10/03/15

Orbital angular momentum

Now we will turn to orbital angular momentum.

$$\vec{L} = \vec{r} \times \vec{p} \qquad (1)$$

Note that, in this case we do not use $\vec{L}$ in units of $\hbar$.
We already know the eigenvalues of $\hat{L}_z$ and $\hat{L}^2$. They are

$$\hat{L}_z |l,m\rangle = \hbar m |l,m\rangle \qquad (2)$$

$$\hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle \qquad (3)$$

Now what I want to do is find the corresponding eigenfunctions in the coordinate representation. In doing so we will discover something quite interesting: for orbital angular momentum l cannot be a half-integer.
that is

$$l = 0, 1, 2, 3, \dots \qquad (4)$$

Half-integer angular momentum can only occur for spin.
In the coordinate representation we know that

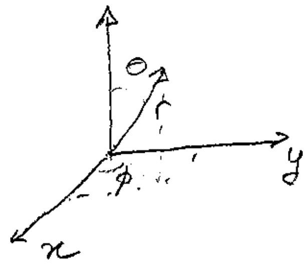
$$\vec{p} = -i\hbar \nabla \qquad (5)$$

so that Eq (1) becomes

$$\vec{L} = -i\hbar\, \vec{r} \times \nabla \qquad (6)$$

It is convenient, as you will see, to do the calculations in spherical coordinates, defined by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad (7)$$
$$z = r\cos\theta$$



The gradient in spherical coordinates is

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad (8)$$

where $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are the unit vectors in the spherical coordinates basis (these guys are <u>not</u> operators!!).

Explicitly we have

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \qquad (9a)$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \qquad (9b)$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \qquad (9c)$$

$\hat{x}, \hat{y}, \hat{z}$ = unit vectors not operators!

We now compute the cross product

$$\vec{r}\times\nabla = (\vec{r}\times\hat{r})\frac{\partial}{\partial r} + (\vec{r}\times\hat{\theta})\frac{1}{r}\frac{\partial}{\partial\theta} + (\vec{r}\times\hat{\phi})\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \qquad (10)$$

The vectors $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ are mutually orthogonal like $\hat{x}$, $\hat{y}$ and $\hat{z}$. Thus

$$\vec{r}\times\hat{r} = 0$$
$$\vec{r}\times\hat{\theta} = r\hat{\phi}$$
$$\vec{r}\times\hat{\phi} = -r\hat{\theta}$$

Hence

$$\boxed{\vec{L} = -i\hbar \left\{ \hat{\phi} \frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin\theta} \frac{\partial}{\partial \phi} \right\}} \tag{11}$$

Now we go backwards: we substitute $\hat{\theta}$ and $\hat{\phi}$ from (9) to obtain $\hat{L}_x$, L_y and L_z:

$$\hat{L}_x = -i\hbar \left\{ -\sin\phi \frac{\partial}{\partial \theta} - \cot\theta \cos\phi \frac{\partial}{\partial \phi} \right\} \tag{12a}$$

$$\hat{L}_y = -i\hbar \left\{ \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi} \right\} \tag{12b}$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \tag{12c}$$

Yay! We now have the orbital angular momentum operators in the coordinate representation. Note that L_z is particularly simple.

What I want to do now is find L^2. To do this I will write

$$L^2 = L_x^2 + L_y^2 + L_z^2 = \frac{L_+ L_- + L_- L_+}{2} + L_z^2$$

since

$$L_+ L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i\,[L_x, L_y]$$

$$L_- L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + L_y^2 + i\,[L_x, L_y]$$

$$\therefore \quad L_+ L_- + L_- L_+ = 2(L_x^2 + L_y^2)$$

Moreover, we have

$$[L_+, L_-] = 2\hbar L_z$$

so that

$$L_+ L_- + L_- L_+ = L_+ L_- + (L_+ L_- - 2\hbar L_z)$$
$$= 2 L_+ L_- - 2\hbar L_z$$

thus

$$\boxed{L^2 = L_z^2 - \hbar L_z + L_+ L_-} \qquad (13)$$

This is good because the first two terms are easy to compute

$$L_z^2 - L_z = -\hbar^2 \frac{\partial^2}{\partial\phi^2} + i\hbar \frac{\partial}{\partial\phi}$$

