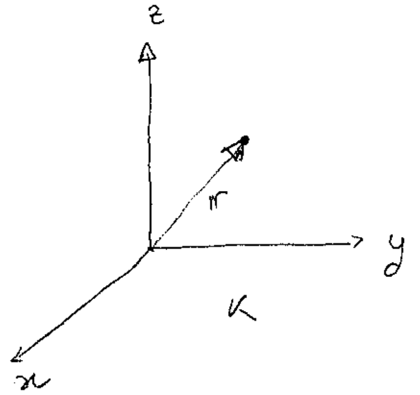
Non-inertial reference frames

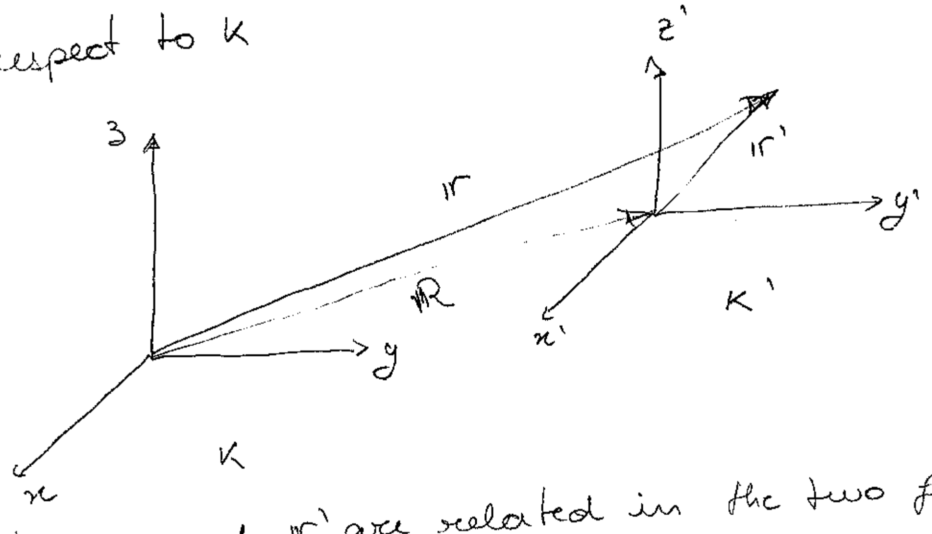
We will now consider how to describe the motion of system using an accelerating reference frame. Let us start simple: assume you have a system with Lagrangian

$$\mathcal{L} = \frac{1}{2} m v^2 - U(\mathbf{r}) \tag{1}$$

where $\mathbf{r}$ and $v = \dot{\mathbf{r}}$ are both measured from an inertial frame K.



We now consider another reference frame K' which is moving with respect to K in some arbitrary way. Let $\mathbb{R}(t)$ be the position of K' with respect to K



The coordinates $\mathbf{r}$ and $\mathbf{r}'$ are related in the two frames as

$$\mathbf{r} = \mathbf{r}' + \mathbb{R}(t) \tag{2}$$

as can be seen geometrically.

Here $\mathbf{R}(t)$ is some pre-specified function of time. In an inertial reference frame we have

$$\mathbf{V} = \dot{\mathbf{R}} = \text{constant} \qquad (\text{inertial frame}) \tag{3}$$

But what we want is precisely to consider more general situation where K' is accelerating.

In terms of $\mathbf{r}'$ the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} m (\mathbf{v}' + \mathbf{V})^2 - U(\mathbf{r}' + \mathbf{R})$$

$$= \frac{1}{2} m (\mathbf{v}'^2 + 2\mathbf{v}' \cdot \mathbf{V} + \mathbf{V}^2) - U(\mathbf{r}' + \mathbf{R})$$

The term $\frac{1}{2} m \mathbf{V}^2$ is just a function of time so it does not affect the equations of motion. Neglecting it we then find

$$\mathcal{L} = \frac{1}{2} m \mathbf{v}'^2 + m \mathbf{v}' \cdot \mathbf{V} - U(\mathbf{r}' + \mathbf{R}) \tag{4}$$

We now compute the Euler - Lagrange equations in terms of the generalized coordinates $\mathbf{r}'$. We have

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}'} = m\mathbf{v}' + m\mathbf{V}$$

So

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{v}'}\right) = m\dot{\mathbf{v}}' + m\dot{\mathbf{V}}$$

If we let

$$\mathbf{F}' = -\frac{\partial U}{\partial \mathbf{r}'} \tag{5}$$

be the force in terms of $\mathbf{r}'$ then we see that

$$\boxed{m\frac{d\mathbf{v}'}{dt} = -m\mathbf{A} + \mathbf{F}'} \qquad (6)$$

where $\mathbf{A} = \dot{\mathbf{V}} = \ddot{\mathbf{R}}$ is the acceleration of the reference frame.

