Spontaneous Symmetry Breaking

Part 1

- Curie - Weiss model
- Mean - field approximation
- Landau theory
- Hysteresis
- Relaxation to equilibrium.

Mean-field approximation

Curie and Weiss introduced a phenomenological model of magnetism using a very clever idea. Consider a single spin 1/2 particle subject to a magnetic field. The Hamiltonian is

$$H = -h\,\sigma_z \tag{1}$$

In thermal equilibrium the partition function will be

$$Z = \mathrm{tr}\, e^{-\beta H} = 2\cosh(\beta h) \tag{2}$$

Thus, the average magnetization will be

$$m = \langle \sigma_z \rangle = \mathrm{tr}\left(\sigma_z \frac{e^{-\beta H}}{Z}\right) \tag{3}$$

or

$$m = \tanh(\beta h) \tag{4}$$

Now comes Curie and Weiss's idea: in a ferromagnetic material the system will feel, in addition to the external field h, an effective field due to the magnetization of all its neighbors.

But the magnetization of it's neighbors will be very similar to its own magnetization m. Thus, the total field a spin will feel will be

$$h_{eff} = h + \lambda m \tag{5}$$

where λ is some constant (which they called the <u>molecular field constant</u>).

Eq (4) then becomes

$$\boxed{m = \tanh\left(\frac{h + \lambda m}{T}\right)} \tag{6}$$

which is known as the <u>Curie - Weiss</u> Eq. It is an implicit equation for m.

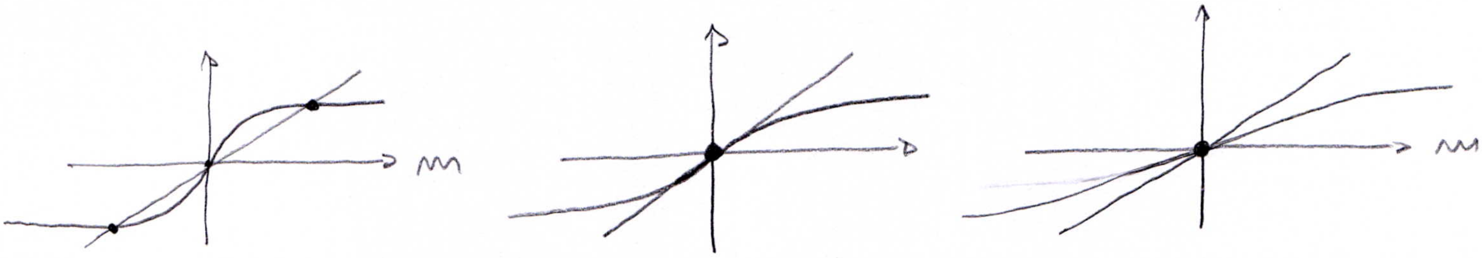
Let us first consider the case $h = 0$. then we get

$$m = \tanh\left(\frac{\lambda m}{T}\right) \tag{7}$$

To solve this equation we may compare graphically the left and right-hand sides. We can have 3 possibilities depending on the value of J/T. It is useful to know that

$$\tanh(x) \simeq x - \frac{x^3}{3} \tag{8}$$

Thus, close to $m = 0$ the function $\tanh\left(\frac{\lambda m}{T}\right)$ will be a straight line with a slope that may be either larger or smaller than 1, depending on λ/T. Thus we may have one of 3 possibilities