But now we need to find the last term and it is a bit harder. We start by finding $L_\pm$:

$$L_+ = L_x + i L_y$$
$$= -i\hbar \left\{ (-\sin\phi + i\cos\phi) \frac{\partial}{\partial\theta} - \cot\theta\,(\cos\phi + i\sin\phi) \frac{\partial}{\partial\phi} \right\}$$
$$= -i\hbar \left\{ i e^{i\phi} \frac{\partial}{\partial\theta} - \cot\theta\, e^{i\phi} \frac{\partial}{\partial\phi} \right\}$$

$$\therefore \quad \boxed{L_+ = -i\hbar\, e^{i\phi} \left\{ i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi} \right\}} \qquad (14)$$

the procedure is similar for L_-

$$L_- = L_x - i L_y$$

$$= -i\hbar \left\{ (-\sin\phi - i\cos\phi) \frac{\partial}{\partial\theta} - \cot\theta\, (\cos\phi - i\sin\phi) \frac{\partial}{\partial\phi} \right\}$$

$$= -i\hbar \left\{ -i\, e^{-i\phi} \frac{\partial}{\partial\theta} - e^{-i\phi} \cot\theta \frac{\partial}{\partial\phi} \right\}$$

$$\therefore \quad \boxed{L_- = -i\hbar\, e^{-i\phi} \left\{ -i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi} \right\}} \qquad (15)$$

Of course, we could also have taken the dagger. For this recall that

$$\left(\frac{\partial}{\partial x}\right)^{+} = -\frac{\partial}{\partial x}$$

for any variable x. Then

$$L_+^{+} = i\hbar\, e^{-i\phi} \left\{ (-i)\left(-\frac{\partial}{\partial\theta}\right) - \cot\theta \left(-\frac{\partial}{\partial\phi}\right) \right\}$$

$$= -i\hbar\, e^{-i\phi} \left\{ -i\frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi} \right\}$$

Phew!

Now to compute (13) we need $L_+ L_-$ and this is the trickiest part: it helps if you apply this to some function $f(\theta,\phi)$. So first we find

$$L_- f = -i\hbar\, e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) f$$

$$= -i\hbar\left[-i\, e^{-i\phi}\frac{\partial f}{\partial\theta} - e^{-i\phi}\cot\theta\,\frac{\partial f}{\partial\phi}\right]$$

Now we apply L_+

$$L_+L_-f = (-i\hbar)^2\, e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\left[-i\, e^{-i\phi}\frac{\partial f}{\partial\theta} - e^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right]$$

$$= -\hbar^2 e^{i\phi}\left\{\frac{\partial}{\partial\theta}\left(e^{-i\phi}\frac{\partial f}{\partial\theta}\right) - i\frac{\partial}{\partial\theta}\left(e^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right)\right.$$
$$\left. + i\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\frac{\partial f}{\partial\theta}\right) + \cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\cot\theta\frac{\partial f}{\partial\phi}\right)\right\}$$

$$= -\hbar^2 e^{i\phi}\left\{e^{-i\phi}\frac{\partial^2 f}{\partial\theta^2} - i\, e^{-i\phi}\frac{\partial}{\partial\theta}\left(\cot\theta\frac{\partial f}{\partial\phi}\right) + \right.$$
$$\left. + i\cot\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\frac{\partial f}{\partial\theta}\right) + \cot^2\theta\frac{\partial}{\partial\phi}\left(e^{-i\phi}\frac{\partial f}{\partial\phi}\right)\right\}$$

