07/03/15

Eigenvalues and eigenvectors of angular momentum

In these notes we will do something quite amazing: we will completely diagonalize the angular momentum operators using only the commutation relations

$$[\hat{J}_x, \hat{J}_y] = i J_z \qquad [\hat{J}_y, \hat{J}_z] = i J_x \qquad [\hat{J}_z, \hat{J}_x] = i J_y \quad ($$

Of course, since the $\hat{J}_i$ do not commute, we cannot diagonalize them simultaneously. thus we need to choose one to diagonalize, It is a tradition to choose $\hat{J}_z$. We will then write $\hat{J}_x$ and $\hat{J}_y$ in the eigenbasis of $\hat{J}_z$. You've already seen this with the Pauli matrices.

We will also consider the following operator

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \tag{2}$$

Physically, this is like the magnitude of the angular momentum squared. It is mathematically awkward to work with the $\sqrt{\ }$ of operators, so we rather work with J^2.

I will leave for you as an exercise to show that J^2 commutes with everyone!

$$[\hat{J}^2, \hat{J}_x] = [\hat{J}^2, \hat{J}_y] = [\hat{J}^2, \hat{J}_z] = 0 \tag{3}$$

Tip: if you show that $[\hat{J}^2, \hat{J}_z] = 0$, then the other two follow immediately since $\hat{J}^2$ is <u>isotropic</u>

What Eq (3) says is that we may simultaneously diagonalize $\hat{J}^2$ and $\hat{J}_z$. Thus we are looking for

$$\hat{J}^2 |a,b\rangle = a |a,b\rangle \tag{4a}$$

$$\hat{J}_z |a,b\rangle = b |a,b\rangle \tag{4b}$$

The only things we know so far is that b is real because J_z is Hermitian and a is real and non-negative because J^2 is a positive operator.

Now comes the big trick. Define

$$\hat{J}_+ = \hat{J}_x + i J_y \tag{5a}$$

$$\hat{J}_- = \hat{J}_x - i J_y \tag{5b}$$

Note that these operators are not Hermitian. In fact

$$J_+^\dagger = J_- \tag{6}$$

The inverse relations are

$$\hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \tag{7a}$$

$$\hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2} \tag{7b}$$

whenever we define new operators, it is important that we understand the algebraic structure behind them.

Let us also look at some commutation relations:

$$[\hat{J}_+, \hat{J}_-] = [\hat{J}_x + i J_y,\ \hat{J}_x - i J_y]$$

$$= -i[\hat{J}_x, \hat{J}_y] + i[\hat{J}_y, \hat{J}_x]$$

$$= (-i)\, i J_z + i(-i)\hat{J}_z$$

$$\therefore \quad \boxed{[\hat{J}_+, \hat{J}_-] = 2\hat{J}_z} \qquad (8a)$$

Next:

$$[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z,\ \hat{J}_x \pm i\hat{J}_y] = [\hat{J}_z, \hat{J}_x] \pm i[\hat{J}_z, \hat{J}_y]$$

$$= i\hat{J}_y \pm i(-i)\hat{J}_x$$

$$= i\hat{J}_y \pm \hat{J}_x$$

$$= \pm(\hat{J}_x \pm i\hat{J}_y)$$

$$\therefore \quad \boxed{[\hat{J}_z, \hat{J}_\pm] = \pm\hat{J}_\pm} \qquad (8b)$$

Finally, since $\hat{J}^2$ commutes with $\hat{J}_x$ and $\hat{J}_y$ we must have

$$[\hat{J}^2, \hat{J}_\pm] = 0 \qquad (8c)$$

Let us also note the following useful relations

$$\hat{J}_+ \hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 - i(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x)$$

$$= \hat{J}_x^2 + \hat{J}_y^2 - i(i)\hat{J}_z$$

$$= (\hat{J}^2 - \hat{J}_z^2) + \hat{J}_z$$

$$\therefore \quad \boxed{\hat{J}_+ \hat{J}_- = \hat{J}^2 - \hat{J}_z(\hat{J}_z - 1)} \qquad (8d)$$

Since $\hat{J}_- \hat{J}_+ = \hat{J}_+ J_- - 2\hat{J}_z$

$$\boxed{\hat{J}_- \hat{J}_+ = \hat{J}^2 - J_z(J_z + 1)} \qquad (8e)$$

These relations are all we will need to diagonalize $\hat{J}_z$ and $\hat{J}^2$.

