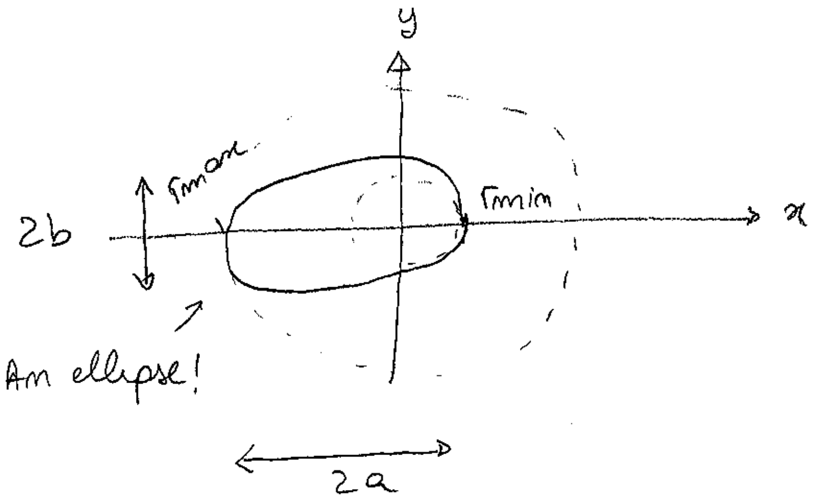
Define

$$\boxed{\begin{aligned} a &= \frac{p}{1-\epsilon^2} = \frac{\alpha}{2|E|} \\ b &= \frac{p}{\sqrt{1-\epsilon^2}} = \frac{J}{\sqrt{2m|E|}} \end{aligned}} \qquad (45)$$

Then we finally arrive at

$$\boxed{\frac{(x+x_0)^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (46)$$

which is the equation of an ellipse.



The values of r_{min} and r_{max} obtained in (43) can be rederived as

$$r_{min} = a - x_0$$

$$r_{max} = a + x_0$$

or

$$r_{min} = a(1-\epsilon) \qquad r_{max} = a(1+\epsilon) \qquad (47)$$

which is the same as (43) because $a = \alpha/2|E|$

the point r_{min}, being the closest distance from the center, is usually called the perihelion. Similarly, r_{max} is called the aphelion.

Period of the orbit

There is an easy way to compute the period of the orbit; ie, the time it takes for ϕ to change by 2π. The idea is to use Eq (23):

$$J = 2m\dot{A} = 2m\frac{dA}{dt}$$

or

$$dt = \frac{2m}{J}dA \tag{48}$$

If we integrate over an entire period, the left-hand side will be T, the period. In the right-hand side we will get A, which is simply the area of an ellipse, $A = \pi ab$. Thus

$$T = \frac{2m}{J}\pi ab \tag{49}$$

using (45) we get

$$T = \frac{2m\pi}{J}\frac{p^2}{(1-\epsilon^2)^{3/2}} = \frac{2\pi m}{J}\frac{J^4}{m^2\alpha^2}\left(\frac{1}{-\frac{2J^2E}{m\alpha^2}}\right)^{3/2}$$

The minus sign is not a problem since $E<0$. Simplifying we finally get

$$T = \frac{2\pi J^3}{m\alpha^2}\left(\frac{m^{3/2}\alpha^3}{2^{3/2}J^3|E|^{3/2}}\right)$$

$$T = \frac{\pi m^{1/2}\alpha}{2^{1/2}|E|^{3/2}}$$

or

$$\boxed{T = \pi \alpha \sqrt{\frac{m}{2|E|^3}}} \qquad (50)$$

The period is therefore seen to be a function only of the energy of the orbit. It diverges as $|E| \to 0$ since in this case the orbit becomes unbounded. It has its largest possible value when E achieves the minimum

$$E_{min} = U_{eff,\,min} = -\frac{m\alpha^2}{2J^2}$$

then

$$T_{max} = \pi \alpha \sqrt{\frac{m}{2} \frac{2^3 \quad J^6}{m^3 \alpha^6}}$$

$$T_{max} = \frac{2\pi}{m\alpha^2} J^3 \qquad (51)$$

The period is therefore maximum when the orbit is circular.