Now we evaluate each term separately:

$$\frac{\partial}{\partial\theta}\left[\cot\theta\frac{\partial f}{\partial\phi}\right] = -\csc^2(\theta)\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial^2 f}{\partial\theta\,\partial\phi}$$

$$= -\frac{1}{\sin^2\theta}\frac{\partial f}{\partial\phi} + \cot\theta\frac{\partial^2 f}{\partial\theta\,\partial\phi}$$

$$\frac{\partial}{\partial\phi}\left[e^{-i\phi}\frac{\partial f}{\partial\theta}\right] = -i\,e^{-i\phi}\frac{\partial f}{\partial\theta} + e^{-i\phi}\frac{\partial^2 f}{\partial\phi\,\partial\theta}$$

$$\frac{\partial}{\partial\phi}\left[e^{-i\phi}\frac{\partial f}{\partial\phi}\right] = -i\,e^{-i\phi}\frac{\partial f}{\partial\phi} + e^{-i\phi}\frac{\partial^2 f}{\partial\phi^2}$$

thus

$$L_+L_-f = -\hbar^2\left\{\frac{\partial^2 f}{\partial\theta^2} - i\left[\frac{-1}{\sin^2\theta}\frac{\partial f}{\partial\phi} + \cot\theta\,\frac{\partial^2 f}{\partial\theta\,\partial\phi}\right]\right.$$

$$+\,i\cot\theta\left[-i\,\frac{\partial f}{\partial\theta} + \frac{\partial^2 f}{\partial\phi\,\partial\theta}\right] +$$

$$\left. +\,\cot^2\theta\left[-i\,\frac{\partial f}{\partial\phi} + \frac{\partial^2 f}{\partial\phi^2}\right]\right\}$$

$$= -\hbar^2\left\{\frac{\partial^2 f}{\partial\theta^2} + \frac{i}{\sin^2\theta}\frac{\partial f}{\partial\phi} + \cot\theta\,\frac{\partial f}{\partial\theta} + \cot^2\theta\,\frac{\partial^2 f}{\partial\phi^2} - i\cot^2\theta\,\frac{\partial f}{\partial\phi}\right\}$$

The second and last terms may be joined together.

$$\frac{1}{\sin^2\theta} - \cot^2\theta = \frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} = 1$$

thus we finally conclude that

$$\boxed{L_+L_-f = -\hbar^2\left\{\frac{\partial^2 f}{\partial\theta^2} + \cot\theta\,\frac{\partial f}{\partial\theta} + \cot^2\theta\,\frac{\partial^2 f}{\partial\phi^2} + i\,\frac{\partial f}{\partial\phi}\right\}} \quad (16)$$

Of course, now we don't need the f any more:

$$\boxed{L_+ L_- = -\hbar^2 \left\{ \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \cot^2\theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} \right\}} \tag{17}$$

Now we can finally write L^2 [Eq (13)]:

$$L^2 = -\hbar^2 \frac{\partial^2}{\partial \phi^2} - \hbar \left(-i\hbar \frac{\partial}{\partial \phi}\right) - \hbar^2 \left\{ \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \cot^2\theta \frac{\partial^2}{\partial \phi^2} + i \frac{\partial}{\partial \phi} \right\}$$

$$L^2 = -\hbar^2 \left\{ \frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + (\cot^2\theta + 1) \frac{\partial^2}{\partial \phi^2} \right\}$$

The final magic touch is to write

$$\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} = \frac{1}{\sin\theta} \left[\sin\theta \frac{\partial^2}{\partial \theta^2} + \cos\theta \frac{\partial}{\partial \theta} \right]$$

$$= \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right)$$

Moreover $$1 + \cot^2\theta = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

Thus we finally arrive at

$$\boxed{L^2 = -\hbar^2 \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right\}} \tag{18}$$