We therefore see that the effect of using an accelerating reference frame is the appearance of a "fecticious" force $-m\mathbf{A}$. We say it is fecticious because it is not an actual force, but simply a consequence of the fact that we are moving on a non-inertial frame.

The canonical example is the fall under gravity.

$$m\frac{d\mathbf{v}'}{dt} = -m\mathbf{A} + m\mathbf{g}$$

So if the motion is described through a non-inertial frame, it is as if the particle experienced an effective acceleration $\mathbf{g} - \mathbf{A}$.

Remember that we may choose $\mathbf{A}$ to be whatever you want. In particular, if we set $\mathbf{A} = \mathbf{g}$ we get

$$m\frac{d\mathbf{v}'}{dt} = 0$$

So in a reference frame accelerating with acceleration $\mathbf{g}$, it is as if the particle experienced no force at all.

In order to align these ideas with what you are used to, for instance when you are in an elevator or airplane, we need to include the normal force.

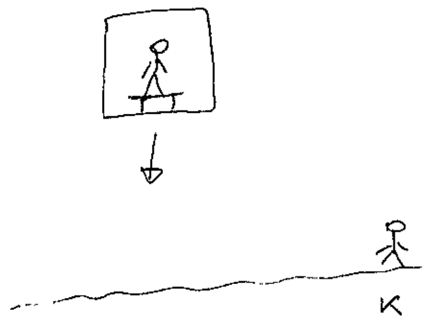
For instance, when you are weighting yourself, the weight pushes you up to prevent you from falling through the floor. So, since you are not moving, the normal force done by the scale is

$$N = mg - mA$$

If you are going up, $A < 0$ and $m(g-A) > mg$. So when the elevat is accelerating up you feel heavier. When it is going down $A >$ so $m(g-A) < mg$, and you feel lighter.

We can also analyze this in an inertial frame. then you would have

$$m\frac{dv}{dt} = mg - N$$

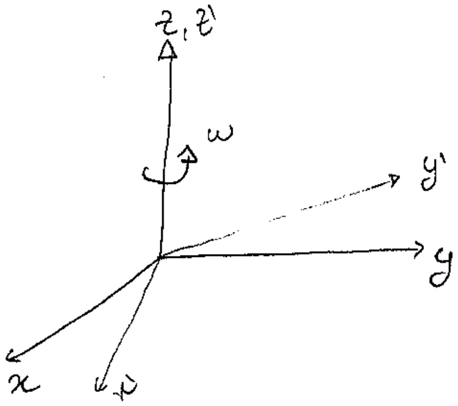


Now $m\dot{v} \neq 0$ so the normal will be smaller than mg. the result is the same as before of course.

Rotating frames

Up to now we have considered a non-inertial frame which is moving through space with a certain acceleration. But the orientation of the axes is fixed.

Now we will consider the description of a system from the perspective of a frame K' which is rotating with a certain angular velocity ω. For simplicity we assume that the origin of both frames coincide, and that the rotation is around the z axis



As we have seen before, we are talking here about a passive rotation. If (x, y, z) and (x', y', z') are the coordinates in the two frames, we have seen that

$$x' = x\cos\omega t + y\sin\omega t$$
$$y' = -x\sin\omega t + y\cos\omega t \qquad (7)$$
$$z' = z$$

Consider again the Lagrangian

$$\mathcal{L} = \frac{1}{2} m v^2 - U \qquad (8)$$

To express the kinetic energy in terms of the K' coordinates we first invert (7) (simply change $\omega t \to -\omega t$):

$$x = x'\cos\omega t - y'\sin\omega t \qquad (9)$$
$$y = x'\sin\omega t + y'\cos\omega t$$
$$z = z'$$