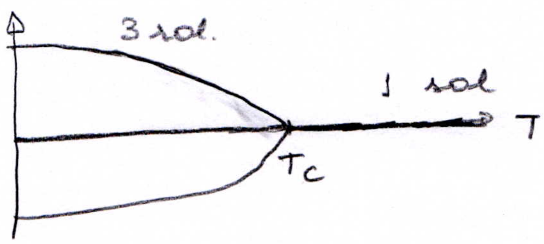


$\frac{\lambda}{T} > 1$ $\qquad\qquad$ $\frac{\lambda}{T} = 1$ $\qquad\qquad$ $\frac{\lambda}{T} < 1$

when $\frac{\lambda}{T} \leqslant 1$ there will only be one solution: $m = 0$. But when $\frac{\lambda}{T} > 1$, two new solutions appear at non-zero values m^* and $-m^*$. This therefore signals a phase transition from a disordered (paramagnetic) to an ordered (ferromagnetic) phase. The transition occurs at $\frac{\lambda}{T} = 1$, which defines the critical temperature

$$\boxed{T_c = \lambda} \tag{9}$$

The solutions will look like this



We will learn later that when $T < T_c$ the solution with $m = 0$ is actually unstable. Thus at $T < T_c$ the magnetization is non-zero.

The Curie-Weiss model is purely phenomenological and introduces an arbitrary constant λ without explanation.

We can derive the Curie-Weiss model in a more controlled way using the mean-field approximation. This approximation can be used for many models, but we will do it specifically for the Ising model:

$$\boxed{H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^z} \tag{10}$$

We assume the model is defined in a hypercubic lattice with dimension d and interaction over nearest-neighbors only.

The MF approx. is based on the following idea. Write

$$\sigma_i^z = \langle \sigma_i^z \rangle + (\sigma_i^z - \langle \sigma_i^z \rangle) \tag{11}$$

This can always be done for any operator: we simply split it into its mean value and the fluctuations around the mean. Due to translation invariance, $\langle \sigma_i^z \rangle = m$ for all i. Thus

$$\boxed{\sigma_i^z = m + (\sigma_i^z - m)} \tag{12}$$

Now we write

$$\sigma_i^z \sigma_j^z = \left[m + (\sigma_i^z - m)\right]\left[m + (\sigma_j^z - m)\right]$$

$$= m^2 + m(\sigma_i^z - m) + m(\sigma_j^z - m) +$$

$$+ \underbrace{(\sigma_i^z - m)(\sigma_j^z - m)}_{\approx 0}$$

The MF approx. consists in neglecting the last term. The justification is that this term is quadratic in the fluctuations and should thus be small if the fluctuations are small. Whether or not this approximation is justified will be discussed later.

Thus, we approximate

$$\boxed{\sigma_i^z \sigma_j^z \simeq -m^2 + m(\sigma_i^z + \sigma_j^z)} \qquad (13)$$

Now we plug this in (10). We then get

$$-J \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z = J \sum_{\langle i,j \rangle} m^2 - J m \sum_{\langle i,j \rangle} (\sigma_i^z + \sigma_j^z)$$

$$= J m^2 (dN) - J m (2d) \sum_i \sigma_i^z$$

Thus we get

$$\boxed{H = N d J m^2 - (h + 2dJm)\sum_i \sigma_i^z} \qquad (14)$$

This is now the Hamiltonian for N independent spins in a magnetic field

$$h_{eff} = h + 2dJm \qquad (15)$$

Comparing with (5), we see that this is just like the Curie-Weiss model, with a molecular constant

$$\lambda = 2Jd \qquad (16)$$

This makes sense: the higher is the exchange constant J and the higher is the dimension d, the stronger will be the FM interaction.

From (9) we know that $\lambda = T_c$. Plugging back k_B we then get the relation

$$\boxed{T_c = \frac{2Jd}{k_B}} \qquad (17)$$

I find it useful to remember that $T_c \propto J$: by looking at the T_c of different materials, we get an idea of the exchange constant J.

Behavior in the vicinity of T_c

Now let's go back to the Curie-Weiss Eq, which I will write as

$$m = \tanh\left(\frac{h + T_c m}{T}\right) \quad (18)$$

[I replaced λ by T_c]. We can also write this as

$$h + T_c m = T \tanh^{-1}(m) \quad (19)$$

Close to T_c the magnetization m will be small, so we may expand

$$\tanh^{-1}(m) \simeq m + \frac{m^3}{3} \quad (20)$$

We then get

$$\boxed{h = (T - T_c) m + T \frac{m^3}{3}} \quad (21)$$

This is the equation of state (like $PV = N k_B T$) close to the critical point.