Let us also write (50) as a function of a only (the semimajor axis of the ellipse). We have, from (45)

$$|E| = \frac{\alpha}{2a}$$

then (50) becomes

$$T = \pi \alpha \sqrt{\frac{m}{2}} \left(\frac{2a}{\alpha}\right)^{3/2}$$

or

$$\boxed{T = 2\pi a^{3/2} \sqrt{\frac{m}{\alpha}}} \qquad (52)$$

This formula shows that

$$\boxed{\frac{T^2}{a^3} = \text{constant}} \tag{53}$$

which is called Kepler's third law. The constant is

$$\frac{T^2}{a^3} = (2\pi)^2 \frac{m}{\alpha} \tag{54}$$

We can also write it in terms of the actual gravitational constants:

$$U = -\frac{\alpha}{r} = -\frac{G M m_0}{r}$$

Here we need to be careful. The mass m in (54) is the mass that appears in the kinetic energy of (2). As we have discussed this is not the mass of the planet, which I am calling m_0. This m is the reduced mass

$$m = \frac{m_0 M}{m_0 + M}$$

Thus, Eq (54) becomes

$$\frac{T^2}{a^3} = 4\pi^2 \frac{m_0 M}{m_0 + M} \frac{1}{G M_0 m}$$

or

$$\boxed{\frac{T^2}{a^3} = \frac{4\pi^2}{G(m_0 + M)}} \tag{55}$$

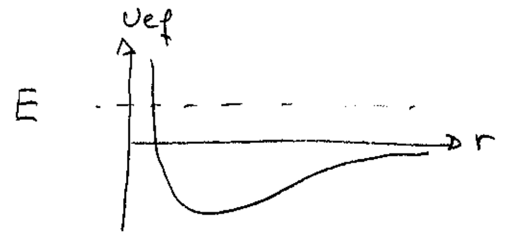
For the planets in our solar system, $m \ll M$, which leads to an approximate universal behavior

$$\boxed{\frac{T^2}{a^3} \approx \frac{4\pi^2}{GM}} \tag{56}$$

this is a quick formula to estimate the period of a planet from its distance from the sun.

Parabolic and hyperbolic motions

When $E \geq 0$ the motion becomes infinite. We can see this from a plot of the effective potential



In this case there is only one turning point. So the particle may approach the center, but it will eventually be deflected back to infinity

This is what happens to comets that visit our solar system. At infinite ($r \to \infty$) the energy is

$$E = \frac{1}{2} m \dot{r}^2 \geq 0$$

so any body comming from infinity will not be bound to our solar system. He will come, visit and then leave.

The main solution (41) remains valid in this case:

$$\frac{p}{r} = 1 + \epsilon \cos\phi \qquad (57)$$

where, recall,

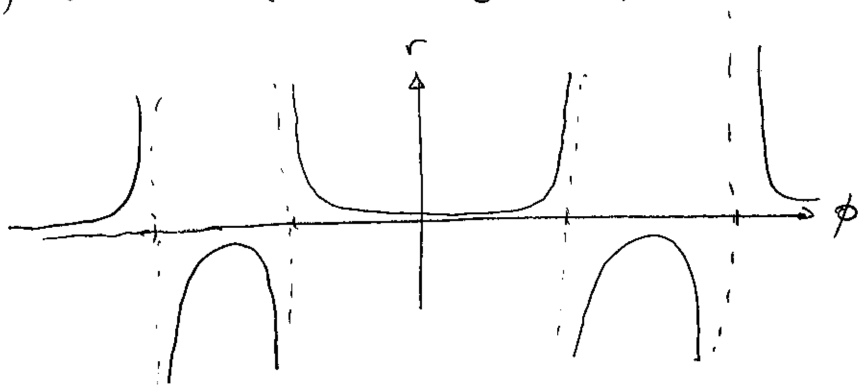
$$p = \frac{J^2}{m\alpha} \qquad \epsilon = \sqrt{1 + \frac{2J^2E}{m\alpha^2}} \qquad (58)$$

when $E \geq 0$, the eccentricity ϵ will be ≥ 1. So if we write (57) as

$$r = \frac{p}{1+\epsilon \cos\phi} \tag{59}$$

we find that there are certain values of ϕ for which $r \to \infty$. this did not happen when $\epsilon \in [0,1]$ and is another way of seeing that the motion is infinite.