We then have

$$\dot{x} = \dot{x}'\cos\omega t - \dot{y}'\sin\omega t - \omega x'\sin\omega t - \omega y'\cos\omega t$$
$$= (\dot{x}' - \omega y')\cos\omega t - (\dot{y}' + \omega x')\sin\omega t$$
$$\dot{y} = \dot{x}'\sin\omega t + \dot{y}'\cos\omega t + \omega x'\cos\omega t - y'\omega\sin\omega t$$
$$= (\dot{y}' + \omega x')\cos\omega t + (\dot{x}' - \omega y')\sin\omega t$$

Thus

$$\dot{x}^2 + \dot{y}^2 = (\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2$$

we can write in a compact way

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}')^2 \qquad (10)$$

where $\boldsymbol{\omega} = (0, 0, \omega)$. Thus, in terms of the K' coordinates the Lagrangian (8) becomes

$$\boxed{\mathcal{L} = \frac{1}{2} m (\dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}')^2 - U(\mathbf{r}', t)} \qquad (11)$$

where $U(\mathbf{r}', t)$ is the potential expressed using (7), which may cause it to depend on time explicitly

Now let us find the equations of motion. We have

$$(\mathbf{\dot{r}} + \boldsymbol{\omega} \times \mathbf{r})^2 = \mathbf{\dot{r}}^2 + 2\,\mathbf{\dot{r}}' \cdot (\boldsymbol{\omega} \times \mathbf{r}') + (\boldsymbol{\omega} \times \mathbf{r}')^2$$

So

$$\mathcal{L} = \frac{1}{2} m \mathbf{\dot{r}}'^2 + m\,\mathbf{\dot{r}}' \cdot (\boldsymbol{\omega} \times \mathbf{r}') + \frac{1}{2} m (\boldsymbol{\omega} \times \mathbf{r}')^2 - U \qquad (12)$$

Now:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{\dot{r}}'} = m\mathbf{\dot{r}}' + m(\boldsymbol{\omega} \times \mathbf{r}')$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{\dot{r}}'}\right) = m\mathbf{\ddot{r}}' + m(\boldsymbol{\omega} \times \mathbf{\dot{r}}') \qquad (13)$$

where I assumed that $\dot{\omega} = 0$. This is not really necessary (it would just contribute yet another term $m(\boldsymbol{\dot{\omega}} \times \mathbf{r})$ to the final equation). On the other hand, we have

$$\frac{\partial \mathcal{L}}{\partial \mathbf{r}'} = -\frac{\partial U}{\partial \mathbf{r}'} + \frac{\partial}{\partial \mathbf{r}'}\left[m\mathbf{\dot{r}}' \cdot (\boldsymbol{\omega} \times \mathbf{r}')\right] + \frac{\partial}{\partial \mathbf{r}'}\left[\frac{1}{2} m (\boldsymbol{\omega} \times \mathbf{r})^2\right] \qquad (14)$$

For the second term we use the cyclic property

$$\mathbf{\dot{r}}' \cdot (\boldsymbol{\omega} \times \mathbf{r}') = \mathbf{r}' \cdot (\mathbf{\dot{r}}' \times \boldsymbol{\omega})$$

then

$$\frac{\partial}{\partial \mathbf{r}'}\left[m\mathbf{r}' \cdot (\mathbf{\dot{r}}' \times \boldsymbol{\omega})\right] = m\,\mathbf{\dot{r}}' \times \boldsymbol{\omega}.$$

In the third term of (14) we use the formula

$$(\boldsymbol{\omega} \times \mathbf{r}')^2 = \omega^2 |\mathbf{r}'|^2 - (\boldsymbol{\omega} \cdot \mathbf{r}')^2$$

then

$$\frac{\partial}{\partial \mathbf{r}'}\left[\frac{1}{2} m (\boldsymbol{\omega} \times \mathbf{r}')^2\right] = \frac{1}{2} m \omega^2 \frac{\partial}{\partial \mathbf{r}'} |\mathbf{r}'|^2 - \frac{1}{2} m \frac{\partial}{\partial \mathbf{r}'} (\boldsymbol{\omega} \cdot \mathbf{r}')^2$$

$$= m\omega^2 \mathbf{r}' - m(\boldsymbol{\omega} \cdot \mathbf{r}')\, \boldsymbol{\omega}$$

On the other hand, there is the identity

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') = \boldsymbol{\omega}\, (\boldsymbol{\omega} \cdot \mathbf{r}') - \omega^2 \mathbf{r}'$$

So we conclude that

$$\frac{\partial}{\partial \mathbf{r}'}\left[\frac{1}{2} m (\boldsymbol{\omega} \times \mathbf{r}')^2\right] = - m\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$