Now we have a formula for L^2 so we may look for the eigenfunctions (we already know the eigenvalues!), thus, what we want is

$$\boxed{\begin{aligned} L^2 Y_{em}(\theta,\phi) &= \hbar^2 \ell(\ell+1)\, Y_{em}(\theta,\phi) \\ L_z Y_{em}(\theta,\phi) &= \hbar m\, Y_{em}(\theta,\phi) \end{aligned}} \qquad (19)$$

the eigenfunctions, which we call $Y_{em}(\theta,\phi)$ will be functions of θ and ϕ only, since L^2 and L_z only depend on θ and ϕ. We will soon learn that they are the spherical harmonics which you may have already seen in electrostatics.

But before we do that I want to point out something very interesting. the kinetic energy operator is

$$\frac{\vec{p}^{\,2}}{2m} = -\frac{\hbar^2}{2m}\nabla^2 \qquad (20)$$

and, in spherical coordinates the Laplacian is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\} \qquad (21)$$

Look! the second term is L^2.

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L^2}{r^2\hbar^2}} \qquad (22)$$

that is, L^2 is the <u>angular part of the Laplacian</u>

Random challenge: show that

$$\boxed{L^2 = (\vec{r} \times \vec{p})^2 = r^2 p^2 - (\vec{r} \cdot \vec{p})^2 + i\hbar\, \vec{r} \cdot \vec{p}} \tag{23}$$

The first two terms are classical. The third one is purely quantum mechanical

Spherical Harmonics

Our goal is now to solve (19). The 2^{nd} Eq. is quite easy: It reads

$$-i\hbar \frac{\partial}{\partial\phi} Y_{\ell m} = \hbar m Y_{\ell m}$$

or

$$\frac{\partial Y_{\ell m}}{\partial\phi} = i m Y_{\ell m}$$

The solution is

$$Y_{\ell m} = e^{im\phi} P_\ell^m(\theta) \qquad (24)$$

The integration constant $P_\ell^m(\theta)$, will be a function of θ.

The other Eq in (19) reads

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \qquad (25)$$

The part which depends on ϕ, we already know how to act:

$$\frac{\partial^2}{\partial\phi^2} Y_{\ell m} = -m^2 e^{im\phi} P_\ell^m(\theta)$$

Thus we arrive at

$$\boxed{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\right] P_\ell^m(\theta) = -\ell(\ell+1) P_\ell^m(\theta)} \qquad (26)$$

Next define

$$z = \cos\theta$$

then

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}$$

If f is some arbitrary function

$$\frac{\partial f}{\partial\theta} = \frac{\partial z}{\partial\theta}\frac{\partial f}{\partial z} = -\sin\theta\frac{\partial f}{\partial z}$$

$$\frac{\partial^2 f}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left[-\sin\theta\frac{\partial f}{\partial z}\right] = -\cos\theta\frac{\partial f}{\partial z} - \sin\theta\frac{\partial}{\partial\theta}\frac{\partial f}{\partial z}$$

$$= -\cos\theta\frac{\partial f}{\partial z} + \sin^2\theta\frac{\partial^2 f}{\partial z^2}$$

thus

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = -\cos\theta\frac{\partial f}{\partial z} + \sin^2\theta\frac{\partial^2 f}{\partial z^2}$$
$$-\cot\theta\sin\theta\frac{\partial f}{\partial z}$$

$$= (1-z^2)f'' - 2zf'$$

Hence Eq (26) becomes

$$\boxed{(1-z^2)P'' - 2zP' + \left[l(l+1) - \frac{m^2}{1-z^2}\right]P = 0} \qquad (27)$$

The solution will give us $P(z) = P(\cos\theta)$

Eq (27) is well known. It is called the associated Legendre equation and the solutions are called associated Legendre functions. What is interesting is that this Eq. only has solutions for ℓ being an integer

$$\ell = 0, 1, 2, 3, \ldots \qquad (28)$$

Thus, orbital angular momentum only comes in integers, never 1/2 integers.