Eq (59) has the following shape:



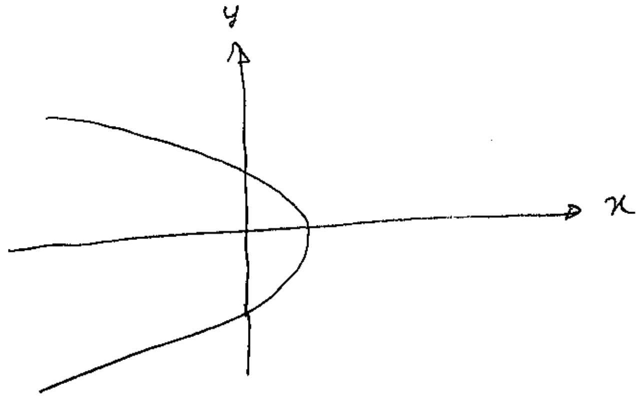
the radius r is only positive (as it must) between

$$-\phi^* \leq \phi \leq \phi^*$$

where

$$\phi^* = \arccos(-1/\epsilon) \tag{60}$$

So this is the interval where Eq (59) has physically meaningful solutions. In this interval the motion is a hyperbole.



The perihelion, which is the point where the particle is closest to the center, is obtained by minimizing (59), which corresponds to $\phi = 0$:

$$r_{min} = \frac{p}{1+\epsilon} \qquad (61)$$

It is also convenient to define, in analogy with (45),

$$a = \frac{p}{\epsilon^2 - 1} = \frac{\alpha}{2E} \qquad (62)$$

as the semi-axis of the hyperbola. Then we get

$$r_{min} = a(\epsilon - 1) \qquad (63)$$

It is also interesting to find the path $(x(t), y(t))$ as a function of time. This can be done using (32):

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left[E - \frac{p\alpha}{2r^2} + \frac{\alpha}{r}\right]}}$$

$$= \sqrt{\frac{m}{2E}} \int \frac{r\,dr}{\sqrt{r^2 - \frac{p\alpha}{2E} + \frac{\alpha r}{E}}}$$

(I'm looking for $E>0$ so I don't have to worry about $|E|$).
We now write $\alpha/E = 2a$ and complete squares

$$r^2 + \frac{\alpha}{E}r = r^2 + 2ar = (r+a)^2 - a^2$$

then we get

$$t = \sqrt{\frac{ma}{\alpha}} \int \frac{r\,dr}{\sqrt{(r+a)^2 - a^2 - \frac{p\alpha}{2E}}}$$

the constant which remains is

$$a^2 + \frac{p\alpha}{2E} = a^2 + pa = a^2 + a^2(\epsilon^2 - 1)$$

$$= a^2\epsilon^2$$

Hence

$$t = \sqrt{\frac{ma}{\alpha}} \int \frac{r\,dr}{\sqrt{(r+a)^2 - a^2\epsilon^2}} \tag{64}$$

we now substitute

$$r + a = a\epsilon \cosh\xi \tag{65}$$

$$dr = a\epsilon \sinh\xi \, d\xi$$

$$t = a\epsilon\sqrt{\frac{ma}{\alpha}} \int \frac{(a\epsilon\cosh\xi - a)\sinh\xi \, d\xi}{\sqrt{(a\epsilon)^2(\cosh^2\xi - 1)}}$$

$$= \sqrt{\frac{ma^3}{\alpha}} \int (\epsilon\cosh\xi - 1)\,d\xi$$

$$= \sqrt{\frac{ma^3}{\alpha}} \left[\epsilon\sinh\xi - \xi\right] \tag{66}$$

If we look at (65) and (66), we see a system of parametri equations for r and t, with ξ as a parameter:

$$\boxed{\begin{aligned} t &= \tau_0 \, (\epsilon \sinh \xi - \xi) \\ r &= a \, (\epsilon \cosh \xi - 1) \\ \tau_0 &= \sqrt{\frac{m a^3}{\alpha}} \end{aligned}} \qquad (67)$$

the parameter ξ may range from $-\infty$ to ∞.