Hence

$$\frac{\partial L}{\partial \mathbf{r}'} = -\frac{\partial U}{\partial \mathbf{r}'} + m\left[\dot{\mathbf{r}}' \times \boldsymbol{\omega} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')\right] \qquad (15)$$

Letting $\mathbf{F}' = -\partial U/\partial \mathbf{r}'$ we finally obtain the equations of motion by combining (13) and (15):

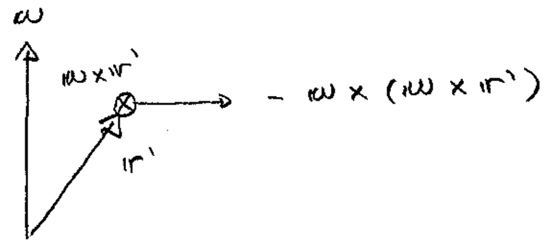
$$\boxed{m\ddot{\mathbf{r}}' = \mathbf{F}' - 2m\, \boldsymbol{\omega} \times \mathbf{v}' - m\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')} \qquad (16)$$

We see that the effect of describing the motion in a rotating frame is therefore to add two new "fictitious" forces to the equations of motion. These terms have famous names

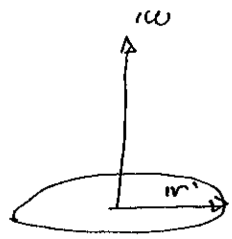
$$\boxed{\begin{aligned} &\text{Centrifugal force: } -m\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \\ &\text{Coriolis force: } -2m\, \boldsymbol{\omega} \times \mathbf{v}' \end{aligned}} \qquad (17)$$

<u>Centrifugal force</u> : $- m \omega \times (\omega \times r')$

The centrifugal force is always directed outwards



A body moving in a rotating frame feels a force outward due to the rotation.



when $\omega \perp r'$ we get

$$| m \omega \times (\omega \times r') | = m \omega^2 r' \qquad (18)$$

The centrifugal force should not be confused with the centripetal force.

$$\text{Centripetal} = \text{centrum} + \text{petere } (\text{"to seek"})$$

"Centripetal force" means any force which pulls you toward the center. So in the central force problem, gravity acts as a centripetal force because it pulls you to the center.

A centrifugal force, on the other hand, is a fictitious force which appears when you measure the position of a body in a non-inertial reference frame.

Example : apparent gravity

The earth is rotating around its own axis so any experiments we do at the earth are in fact being done at a non-inertial frame.

The velocity of rotation of the earth is

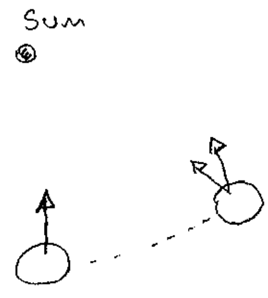
$$\omega = 7.292 \times 10^{-5} \text{ rad/s} \tag{19}$$

which corresponds to a period of

$$T = \frac{2\pi}{\omega} = 86165.5 \text{ s} \tag{20}$$

$$= 23 \text{ days and } 56 \text{ minutes}$$

This is called a sidereal day. It represents the actual rotation period with respect to the fixed stars (ie, with respect to an inertial frame). You see, the earth is not only rotating around itself, but is also rotating around the sun. what we call "1 day" is the time it takes to face the sun twice.

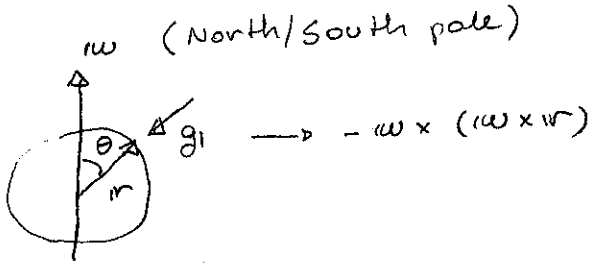


One day is slightly bigger than a sidereal day because the earth has to rotate a tiny bit more to face the sun again.