First we set $h=0$. then we get the Eq

$$\frac{Tm^3}{3} + (T-T_c)m = 0 \tag{22}$$

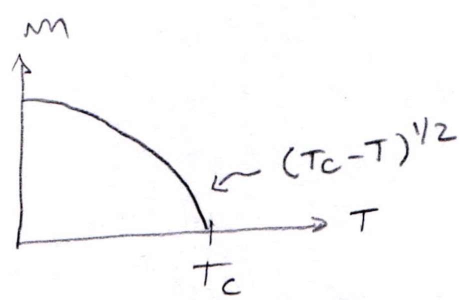
One solution is $m=0$. the other is

$$m^2 = \frac{3(T_c - T)}{T} \tag{23}$$

If $T > T_c$ then $T_c - T < 0$, which means m must be imaginary. thus, when $T > T_c$ the only real solution is $m=0$. But for $T < T_c$ the solution (23) will exist and we see that

$$\boxed{m \propto (T_c - T)^{1/2}} \tag{24}$$

this looks like this:



Back to (21), exactly at $T = T_c$ we get

$$\boxed{m = \left(\frac{3h}{T_c}\right)^{1/3} \sim h^{1/3}} \qquad (25)$$

we can also look at the susceptibility,

$$\chi = \frac{\partial m}{\partial h} \qquad (26)$$

Differentiating (21) with respect to h, we get

$$1 = (T - T_c)\,\chi + \frac{T}{3}\,3m^2\,\chi$$

Thus

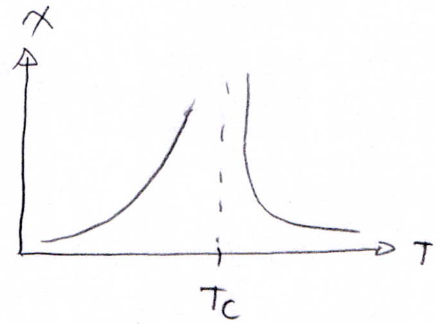
$$\chi = \frac{1}{T - T_c + Tm^2} \qquad (27)$$

If $T > T_c$ then $m = 0$. But if $T < T_c$ then $m^2 = \dfrac{3(T_c - T)}{T}$

so we get

$$\boxed{\chi = \begin{cases} \dfrac{1}{|T - T_c|} & T > T_c \\[2ex] \dfrac{1}{2(T_c - T)} & T < T_c \end{cases}} \qquad (28)$$

On both sides χ diverges at T_c



and it does so with the same exponent

$$\boxed{\chi \sim \frac{1}{|T-T_c|}} \tag{29}$$

Finally, we can look at the internal energy and specific heat. Averaging Eq (14) we get

$$U = \langle H \rangle = N d J m^2 - (h + 2dJm) \sum_i \langle \sigma_i^z \rangle$$

But $\langle \sigma_i^z \rangle = m$. We can also replace $2dJ = T_c$. Then we get

$$\frac{U}{N} = \frac{T_c}{2} m^2 - (h + T_c m) m$$

or

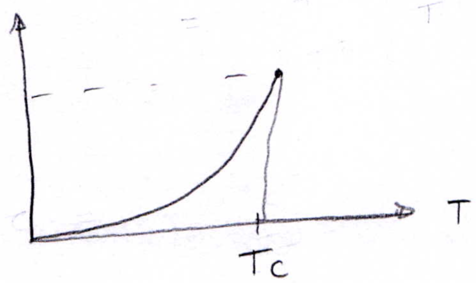
$$\boxed{\frac{U}{N} = -hm - \frac{T_c}{2} m^2} \tag{30}$$

If we set $h=0$ we get, $\frac{U}{N} = -\frac{T_c}{2} m^2$. If $T > T_c$ then

$U=0$, but if $T < T_c$,

$$\frac{U}{N} = -\frac{T_c}{2} m^2$$

To compute c we need the slope of m vs. T. Our approximate solution (23) does not give a good answer. Numerically, we find the following



It does not diverge at T_c, but has a discontinuity.