We now have $r(\xi)$ and we know $\phi(r)$ from (57), so we have all the ingredients to compute $x(\xi)$ and $y(\xi)$. From (57),

$$\frac{p}{r} = 1 + \frac{\epsilon x}{r}$$

so

$$\begin{aligned} \epsilon x &= p - r \\ &= a(\epsilon^2 - 1) - a(\epsilon \cosh \xi - 1) \end{aligned}$$

$$\therefore \quad x = a(\epsilon - \cosh \xi)$$

Since $x^2 + y^2 = r^2$, we then get

$$\begin{aligned} y^2 &= a^2 \left[(\epsilon \cosh \xi - 1)^2 - (\epsilon - \cosh \xi)^2 \right] \\ &= a^2 \Big[\cosh^2 \xi \, (\epsilon^2 - 1) + \cosh \xi \, (-2\epsilon + 2\epsilon) \\ &\qquad + 1 - \epsilon^2 \Big] \\ &= a^2 (\epsilon^2 - 1)(\cosh^2 \xi - 1) \end{aligned}$$

Hence

$$y = a\sqrt{\epsilon^2-1}\,\sinh\xi$$

where the correct sign of y is already adjusted from the sign of $\sinh\xi$.

In summary

$$\boxed{\begin{aligned} x &= a(\epsilon - \cosh\xi) \\ y &= a\sqrt{\epsilon^2-1}\,\sinh\xi \\ t &= b_0(\epsilon\sinh\xi - \xi) \end{aligned}} \qquad \xi \in [-\infty, \infty] \qquad (68)$$

this gives the path as a function of time. the constants are such that $t=0$ when $\xi = 0$. this is the perihelion, where

$$x = a(\epsilon - 1) = r_{min}$$

(see (61)).

The Wilson - Sommerfeld quantization rules

In 1915, 11 years before Schrödinger published his famous equation, W. Wilson and A. Sommerfeld independently discovered a quantization method applicable to closed orbits.

According to their rule, given a set of generalized coordinates $q_1, q_2, \ldots$, and the corresponding momenta $p_1, p_2, \ldots$, in quantum mechanics the only allowed orbits are those for which

$$\oint p_i \, dq_i = 2\pi n_i \hbar \tag{69}$$

where n_i is an integer and the integral is over one period of the orbit. This type of integral is called an action integral.

Let us see how this works. Consider first the harmonic oscillator

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x}$$

$$x(t) = A \cos(\omega t - \phi)$$

$$p(t) = -m\omega A \sin(\omega t - \phi)$$

Eq (69) then gives

$$\int_0^{2\pi/\omega} p \frac{dx}{dt} dt = \int_0^{2\pi/\omega} m[\omega^2 A^2 \cos^2(\omega t - \phi)] dt$$

$$= m\omega^2 A^2 \left(\frac{\pi}{\omega}\right) = 2\pi n \hbar$$

Thus

$$m\omega^2 A^2 = 2n\hbar\omega$$

The energy of the harmonic oscillator is

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m\omega^2 A^2$$

Hence, we find that

$$\boxed{E = n\hbar\omega}$$

which is the energy of the quantum harmonic oscillator, as you may recall. Since $m\omega^2 A^2 \geq 0$, we must have

$$n = 0, 1, 2, \ldots$$

Now let us apply it to the hydrogen atom. Since the electrostatic force has the same structure as the gravitational force, we may use everything we have done so far, using

$$U(r) = -\frac{e^2}{4\pi \epsilon_0 r^2}$$

ie, using

$$\alpha = \frac{e^2}{4\pi \epsilon_0} \tag{70}$$

The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

so

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

$$p_\phi = m r^2 \dot{\phi} = J$$

Eq (69) then gives

$$\oint p_r \, dr = 2\pi M_r \hbar \tag{71a}$$

$$\oint p_\phi \, d\phi = 2\pi M_\phi \hbar \tag{71b}$$

the second equation is easy to compute because $p_\phi = J$ is a constant of the motion, so

$$\oint p_\phi \, d\phi = J \int_0^{2\pi} d\phi = 2\pi J = 2\pi M_\phi \hbar$$

Historically, we call n_ϕ by the letter l. We then get

$$\boxed{J = l\hbar} \qquad l = 0, 1, 2, \ldots \tag{72}$$

The values of l cannot be negative because we have chosen our reference frame in such a way that $J > 0$.