Let us apply the equations of motion (16) to the problem of a body being attracted by gravity.

$$m\ddot{\mathbf{r}}' = m\mathbf{g}_1 - 2m\boldsymbol{\omega}\times\mathbf{v}' - m\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') \qquad (21)$$

Forget about the Coriolis force for now: it depends on the velocity of the particle. We have



We see from the (awesomely drawn) figure and from (21) that due to the rotation of the earth, a body experiences an <u>apparent</u> <u>gravity</u>

$$\mathbf{g}_1^* = \mathbf{g}_1 - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}') \qquad (22)$$

Using the angle θ defined in the figure we have

$$|\boldsymbol{\omega}\times\mathbf{r}'| = \omega R\sin\theta$$

where R is the radius of the earth. Moreover

$$|\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')| = \omega^2 R\sin\theta \qquad (23)$$

the "vertical" component of the gravitational pull (i.e, the one pointing to the ground) is the radial component.

$$-\mathbf{w} \times (\mathbf{w} \times \mathbf{r}) = \omega^2 R \sin\theta \left[\sin\theta \, \hat{e}_r - \cos\theta \, \hat{e}_\theta\right]$$

thus

$$g_v^* = g - \omega^2 R \sin^2\theta$$
$$g_h^* = \omega^2 R \sin\theta \cos\theta \qquad (24)$$

If we plug the value of ω from (19) and $R = 6371$ km we find

$$\omega^2 R = 34 \ \mathrm{mm/s^2} \qquad (25)$$

At the poles ($\theta = 0$ or $\theta = \pi$) we see no effect : $g_v^* = g$, $g_h^* = 0$.

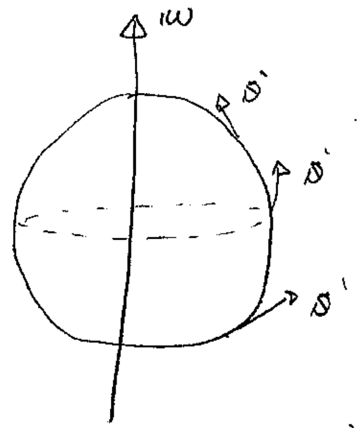
At the equatorial line ($\theta = \pi/2$) we get a deviation of g_v^* of $34 \ \mathrm{mm/s^2}$. Thus

$$\Delta g_v^* = g_v^*(\text{pole}) - g_v^*(\text{equator})$$
$$= 34 \ \mathrm{mm/s^2}$$

The experimental value is $52 \ \mathrm{mm/s^2}$. This discrepancy is due to the fact that the earth is not really spherical, but flatened at the poles. So even if we exclude the centrifugal term, the gravitational pull is already larger at the poles than at the equator.

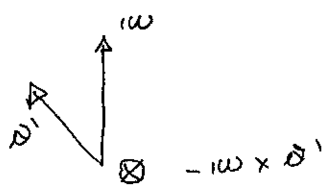
The Coriolis force : $-m\,\omega \times v'$

The Coriolis force depends on v' and therefore only appears due to the motion of the particles. It also has the very interesting effect of acting differently if you are on the northern or southern hemisphere. We can see this as follows. Suppose you are moving north:



At the equator $\omega \parallel v'$ and $\omega \times v' = 0$. At the northern hemisphere

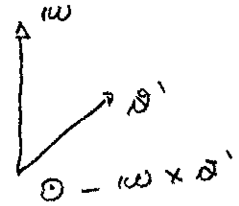
- Northern hemisphere:



$\Rightarrow$ Coriolis pushes you to the right

On the other hand

- Southern hemisphere:



$\Rightarrow$ Coriolis pushes you to the left.

The Coriolis force therefore pushes you to different directions whether you are on the northern and southern hemispheres.

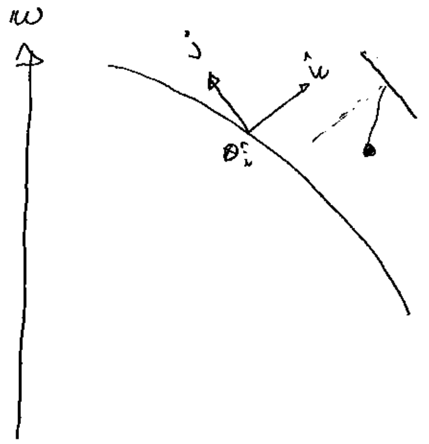
This effect can be observed in cyclones: in the northern hemisphere their rotation is counterclockwise and in the southern hemisphere it is clockwise.

It used to be thought that this also determined the direction of rotation of water when you flush the toilet. But this is not the case: external effects such as the shape of the toilet and how the water is pushed predominate over the (weak) Coriolis force.