Critical exponents

We have seen that different quantities behave as power laws in $T_c - T$. The exponents of these power laws are called critical exponents:

$$m \sim (T_c - T)^{\beta}$$

$$\chi \sim (T_c - T)^{-\gamma}$$

$$c \sim (T_c - T)^{\alpha}$$

$$m \sim h^{1/\delta} \qquad \text{(at } T_c\text{)}$$

$$\xi \sim (T_c - T)^{1/\nu} \qquad \text{correlation length}$$

These exponents are robust: they don't depend on the silly details of your model, but only on the symmetries and the dimensionality. This is the idea of universality. For instance, all mean-field models will always have the same critical exponents.

what changes for higher spin models

Answer: almost nothing.

The general paramagnetic Hamiltonian for a spin J system may be written as

$$H = - g \mu_B H J_z \qquad (31)$$

where μ_B is the Bohr magneton and g is called the Landé factor. In general J_z mixes spin and orbital angular momentum. The Landé factor weigths the two as

$$g = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)} \qquad (32)$$

which means you must know S, L and J. I will not worry about this here.

The magnetization m can be computed as before. It is a simple exercise in statistical physics. The result is

$$\boxed{m = g \mu_B J \, B_J \left(\frac{g \mu_B J H}{k_B T} \right)} \qquad (33)$$

where $B_J(x)$ is called the Brillouin function

$$B_J(x) = \frac{2J+1}{2J} \coth\left(\frac{2J+1}{2J} x\right) - \frac{1}{2J} \coth\left(\frac{x}{2J}\right) \tag{34}$$

Two cases are important. If $J = 1/2$ this simplifies to

$$B_{1/2}(x) = \tanh(x) \tag{35}$$

so we recover what we were doing before. If $J \to \infty$ we get

$$\lim_{J \to \infty} B_J(x) := \mathcal{L}(x) = \coth(x) - \frac{1}{x} \tag{36}$$

This is called the Langevin function.

Another thing that may be useful to know is that, if $x \ll 1$

$$B_J(x) \simeq \frac{J(J+1)}{3J^2} x \tag{37}$$

Since we are talking about paramagnetism, I may as well show you something useful. When is small we may use (37) to write

$$m \simeq (g\mu_B J)\ \frac{J(J+1)}{3J^2}\ \frac{g\mu_B J H}{k_B T}$$

or

$$m = \frac{(g\mu_B)^2 J(J+1)}{3k_B T}\ H \qquad (38)$$

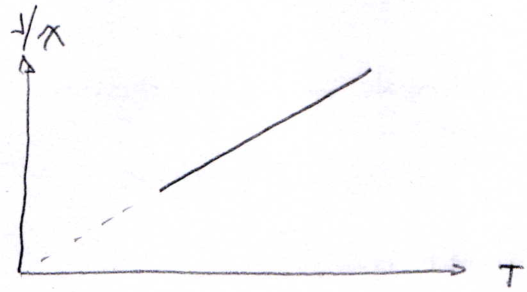
The magnetization is linear in the field. The slope is the susceptibility

$$\chi = \frac{\partial m}{\partial H} = \frac{(g\mu_B)^2 J(J+1)}{3k_B T} \qquad (39)$$

Note how this has the form

$$\boxed{\chi = \frac{C}{T} \qquad C = \frac{(g\mu_B)^2 J(J+1)}{3k_B T}} \qquad (40)$$

This is known as <u>Curie's law</u> and C is <u>Curie's constant</u>. Usually, what we do experimentally is plot $1/\chi$ vs T. This should then give a straight line with slope $1/C$.



It is common to use this to compute the curie constant. Since it is proportional to $g\, J(J+1)$, it tells you about the magnetic moment in your sample.