First, since $\hat{J}^2$ and $\hat{J}_\pm$ commute then

$$J^2 \left(\hat{J}_\pm |a,b\rangle \right) = \hat{J}_\pm \hat{J}^2 |a,b\rangle$$
$$= a \left(\hat{J}_\pm |a,b\rangle \right)$$

Thus, $\hat{J}_\pm |a,b\rangle$ is also an eigenvector of $\hat{J}^2$ with the same eigenvalue.

For $\hat{J}_z$ things get more interesting. From (8b)

$$\hat{J}_z \hat{J}_\pm = \hat{J}_\pm \hat{J}_z \pm \hat{J}_\pm$$
$$= \hat{J}_\pm (\hat{J}_z \pm 1)$$

Thus

$$\hat{J}_z \left[\hat{J}_\pm |a,b\rangle \right] = \hat{J}_\pm (\hat{J}_z \pm 1) |a,b\rangle$$
$$= \hat{J}_\pm (b \pm 1) |a,b\rangle$$
$$= (b \pm 1) \left[\hat{J}_\pm |a,b\rangle \right]$$

This is just like in the Harmonic oscillator! We find that $\hat{J}_\pm |a,b\rangle$ is also an eigenvector of $\hat{J}_z$ but with eigenvalue $b \pm 1$. Thus we must have

$$J_\pm |a,b\rangle = c_\pm |a, b \pm 1\rangle \qquad (9)$$

where $c_\pm$ is just some constant to ensure that the eigenvectors are properly normalized.

The next thing we do is relate a and b. To do this first note that

$$\langle a,b|(\hat{J}^2 - \hat{J}_z^2)|a,b\rangle = a - b^2$$

On the other hand,

$$\hat{J}^2 - \hat{J}_z = \hat{J}_x^2 + \hat{J}_y^2$$

so that this must be a positive operator with positive eigenvalues. Whence

$$\boxed{a \geqslant b^2} \tag{10}$$

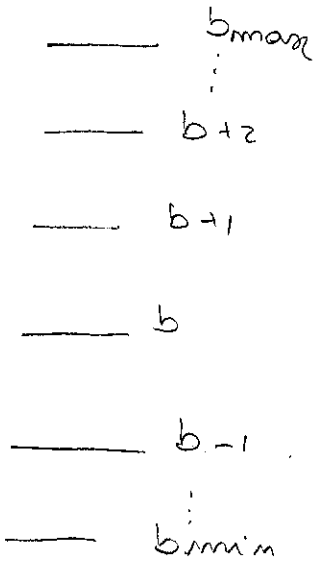
Well, this is weird. From (9) we know that if we apply J_+ k times we should obtain a state with eigenvalue $b+k$; ie,

$$J_+^k |a,b\rangle \propto |a, b+k\rangle$$

But this can't go on for any k since we will eventually violate Eq. (10). The only possibility is that there must be some value of b_{max} such that

$$J_+ |a, b_{max}\rangle = 0 \tag{11}$$

In other words, the spectrum of $\hat{J}_z$ must be a ladder, but this ladder must terminate eventually



We may find b_{max} using Eq (8e). According to (11) we must have

$$\hat{J}_- \hat{J}_+ |a, b_{max}\rangle = 0$$

But using (8e) this becomes

$$\left[\hat{J}^2 - \hat{J}_z(J_z + 1)\right]|a, b_{max}\rangle = \left[a - b_{max}(b_{max}+1)\right]|a, b_{max}\rangle$$