Focault's pendulum

In 1851 Léon Focault introduced an experiment to demonstrate that the earth is indeed rotating. Focault's pendulum is just a very long free pendulum (so that it may swing freely for a long time) and perfectly symmetrical (so that the period of oscillation is the same in all directions).

Let us choose our coordinate axes as follows



$\hat{i}$ = east
$\hat{j}$ = north
$\hat{k}$ = up.

So $\hat{k}$ is pointing "up" with respect to someone sitting on the surface of the earth. The angular velocity vector is, in this frame

$$\omega = (0,\ \omega\sin\theta,\ \omega\cos\theta) \tag{26}$$

The effect of the centrifugal force is simply to change the gravitational pull. And since it does not change significantly when compared to the radius of the earth, we can approximate the effect of the centrifugal force as simply saying that it changes g to g^*. We may also neglect the horizontal part g_h^* because it is much smaller than g_v^*.

We now have

$$\omega \times v' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\sin\theta & \omega\cos\theta \\ v_x' & v_y' & v_z' \end{vmatrix}$$

$$= \hat{i}\left(\omega \sin\theta\, v_z' - \omega\cos\theta\, v_y'\right) +$$
$$+ \hat{j}\left(\omega\cos\theta\, v_x'\right)$$
$$+ \hat{k}\left(-\omega\sin\theta\, v_x'\right)$$

we will finally also assume that the oscillations are small so that v_z' may be neglected and the problem may be treated as that of a simple harmonic motion.

We then have

$$\ddot{x} = -\frac{g}{l}x + 2\omega \dot{y}\cos\theta \tag{27}$$
$$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x}\cos\theta$$

We can solve these equations using a trick that Landau used very oftenly. Let

$$z(t) = x + iy \tag{28}$$

Then if we multiply the second equation by i and add it to the first we get

$$\ddot{z} = -\frac{g}{l}z + \omega\cos\theta\underbrace{\left(\dot{y} - i\dot{x}\right)}_{-i(\dot{x}+i\dot{y}) = -i\dot{z}}$$

$$\therefore \quad \ddot{z} = -\frac{g}{l}z - i\dot{z}\,\omega\cos\theta \tag{29}$$

This is just like the equation for a damped harmonic oscillator, but with a complex coefficient. Let

$$\omega_0^2 = g/\ell \qquad \Omega = \omega \cos\theta \tag{30}$$

We are then left with

$$\ddot{z} + 2i\Omega\dot{z} + \omega_0^2 z = 0 \tag{31}$$

To solve it we try the ansatz

$$z = A e^{pt} \tag{32}$$

We then get

$$p^2 + 2i\Omega p + \omega_0^2 = 0$$

with solution

$$p = -i\Omega \pm \sqrt{-\Omega^2 - \omega_0^2}$$

$$p = -i\Omega \pm i\Omega_1 \qquad \Omega_1^2 = \Omega^2 + \omega_0^2 \tag{33}$$

The general solution is then

$$z(t) = A e^{-i(\Omega+\Omega_1)t} + B e^{-i(\Omega-\Omega_1)t} \tag{34}$$

where A and B are two complex constants determined by $x(0)$, $y(0)$, $\dot{x}(0)$, $\dot{y}(0)$.

For definiteness suppose we adjust the initial conditions such that $A = B = a/2$, where a is some arbitrary constant. This corresponds to roughly (but not exactly) starting from rest at $(a, 0)$. We then get

$$z = a\, e^{-i\Omega t} \cos \Omega_1 t$$

and, therefore

$$\boxed{\begin{aligned} x(t) &= a \cos \Omega t \cos \Omega_1 t \\ y(t) &= -a \sin \Omega t \cos \Omega_1 t \end{aligned}} \qquad (35)$$

Choose a pendulum with $\ell = 2\,m$. Then

$$\omega_0 = \sqrt{\frac{g}{\ell}} \simeq 2.21 \text{ rad/s.} \qquad (36)$$

This is to be compared with $\Omega = \omega \cos\theta$, where $\omega \simeq 10^{-4} \frac{\text{rad}}{s}$ [cf. Eq (19)]. Thus $\Omega << \omega_0$ and hence $\Omega_1 = \sqrt{\omega_0^2 + \Omega^2} \approx \omega_0$.

Eq (35) therefore shows that the fast "natural" pendulum oscillations with frequency ω_0 is superimposed on a very low oscillation with frequency Ω.