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Now that we have the general formula (33) for the magnetization, we can write down a curie-weiss equation, like (6), by changing the field H into an effective field $H + \lambda m$. Eq (33) then becomes

$$m = g\mu_B J\, B_J\left(\frac{g\mu_B J}{k_B T} (H + \lambda m)\right) \qquad (41)$$

From here onwards, everything is exactly like before, except that we get a bunch of ugly coefficients to carry around. In particular, all critical exponents remain the same.

The Landau free energy

Landau came up with a phenomenological theory that is extremely useful in understanding critical phenomena.

The free energy of a system is a function of the temperature T and the field, $f(T,h)$. Moreover, a given (T,h) fixes the magnetization.

Landau's idea is to consider a free energy which is itself a function of m, $f(T,h,m)$. We then postulate that the equilibrium state will be that m which minimizes f; ie

$$\frac{\partial f}{\partial m} = 0 \tag{42}$$

We may then recover the usual free energy as

$$f(T,h) = f(T,h,m(T,h)) \tag{43}$$

In the case of the mean-field approximation this occurs naturally. Recall the Hamiltonian (14).

$$H = \frac{N T_c}{2} m^2 - (h + T_c m) \sum_i \sigma_i^z \tag{44}$$

the partition function will be

$$Z = \operatorname{tr} e^{-\beta H} = e^{-\beta \frac{N T_c}{2} m^2} \left[\operatorname{tr}\left(e^{-\beta h_{eff} \sigma_z}\right)\right]^N$$

using (2) this gives

$$Z = e^{-\beta \frac{N T_c m^2}{2}} \left[2\cosh\left(\beta(h + T_c m)\right)\right]^N$$

Thus the free energy will be

$$F = -T \ln Z = \frac{N T_c m^2}{2} - N T \ln\left[2\cosh\left(\beta(h + T_c m)\right)\right]$$

and the free energy per particle will be $f = F/N$ or

$$\boxed{f = \frac{T_c m^2}{2} - T \ln\left\{2\cosh\left(\frac{h + T_c m}{T}\right)\right\}} \qquad (45)$$

If we now apply (42) we get

$$\frac{\partial f}{\partial m} = T_c m - \tanh\left(\frac{h + T_c m}{T}\right) = 0 \qquad (46)$$

which is exactly the Curie-Weiss Eq (6). Thus, by looking at the minima of the free energy, we find the possible solutions for m

The real use for the Landau free energy, however, is in analyzing the behavior close to the critical point. The main argument of Landau is that $f(m)$ should be a smooth function of m. Hence, it may be expanded as a power series:

$$f(m) = a_0 + a_1 m + a_2 m^2 + a_3 m^3 + \dots \tag{47}$$

But the free energy must respect the symmetries of the model. The Ising Hamiltonian we started with, for instance, will have up-down symmetry when $h=0$. This means that $f(-m) = f(m)$. Consequently, the expansion (47) may contain only even powers of m

$$f(m) = a_0 + a_2 m^2 + a_4 m^4 + \dots \tag{48}$$

If we a put on a field, we should break up-down symmetry. Thus, if h is tiny, this could introduce a term $-hm$.

The typical Landau free energy will then look like

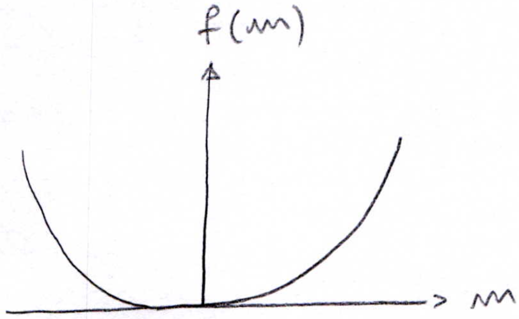
$$\boxed{f \simeq \frac{a}{2} m^2 + \frac{b}{4} m^4 - hm} \tag{49}$$

where I neglected the constant term a_0.