The first few solutions are

$$P_0^0 = 1 \qquad P_2^0 = \frac{3z^2-1}{2}$$

$$P_1^0 = 3 \qquad P_2^1 = -3z\sqrt{1-z^2}$$

$$P_1^1 = -\sqrt{1-z^2} \qquad P_2^2 = 3\sqrt{1-z^2}$$

$$P_1^{-1} = \frac{\sqrt{1-z^2}}{2} \qquad P_2^{-1} = \frac{z}{2}\sqrt{1-z^2}$$

$$P_2^0 = \frac{3z^2-1}{2} \qquad P_2^{-2} = \frac{1-z^2}{8}$$

and etc. We will learn about a formula for generating them later. You can also compute them in Mathematica using LegendreP[ℓ, m, z].

The general solution to our eigenvalue problem is, therefore

$$\boxed{Y_{\ell m}(\theta,\phi) = C_{\ell m}\, e^{im\phi}\, P_{\ell m}(\cos\theta)} \qquad (29)$$

These functions are called spherical harmonics.

I also include in this solution a normalization constant $c_{\ell m}$ because the $P_{\ell m}$ are not normalized. In fact, the satisfy

$$\int_{-1}^{1} P_{\ell m}(z)\, P_{\ell' m'}(z)\, dz = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}\, \delta_{\ell\ell'}\, \delta_{mm'} \tag{30}$$

the functions $Y_{\ell m}$ are normalized in the unit sphere

$$\int |Y_{\ell m}|^2\, d\Omega = 1 \tag{31}$$

where

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

is an element of solid angle. Written explicitly

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, |Y_{\ell m}|^2 = 1$$

From (29) we get

$$|c_{\ell m}|^2\, 2\pi \int_0^{\pi} d\theta\, \sin\theta\, |P_{\ell m}(\cos\theta)|^2 = 1$$

Making $z = \cos\theta$ we get

$$|c_{\ell m}|^2\, 2\pi \underbrace{\int_{-1}^{1} dz\, |P_{\ell m}(z)|^2}_{\frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!}} = 1$$

or

Thus

$$c_{\ell m} = \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}} \qquad (32)$$

For historical reasons some people multiply $c_{\ell m}$ by $(-1)^m$. This in irrelevant: wavefunctions are only defined up to a complex phase.

The first few spherical harmonics are

$$Y_0^0 = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$$

$$Y_1^1 = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{i\phi}\sin\theta$$

$$Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{-i\phi}\sin\theta$$

Reason why people multiply $c_{\ell m}$ by $(-1)^m$

$$Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1)$$

$$Y_2^{\pm 1} = \mp\frac{1}{2}\sqrt{\frac{15}{2\pi}}\, e^{\pm i\phi}\sin\theta\cos\theta$$

$$Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, e^{\pm 2i\phi}\sin^2\theta$$

etc.

Completeness of the spherical harmonics

We may define an angular ket $|\theta,\phi\rangle$ which means "particle is at orientation (θ,ϕ)". They satisfy

$$\int |\theta,\phi\rangle\langle\theta,\phi|\, d\Omega = 1 \qquad (33)$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$. The spherical harmonics are then, by definition

$$\boxed{Y_\ell^m(\theta,\phi) = \langle\theta,\phi|\ell,m\rangle} \qquad (34)$$

Since $|\ell,m\rangle$ forms an orthonormal basis, so will the Y_ℓ^m:

$$\delta_{\ell\ell'}\delta_{mm'} = \langle\ell,m|\ell',m'\rangle = \int d\Omega\, \langle\ell,m|\theta,\phi\rangle\langle\theta,\phi|\ell',m'\rangle$$

$$\boxed{\therefore \int [Y_\ell^m(\theta,\phi)]^* Y_{\ell'}^{m'}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}} \qquad (35)$$

The spherical harmonics form a basis for the space of functions defined in the unit sphere.