To evaluate (71a) we write

$$\frac{1}{2}m\dot{r}^2 = E - U_{eff}$$

or

$$\dot{r}^2 = \frac{2}{m}\left[E - \frac{J^2}{2mr^2} + \frac{\alpha}{r}\right]$$

We are interested in periodic orbits so $E < 0$, so we write this equation in terms of

$$a = \frac{\alpha}{2|E|}, \qquad p = \frac{J^2}{m\alpha^2} = a(1-\epsilon^2)$$

we get

$$\dot{r}^2 = \frac{2|E|}{m}\left\{-1 - \frac{p\alpha}{2|E|r^2} + \frac{\alpha}{|E|r}\right\}$$

$$= \frac{2|E|}{m}\frac{1}{r^2}\left\{-r^2 - pa + 2ar\right\}$$

$$= \frac{2|E|}{m}\frac{1}{r^2}\left\{-pa + a^2 - (r-a)^2\right\}$$

$$= \frac{2|E|}{m}\left(\frac{a^2\epsilon^2 - (r-a)^2}{r^2}\right)$$

Thus

$$p_r = \sqrt{2m|E|}\,\frac{\sqrt{a^2\epsilon^2 - (r-a)^2}}{r} \qquad (73)$$

we now evaluate (71b). A period of motion means going from r_{min} to r_{max} and back. So we just go once and write

$$\oint p_r\,dr = 2\int_{r_{min}}^{r_{max}} \sqrt{2m|E|}\,\frac{\sqrt{a^2\epsilon^2-(r-a)^2}}{r}\,dr$$

where, from (47)

$$r_{min} = a(1-\epsilon)$$

$$r_{max} = a(1+\epsilon)$$

this integral was first evaluated by Sommerfeld. It has the value

$$\oint p_r dr = 2\sqrt{2m|E|}\;a\pi\left(1-\sqrt{1-\epsilon^2}\right) \qquad (74)$$

Since $a = \frac{\alpha}{2|E|}$ and $\epsilon = \sqrt{1+\frac{2J^2E}{m\alpha^2}}$ we may also write

$$\oint p_r dr = 2\pi\sqrt{2m|E|}\;\frac{\alpha}{2|E|}\left[1-\sqrt{\frac{2J^2|E|}{m\alpha^2}}\right]$$

$$= 2\pi\sqrt{\frac{m\alpha^2}{2|E|}}\left[1-\sqrt{\frac{2J^2|E|}{m\alpha^2}}\right]$$

$$= 2\pi\, m_r \hbar$$

Thus

$$1 - \sqrt{\frac{2J^2|E|}{m\alpha^2}} = n_r \hbar \sqrt{\frac{2|E|}{m\alpha^2}}$$

For clarity let $x = \sqrt{2|E|/m\alpha^2}$ then

$$1 + Jx = n_r \hbar \, x$$

or

$$x = \frac{1}{n_r \hbar + J}$$

But $J = \hbar l$ so

$$x = \sqrt{\frac{2|E|}{m\alpha^2}} = \frac{1}{\hbar (n_r + l)}$$

Let us define a new quantum number

$$n = n_r + l \tag{75}$$

then we get

$$|E| = \frac{m\alpha^2}{2\hbar^2} \frac{1}{n^2}$$

substituting $\alpha = \frac{e^2}{4\pi\epsilon_0}$ we finally arrive at

$$|E| = \frac{m e^4}{2(4\pi\epsilon_0)^2 \hbar^2} \frac{1}{n^2}$$

which are the famous Bohr levels of the hydrogen atom.

Euler's theorem on homogeneous functions

Before we continue, I want to show you a useful theorem due to Euler. Consider a function of, say, two variables, $f(x,y)$. We say f is a homogeneous function of degree m if

$$f(\alpha x, \alpha y) = \alpha^m f(x,y) \tag{76}$$

For instance

$$f(x,y) = x^2 + y^2 + 7xy$$

is a homogeneous function of degree 2. But

$$f(x,y) = x^2 + 2y$$

is not homogeneous. Similarly,

$$f(x,y) = \frac{1}{x} + \frac{2}{y}$$

is homogeneous of degree -1. Homogeneous functions appear very oftenly, particularly in mechanics and thermodynamics.

Going back to (76) and differentiating both sides with respect to α, we get

$$\frac{\partial f(\alpha x, \alpha y)}{\partial \alpha} = m\alpha^{m-1} f(x,y)$$

In the left side we write

$$\frac{\partial f(\alpha x, \alpha y)}{\partial \alpha} = \frac{\partial f(\alpha x, \alpha y)}{\partial(\alpha x)}\frac{\partial(\alpha x)}{\partial \alpha} + \frac{\partial f(\alpha x, \alpha y)}{\partial(\alpha y)}\frac{\partial(\alpha y)}{\partial \alpha}$$

$$= x\frac{\partial f(\alpha x, \alpha y)}{\partial(\alpha x)} + y\frac{\partial f(\alpha x, \alpha y)}{\partial(\alpha y)}$$