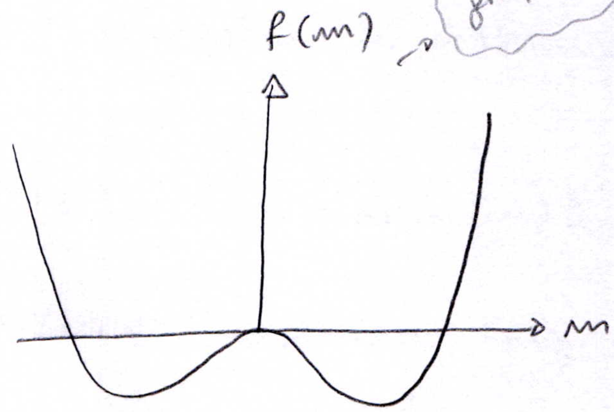
Now comes the magic of all this. Are you ready? I hope you are because here we go: the behavior of $f(m)$ changes completely if $a>0$ or $a<0$.

Assume $h=0$. Then we get

the most important graph in all of physics!



$a>0$



$a<0$

When $a>0$ the free energy has only one minimum at $m=0$. But when $a<0$, two new minima appear at $m \neq 0$. This is the phase transition to the FM phase.

For $a<0$ the energy landscape of the system will thus be that of two wells sepparated by an energy barrier. And this barrier is huge because the actual free energy is $Nf(m)$ and $N \to \infty$.

Thus, the system will spontaneously pick a side and <u>it will stay there</u>. We started out with a Hamiltonian that had up-down symmetry. But the system chose a side. <u>The symmetry is therefore spontaneously broken</u>. BOOM! Mind blowing!

We can verify that all this works like a charm for our mean field free energy (45). Use the Taylor series expansion

$$\ln(\cosh(x)) \simeq \frac{x^2}{2} - \frac{x^4}{12} \tag{50}$$

We then get, for $h=0$,

$$f \simeq T_c \frac{m^2}{2} - \frac{T}{2}\left(\frac{T_c}{T} m\right)^2 + \frac{T}{12}\left(\frac{T_c m}{T}\right)^4$$

$$= \frac{m^2}{2} T_c \left(1 - \frac{T_c}{T}\right) + \frac{T_c^4}{12 T^3} m^4$$

Thus

$$f = \frac{a}{2} m^2 + \frac{b}{4} m^4 \tag{51}$$

where $\quad b = \dfrac{T_c^4}{4T^3} \qquad a = \dfrac{T_c}{T}(T - T_c)$

Note how everything makes sense

$$\begin{aligned} T > T_c &\rightsquigarrow a > 0 \\ T < T_c &\rightsquigarrow a < 0 \end{aligned} \tag{52}$$

Notice also how we can write

$$a = \alpha \left(\frac{T - T_c}{T} \right) \tag{53}$$

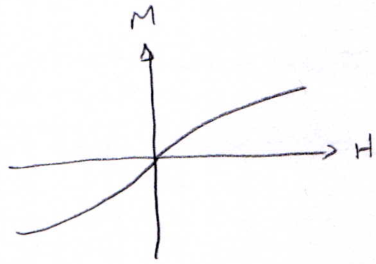
where α is a constant. Thus, a is linear in the dimensionless quantity $(T - T_c)/T$, which changes sign exactly at T_c.

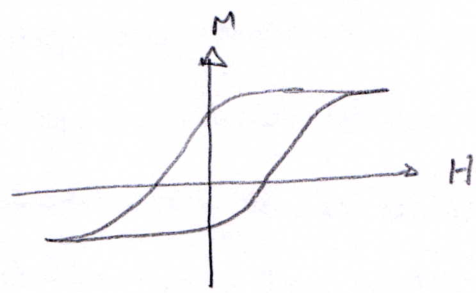
I probably forgot to say, but in terms of a and b the minimum of $f(m)$ occurs at

$$\boxed{m^* = \begin{cases} 0 & \text{if } a > 0 \\ \pm\sqrt{\frac{-a}{b}} & \text{if } a < 0 \end{cases}}$$