Similarly we may look at completeness. The orthogonality relation of the $|\theta,\phi\rangle$ is a bit different since in (33) there is a term $\sin\theta$ coming from $d\Omega$. It reads

$$\langle\theta,\phi|\theta',\phi'\rangle = \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta} \qquad (36)$$

Now:

$$\sum_{l,m} |l,m\rangle\langle l,m| = \mathbb{1} \tag{37}$$

Thus

$$\boxed{\sum_{l,m} Y_l^m(\theta,\phi)\, Y_l^{m}(\theta',\phi') = \frac{\delta(\theta-\theta')\,\delta(\phi-\phi')}{\sin\theta}} \tag{38}$$

Alternative derivation of the Spherical Harmonics

It is also possible to arrive at the spherical harmonics using only the algebraic properties of the angular momentum operators.

The starting point is to note that

$$L_+ \, Y_\ell^\ell = 0 \qquad (39)$$

That is, L_+ annihilates the highest state in the ladder ($m = \ell$)

Using (14) we get

$$\left(i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi} \right) Y_\ell^\ell = 0$$

But

$$\frac{\partial Y_\ell^\ell}{\partial\phi} = i\ell \, Y_\ell^\ell$$

so that we get

$$\frac{\partial}{\partial\theta} P_\ell^m = \ell \cot\theta \, P_\ell^m \qquad (40)$$

Integrating:

$$\log P_\ell^m = \ell \log(\sin\theta) + \text{const}$$

or

$$P_\ell^m = c \sin^\ell\theta \qquad (41)$$

this determines the Y_l^l :

$$Y_l^l = c_{ll}\, e^{il\phi} \sin^l\theta \tag{42}$$

the constant c_{ll} is determined from normalization

$$|c_{ll}|^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta (\sin\theta)^{2l} = 1$$

using that
$$\int_0^{\pi} d\theta (\sin\theta)^{2l+1} = \frac{2\,(2^l l!)^2}{(2l+1)!}$$

we get
$$c_{ll} = \frac{1}{2^l l!}\sqrt{\frac{2l+1}{4\pi}} \tag{43}$$

Now that we have found Y_ℓ^ℓ, we may obtain all other spherical harmonics by repeatedly applying L_-. We have already shown in a previous lecture note that

$$Y_\ell^m = \sqrt{\frac{(\ell+m)!}{(2\ell)!(\ell-m)!}} \; \frac{(L_-)^{\ell-m}}{\hbar^{\ell-m}} Y_\ell^\ell \qquad (44)$$

combining this constant with $c_{\ell\ell}$ we find that

$$c_{\ell\ell}\sqrt{\frac{(\ell+m)!}{(2\ell)!(\ell-m)!}} = \frac{1}{2^\ell \ell!}\sqrt{\frac{2\ell+1}{4\pi}\;\frac{(\ell+m)!}{(2\ell)!\,(\ell-m)!}}$$

this may be compared with $c_{\ell m}$ in Eq. (32). We may

$$c_{\ell\ell}\sqrt{\frac{(\ell+m)!}{(2\ell)!(\ell-m)!}} = \frac{c_{\ell m}}{2^\ell \ell!\sqrt{(2\ell)!}} \qquad (45)$$

Now we need to analyze the effect of $(L_-)^{\ell-m}$ in Y_ℓ^ℓ.