If we now take the limit $\alpha \to 1$ we obtain

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = m f \qquad (77)$$

This is Euler's theorem. It is readily generalized to a function of several variables:

$$\boxed{\begin{array}{lll} \text{Homogeneous of degree } m & : & f(\alpha x_1, \alpha x_2, \dots) = \alpha f(x_1, x_2, \dots) \\ \text{Euler's theorem} & : & \sum_i x_i \frac{\partial f}{\partial x_i} = m f \end{array}} \qquad (78)$$

<u>Example</u>: $T = \frac{1}{2}(a_1 \dot{q}_1^2 + a_2 \dot{q}_2^2 + a_3 \dot{q}_3^2)$

is a homogeneous function of degree 2 in the $\dot{q}_i$

$$\sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = \sum_i \dot{q}_i p_i = 2T$$

This explains why it is so common to have

$$\mathcal{L} = T - U$$

and

$$E = \sum_i \dot{q}_i p_i - \mathcal{L} = 2T - (T - U) = T + U$$

So whenever $\mathcal{L}$ is a homogeneous function of degree 2 in the $\dot{q}_i$, the energy will be given by the familiar formula $T + U$.

Mechanical Similarity

We may use Euler's theorem to obtain useful information about certain systems. Recall that $\mathcal{L}$ and $\alpha\mathcal{L}$ are physically equivalent Lagrangians, in the sense that both give the same equations of motion.

Now suppose the potential energy is a homogeneous function of degree m in the positions $\mathbf{r}_i$

$$U(\alpha\mathbf{r}_1, \alpha\mathbf{r}_2, \dots) = \alpha^m U(\mathbf{r}_1, \mathbf{r}_2, \dots) \qquad (79)$$

Now consider a transformation where

$$\begin{aligned} \mathbf{r}_i &\to \alpha\,\mathbf{r}_i \\ t &\to \beta t \end{aligned} \qquad (80)$$

then

$$v_i = \frac{d\mathbf{r}_i}{dt} \to \frac{\alpha}{\beta} v_i$$

and consequently the kinetic energy changes by

$$T \to \frac{\alpha^2}{\beta^2} T$$

On the other hand, U is multiplied by α^m by the assumption (79):

$$U \to \alpha^m U$$

Now suppose we choose β such that

$$\frac{\alpha^2}{\beta^2} = \alpha^M$$

or

$$\beta = \alpha^{1-M/2} \tag{81}$$

then the final result will be that L will be multiplied by a constant α^M, so the Eqs of motion remain unchanged

A transformation like $r_i \to \alpha r_i$ means that we multiply all paths by the same value α. The result is a geometrically similar path, but with a different size.

Eq (81) then shows that for homogeneous potentials there exist several geometrically similar paths, with times t and scales l related by

$$\boxed{(t/t') = (l/l')^{1-M/2}} \tag{82}$$

what this formula mean is that if l and l' are two lengths and t and t' are the corresponding times, then the ratio t/t' will be related to l/l' by (82).

This is also extensible to other quantities. For instance

$$v/v' = \frac{(l/l')}{(t/t')} = \frac{(l/l')}{(l/l')^{1-M/2}}$$

Thus

$$\boxed{(v/v') = (l/l')^{M/2}} \qquad (83)$$

From this it follows that

$$(T/T') = (l/l')^{M}$$

Since this is also true of U, from (79), we get that

$$\boxed{(E/E') = (l/l')^{M}} \qquad (84)$$

We may also relate the time of the paths with the energy,

$$(t/t') = (l/l')^{1-M/2} = \left[(E/E')^{1/M}\right]^{1-M/2}$$

so

$$\boxed{(t/t') = (E/E')^{\frac{1}{M}-\frac{1}{2}}} \qquad (85)$$

Finally, since $\mathbb{J} = m\, \mathbb{r} \times v$ we get

$$(J/J') = (l/l')(v/v')$$

or

$$(J/J') = (l/l')^{1+M/2} \qquad (86)$$

As an example, consider the harmonic oscillator:

$$U = \frac{1}{2} k \cdot x^2$$

is a homogeneous function of degree $n=2$. Eq (85) then gives

$$(t/t') = 1$$

which means that the period is independent of the amplitude of the oscillation.

This is also true for any quadratic potential:

$$U = \frac{1}{2} \sum_{ij} A_{ij} x_i x_j$$

because this is also of degree 2.