Hysteresis

The most paradigmatic property of a ferromagnetic system is the presence of hysteresis





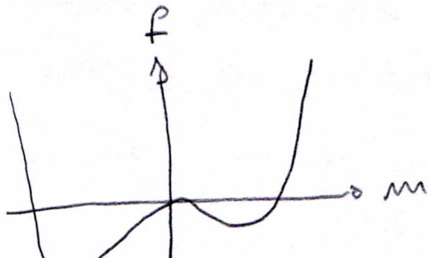
Paramagnetic

- M vs. H is 1-to-1
- there is no memory

Ferromagnetic

- Not 1-to-1
- curve depends on the history of H.

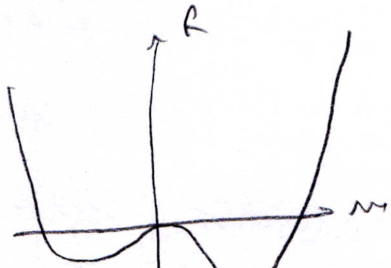
We can understand hysteresis using the Landau free energy. When $h \neq 0$, the general free energy is given by (49). In the FM phase $(a<0)$, $f(m)$ will look like this



$h<0$

$h=0$

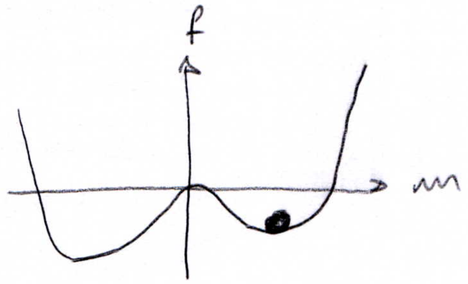
(this should by symmetric!)



$h>0$

The field breaks the degeneracy of the two minima. If $h>0$ then $+m^*$ becomes the true minimum and if $h<0$ then its $-m^*$.

But now suppose you put the system in $+m^*$. For instance we may apply a large positive field. then we lower this field to zero and eventually apply a negative field. We will then get something like this

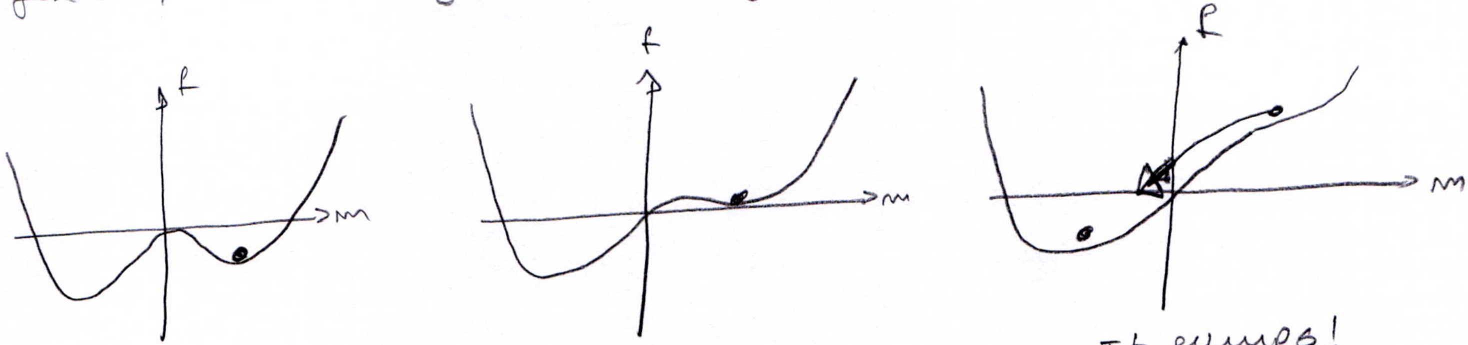


Even though the global minimum is at $-m^*$, the system is stuck at $+m^*$. the question, then, is whether or not it will jump over.