Let $f(\theta)$ be an arbitrary function. then we have

$$(L_-)\, e^{ik\phi} f(\theta) = -i\hbar\, e^{-i\phi}\left\{-i\frac{\partial}{\partial\theta} - \cot\theta\,\frac{\partial}{\partial\phi}\right\} e^{ik\phi} f(\theta)$$

$$= -i\hbar\, e^{-i\phi}\left\{-i\, e^{ik\phi} f'(\theta) - ik\, e^{ik\phi}\cot\theta\, f(\theta)\right\}$$

$$= -\hbar\, e^{i(k-1)\phi}\left[f'(\theta) + k\cot\theta\, f(\theta)\right]$$

he trick is now to write

$$f'(\theta) + k \cot\theta \, f(\theta) = \frac{1}{\sin^k\theta} \frac{d}{d\theta}\left[\sin^k\theta \, f(\theta)\right] \qquad (46)$$

please check this !). Then

$$\frac{L_-}{\hbar} e^{ik\phi} f(\theta) = - \frac{e^{i(k-1)\phi}}{\sin^k\theta} \frac{d}{d\theta}\left[\sin^k\theta \, f(\theta)\right]$$

we can also do another trick and write

$$\frac{d}{d\theta} = \frac{d(\cos\theta)}{d\theta} \frac{d}{d(\cos\theta)} = -\sin\theta \frac{d}{d(\cos\theta)}$$

Then

$$\frac{L_-}{\hbar} e^{ik\phi} f(\theta) = \frac{e^{i(k-1)\phi}}{(\sin\theta)^{k-1}} \frac{d}{d(\cos\theta)}\left[\sin^k\theta \, f(\theta)\right] \qquad (47)$$

what is cool is that this expression now has the same form $e^{i\lambda\phi} g(\theta)$ where $\lambda = k-1$ and

$$g(\theta) = \frac{1}{(\sin\theta)^{k-1}} \frac{d}{d\theta}\left[\sin^k\theta \, f(\theta)\right]$$

Thus, if we apply L_- again we obtain

$$\frac{(L_-)^2}{\hbar^2} e^{ik\phi} f(\theta) = \frac{(L_-)}{\hbar} e^{i\lambda\phi} g(\theta)$$

$$= \frac{e^{i(\lambda-1)\phi}}{(\sin\theta)^{\lambda-1}} \frac{d}{d(\cos\theta)}\left[\sin^\lambda\theta \, g(\theta)\right]$$

Or

$$\frac{(L_-)^2}{\hbar^2} e^{ik\phi} f(\theta) = \frac{e^{i(k-2)\phi}}{(\sin\theta)^{k-2}} \frac{d}{d(\cos\theta)} \left[(\sin\theta)^{k-1} g(\theta) \right]$$

But

$$(\sin\theta)^{k-1} g(\theta) = \frac{(\sin\theta)^{k-1}}{(\sin\theta)^{k-1}} \frac{d}{d(\cos\theta)} \left[\sin^k\theta \, f(\theta) \right]$$

so that

$$\frac{(L_-)^2}{\hbar^2} e^{ik\phi} f(\theta) = \frac{e^{i(k-2)\phi}}{(\sin\theta)^{k-2}} \frac{d^2}{d(\cos\theta)^2} \left[\sin^k\theta \, f(\theta) \right] \quad (4$$

this concludes the problem. If we apply L_- m times we obtain

$$\frac{(L_-)^m}{\hbar^m} e^{ik\phi} f(\theta) = \frac{e^{i(k-m)\phi}}{(\sin\theta)^{k-m}} \frac{d^m}{d(\cos\theta)^m} \left[\sin^k\theta \, f(\theta) \right] \quad (4$$

In our case [Eq (44)] we get

$$\frac{(L_-)^{\ell-m}}{\hbar^{\ell-m}} \left[e^{i\ell\phi} \sin^\ell\theta \right] = \frac{e^{im\phi}}{(\sin\theta)^m} \frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}} \left[\sin^{2\ell}\theta \right]$$

We thus define

$$\boxed{P_\ell^m(z) = \frac{1}{2^\ell \ell! \sqrt{(2\ell)!}} \frac{1}{(1-z^2)^{\frac{m}{2}}} \frac{d^{\ell-m}}{dz^{\ell-m}} \left[(1-z^2)^\ell\right]} \tag{50}$$

hen we recover our solution (29). Thus, this method provided us to a explicit formula for $P_\ell^m(z)$.