When we study 1D systems we considered a potential of the form

$$U = \frac{1}{m!} k |x|^m$$

which is a homogeneous function of degree m. In that occasion we computed the period of oscillation T (not the kinetic energy!) and found that

$$T \propto E^{\frac{1}{m} - \frac{1}{2}}$$

which is exactly (85).

As another example, consider the fall under gravity:

$$U = mg\,x$$

This is a homogeneous function of degree 1. Eq (82) then gives

$$(t/t') = (\ell/\ell')^{1/2}$$

or

$$(\ell/\ell') = (t/t')^2 \qquad (87)$$

This is a well-known law: remember free fall

$$x = x_0 + v_0 t + \frac{1}{2} g t^2$$

Eq (87) then says that the length of a path scales proportionally to t^2.

Another important example is the gravitational (or coulombic) potential

$$U = -\frac{\alpha}{r}$$

which is homogeneous of degree -1. Eq (82) then gives

$$(t/t') = (\ell/\ell')^{3/2} \qquad (88)$$

This is Kepler's third law, Eq (53). Eq (86) also gives

$$(\ell/\ell') = (J/J')^2$$

which is essentially Eq (58).

The Virial theorem

Now suppose that the potential is a homogeneous function and also that the motion occurs in a finite region of space. In this case we may obtain a theorem concerning the time averages of certain mechanical quantities.

Since the kinetic energy T is a homogeneous function of the $\mathbf{v}_i$ with degree 2, we have

$$\sum_i \mathbf{v}_i \cdot \frac{\partial T}{\partial \mathbf{v}_i} = \sum_i \mathbf{v}_i \cdot \mathbf{p}_i = 2T$$

Let us write then

$$\mathbf{v}_i \cdot \mathbf{p}_i = \frac{d}{dt}(\mathbf{r}_i \cdot \mathbf{p}_i) - \mathbf{r}_i \cdot \dot{\mathbf{p}}_i$$

so

$$2T = \frac{d}{dt}\sum_i \mathbf{r}_i \cdot \mathbf{p}_i - \sum_i \mathbf{r}_i \cdot \dot{\mathbf{p}}_i \tag{87}$$

We now define the time average of a function $f(t)$ as

$$\bar{f} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} f(t)\,dt \tag{88}$$

If we assume that the motion is finite, $\mathbf{r}_i \cdot \mathbf{p}_i$ will be bounded so the average of the first term in (87) will be zero:

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \frac{d}{dt} (\mathbf{r}_i \cdot \mathbf{p}_i) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \mathbf{r}_i \cdot \mathbf{p}_i \Big|_0^{\tau} = 0$$

since $\tau \to \infty$.

In the second term of (87) we replace $\dot{\mathbf{p}}_i$ with the force $-\partial U / \partial \mathbf{r}_i$. We then have

$$\boxed{2\overline{T} = \overline{\sum_i \mathbf{r}_i \cdot \frac{\partial U}{\partial \mathbf{r}_i}}} \qquad (89)$$

this is a general theorem, relating the average kinetic energy with the average of $\mathbf{r}_i \cdot \partial U / \partial \mathbf{r}_i$.

If U is a homogeneous function of degree n, then by Euler's theorem

$$\sum_i \mathbf{r}_i \cdot \frac{\partial U}{\partial \mathbf{r}_i} = nU$$

and we get

$$\boxed{2\overline{T} = n\overline{U}} \qquad (90)$$

which is called the Virial theorem (the word virial comes from the latin "vis", which means "force" or "energy". It was first proven by Clausius in 1870.

Since $\bar{T} + \bar{U} = \bar{E} = E$ (because it is a constant) we may also write

$$2(E - \bar{U}) = m\bar{U}$$

$$\Rightarrow \boxed{\begin{aligned} \bar{U} &= \frac{2E}{m+2} \\ \bar{T} &= \frac{mE}{m+2} \end{aligned}} \qquad (91)$$

and then

with these formulas we may find the average kinetic and potential energies simply by knowing the energy.

- $m = 2$ (harmonic oscillator)

$$\bar{T} = \bar{U}$$

- $m = -1$ (gravitational)

$$2\bar{T} = -\bar{U}$$

or $$\bar{U} = 2E \quad \text{and} \quad \bar{T} = -E$$

[Remember that this theorem only holds for finite motion. For the gravitational potential the motion is only finite if $E < 0$, which agrees with $\bar{T} = -E$].