In general it will not because the energy barrier is enormous (recall that it scales with N). thus, the system will stay stuck in $+m^*$. Now comes the catch:

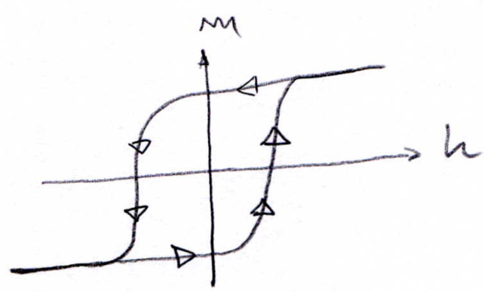
> It will only jump if the barrier separating the two minima is destroyed

If we keep on applying increasingly more negative fields, we will get something like this



It jumps!

When the field is sufficiently large, it kills the barrier and then the system eventually jumps. If we now retrace the behavior of $m(h)$, we will get something like this



voilà! Hysteresis!

The field at which the energy barrier is destroyed is called the <u>coercive</u> <u>field</u>.

Relaxation to equilibrium

Suppose now that we perturb the system slightly and take it away from the equilibrium state m^*. Then it will eventually relax back to equilibrium. Describing this relaxation is a very difficult task. However, if all you want is a rough description, then the Landau free energy offers a nice approximation.

The idea is to consider an equation for $\frac{dm}{dt}$. We want this equation to be such that, at the minimum m^* we get $\frac{dm}{dt} = 0$. Moreover, we expect that the rate of change $\frac{dm}{dt}$ should be proportional to the gradient of the free energy, $\frac{\partial f}{\partial m}$. It should be big when this slope is steep.

A phenomenological way of describing this behavior is by means of an equation like

$$\boxed{\tau \frac{dm}{dt} = -\frac{\partial f}{\partial m}} \qquad (54)$$

where τ is a constant which measures the time-scale of the relaxation process.

Consider the free energy (49) with $h=0$. then we get

$$\tau \frac{dm}{dt} = -am - bm^3 \qquad (55)$$

It is easier to look at m^2:

$$\frac{d}{dt}(m^2) = \frac{2m\,dm}{dt}$$

thus, multiplying (55) by m we get

$$\frac{\tau}{2}\frac{d}{dt}(m^2) = -am^2 - bm^4 \qquad (56)$$

the solution to this equation is

$$\boxed{m^2(t) = \frac{a}{ce^{2at/\tau} - b}} \qquad c = \text{const} \qquad (57)$$

If $a>0$ (PM phase) this will eventually relax to $m^2=0$.
But if $a<0$ this will relax to $m^2 = -a/b$, which is exactly the minimum of $f(m)$. Thus, it makes sense.

A special situation is when $a = 0$, that is, exactly at the critical point. In this case Eq (57) is not valid. Instead we must solve

$$\frac{\gamma}{2}\frac{d(m^2)}{dt} = -b\,m^4 \qquad (58)$$

whose solution is

$$\boxed{m^2(t) = \frac{1}{2bt + c}} \qquad (59)$$

this is actually quite interesting: in both the FM and the PM phases, the relaxation is exponential. But exactly at the critical point the relaxation becomes algebraic. Algebraic is always much slower. this is called critical slowing down.

We also obtain another critical exponent: when t is large we get

$$\boxed{m \sim t^{-1/2}} \qquad (60)$$

The approximate evolution equation (54) does not describe thermal fluctuations. In certain systems, these may play an important role. It is customary to model them by adding a noise term to Eq (54):

$$\boxed{\tau \frac{dm}{dt} = - \frac{\partial f}{\partial m} + \xi(t)} \qquad (61)$$

where $\xi(t)$ satisfies

$$\langle \xi(t) \rangle = 0 \qquad (62)$$

$$\langle \xi(t)\, \xi(t') \rangle = \lambda\, \delta(t-t')$$

where λ is a constant. This is called the Langevin equation. I will not have the time to describe this any further now. I just wanted to mention it so you knew it existed.