Thus we conclude that the maximum value of b is given by

$$a = b_{max}(b_{max}+1) \qquad (12)$$

We now repeat the same analysis to the bottom of the ladder. In order not to violate Eq. (10) we must have

$$\hat{J}_- |a, b_{min}\rangle = 0$$

Thus, using (8a) we get

$$a = b_{min}(b_{min} - 1) \qquad (13)$$

Comparing (12) and (13) we conclude that

$$b_{max}(b_{max} + 1) = b_{min}(b_{min} - 1) \qquad (14)$$

This Eq has two solutions, but only one where $b_{max} > b_{min}$, which must be true by definition. This is

$$b_{max} = -b_{min} \qquad (15)$$

Thus, b will vary from $-b_{max}$ to $+b_{max}$ in steps of 1.

It is customary to define

$$j = b_{max} \qquad (16)$$

so that the eigenvalues of J^2 are then

$$a = j(j+1) \qquad (17)$$

We are almost there. To finish we note that if we are in $|a, b_{min}\rangle$ then we must be able to get to $|a, b_{max}\rangle$ by applying $\hat{J}_+$ an integer number of times. Call this integer n. then

$$b_{max} = -b_{max} + n$$

or

$$\boxed{j = b_{max} = \frac{n}{2}} \qquad (18)$$

This means that the values of j may either be integers or half-integers

$$\boxed{j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots} \qquad (19)$$

It is also customary to call the eigenvalues of $\hat{J}_z$ as m instead of b. We then have

$$\boxed{-j \leq m \leq j \quad \text{in integer steps}} \qquad (20)$$

In this new notation we finally have

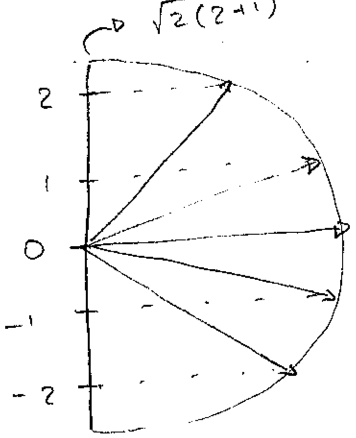
$$\boxed{\begin{aligned} \hat{J}^2 |j,m\rangle &= j(j+1)\,|j,m\rangle \\ \hat{J}_z |j,m\rangle &= m\,|j,m\rangle \end{aligned}}$$

(21)

(22)

We call j the angular momentum and m as the projection onto the z-axis. But note that the eigenvalues of J^2 are not j^2, but $j(j+1)$. When j is large, like in classical system, this difference is immaterial. But when j is small, this is important. But when j is small, to visualize this is, for instance the usual picture drawn

of $j=2$



Note also that the z direction is arbitrary. Thus the eigenvalues of $\hat{J}_x$ and $\hat{J}_y$ must be of the same form as Eq (20). The eigenvectors, of course, will be different.

Two important properties about angular momentum

The eigenvalues of $\hat{J}_i$ are symmetric with respect to zero (Eq (20)). Thus

$$\boxed{\operatorname{tr} \hat{J}_i = 0} \qquad i = x, y, z. \tag{23}$$

The second property is a manifestation of the Cayley-Hamilton theorem. It says that

$$\boxed{\prod_{m=-j}^{j} (\hat{J}_z - m) = 0} \tag{24}$$

To prove this write an arbitrary vector $|v\rangle$ in the basis $|j, m\rangle$:

$$|v\rangle = \sum_{j,m} v_{jm} |j, m\rangle \tag{25}$$

when we apply the product in (24) there will always be a term which gives zero. Thus, for any $|v\rangle$

$$\left[\prod_{m=-j}^{j} (\hat{J}_z - m) \right] |v\rangle = 0$$

Since this is true for any vector, it must be true for the operator itself.

Matrix elements of $\hat{J}_x$ and $\hat{J}_y$

We have previously derived [Eq. (9)] that

$$\hat{J}_+ |j, m\rangle = c_+ |j, m \pm 1\rangle \tag{26}$$

To determine the c_+ we take the absolute value of this Eq.

$$\langle j, m| \hat{J}_- \hat{J}_+ |j, m\rangle = |c_+|^2 \tag{27}$$

Using (8e) we find

$$|c_+|^2 = \langle j, m| [\hat{J}^2 - \hat{J}_z (\hat{J}_z + 1)] |j, m\rangle$$

$$= j(j+1) - m(m+1)$$

Hence, choosing c_+ real and positive,

$$c_+ = \sqrt{j(j+1) - m(m+1)} \tag{28}$$

Another equivalent way is

$$c_+ = \sqrt{(j-m)(j+m+1)} \tag{29}$$

You can check that they are the same thing

Similarly

$$c_- = \sqrt{j(j+1) - m(m-1)} \qquad (30)$$
$$= \sqrt{(j+m)(j-m+1)}$$

Hence

$$\boxed{\hat{J}_\pm |j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\ |j, m\pm 1\rangle} \qquad (31)$$

with this and Eq (7) we may build the matrix entries of $\hat{J}_x$ and $\hat{J}_y$.

If you want a more compact notation, write

$$\boxed{\begin{aligned} \hat{J}_\pm |j,m\rangle &= c^{\pm}_{jm} |j, m\pm 1\rangle \\ c^{\pm}_{jm} &= \sqrt{j(j+1) - m(m\pm 1)} \\ &= \sqrt{(j\mp m)(j\pm m+1)} \end{aligned}} \qquad (32)$$

Note also that J_x and J_y commute with J^2. For this reason, if we write J_x and J_y in the basis $|j,m\rangle$, the resulting matrix will factor into blocks

$$\begin{array}{cccc} & |0,m\rangle & |\tfrac{1}{2},m\rangle & |1,m\rangle \quad \cdots \\ J_x = & \multicolumn{3}{c}{\left[\begin{array}{cccccc} 0 & & & & & \\ & 0 & 1/2 & & & \\ & 1/2 & 0 & & & \\ & & & 0 & 1/\sqrt{2} & 0 \\ & & & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ & & & 0 & 1/\sqrt{2} & 0 \\ & & & & & & \ddots \end{array}\right]} \end{array}$$

How to build the eigenvectors

It is important to know how to relate the eigenvectors among themselves. In particular, we wish to be able to write the $|j,m\rangle$ given only one eigenvector, say $|j,j\rangle$ or $|j,-j\rangle$

For this we use (32)

$$\hat{J}_- |j,j\rangle = c^-_{j,j} \; |j,j-1\rangle$$

$$(\hat{J}_-)^2 |j,j\rangle = c^-_{jj} \; c^-_{jj-1} \; |j,j-2\rangle$$

$$\vdots$$

$$(\hat{J}_-)^k |j,j\rangle = c^-_{jj} \; c^-_{jj-1} \cdots \; c^-_{jj-k+1} \; |j,j-k\rangle$$

Now let us see what this constant is: since

$$c^-_{jk} = \sqrt{(j+k)(j-k+1)}$$

we have (forgetting about the $\sqrt{\ }$ for now)

$$(j+j)(j-j+1)\,(j+j-1)\,(j-j+1+1)\cdots(j+j-k+1)(j-j+k-1+1)$$

$$= \left[(2j)(2j-1)\cdots(2j-k+1)\right]\left[(1)(2)\cdots(k)\right]$$

$$= \frac{(2j)!}{(2j-k)!}\; k!$$

In our case we want to go up to $j - k = m$ so that $k = j - m$. Hence

$$\boxed{|j, m\rangle = \sqrt{\frac{(j+m)!}{(2j)!\,(j-m)!}}\;(J_-)^{j-m}\,|j, j\rangle} \qquad (33)$$