Spontaneous Symmetry Breaking

Part 2

- Ginzburg - Landau Theory
- Correlation functions
- Higgs and Goldstone modes
- The Anderson - Higgs mechanism.

Ginzburg - Landau theory

The idea of the Landau theory was to describe a phase transition by means of a free energy $f(m)$ which depends on the magnetization m and is such that its minima correspond to the equilibrium configurations.

Ginzburg - Landau is the extension of this theory to include also the possibility that m depends on the position $\mathbf{r}$:

$$m \rightarrow m(\mathbf{r}) \tag{1}$$

The idea is to keep this at a phenomenological level. we want to have an approximate expression for the Landau free energy which, in addition to the effects of the FM-PM transition, also contemplates the effects of inhomogeneities in $m(\mathbf{r})$.

We can make some progress by imposing some symmetries (and by being reasonable!). Intuitively, we want a term which increases the energy when the magnetization is inhomogeneous. this means this term must depend on derivatives of $m(\mathbf{r})$.

But we assume that space is isotropic, so there can be no preferred orientation. this means that our term can only depend on quantities like ∇m, since this quantity is invariant under rotations.

We also want the free energy to preserve up-down symmetry. thus, it cannot depend on ∇m alone, but must depend on something like $(\nabla m)^2$.

thus, a proposal for the Ginzburg-Landau free energy could be

$$\boxed{F = \int d^3 r \left\{ \gamma (\nabla m)^2 + \frac{a}{2} m^2 + \frac{b}{4} m^4 - hm \right\}} \qquad (2)$$

where $\gamma > 0$ is a constant.

It is fun to try to think about which other terms could appear. For instance we could have $(\nabla m)^4$. But if we are interested in somewhat smooth configurations, this will be much smaller that $(\nabla m)^2$

Or we could have something like $m(\nabla m)$.

But

$$\int d^3 r \; m (\nabla m) = \frac{1}{2} \int d^3 r \; \nabla (m^2) = \text{surface term.}$$

If we keep going like this we will eventually conclude that Eq (2) is actually a pretty good free energy.

The minimum of F is very easy to find. The term $\chi (\nabla m)^2$ only increases the energy, so to make it as small as possible we set $m(\mathbf{r}) = m$, independent of $\mathbf{r}$. Then we are back to the original Landau problem. In particular, if $h=0$, the configuration which minimizes F will be

$$m^* = \begin{cases} 0 & a > 0 \\ \sqrt{-a/b} & a < 0 \end{cases} \tag{3}$$

The interesting aspect of the Ginzburg-Landau theory is that it allows us to look at fluctuations around this minimum. To do that we expand m as

$$m = m^* + \phi(\mathbf{r}) \tag{4}$$

where ϕ is assumed to be small.

We then have

$$\nabla m = \nabla \phi$$

$$m^2 = (m^*)^2 + 2m^* \phi + \phi^2$$

$$m^4 = (m^*)^4 + 4(m^*)^3 \phi + 6(m^*)^2 \phi^2 + O(\phi)^3$$

where we neglect terms higher than quadratic in ϕ.

Let us suppose that $a<0$. then we get

$$\frac{a}{2} m^2 + \frac{b}{4} m^4 \simeq \frac{a}{2}\left(\frac{-a}{b}\right) + \frac{b}{4}\left(\frac{-a}{b}\right)^2$$

cancels out!

$$+ \left[\frac{a}{2} 2\sqrt{\frac{-a}{b}} + \frac{b}{4} 4 \left(\frac{-a}{b}\right)^{3/2}\right] \phi$$

$$+ \left[\frac{a}{2} + \frac{b}{4} 6 \left(\frac{-a}{b}\right)\right] \phi^2$$

or

$$\frac{a}{2} m^2 + \frac{b}{4} m^4 \simeq -\frac{a^2}{4b} - a\phi^2 \qquad (5)$$

Thus the free energy (2) is approximated as (I'm assuming $h=0$)

$$\boxed{F \simeq F_0 + \int d^3r \left\{ g (\nabla\phi)^2 + |a| \phi^2 \right\}} \qquad (6)$$

which is quadratic (and not quartic) in ϕ. It thus resembles a free system

Now let us move to Fourier space: define

$$\boxed{\phi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \phi_{\mathbf{k}}} \qquad (7)$$

then

$$\int d^3\mathbf{r}\ \phi(\mathbf{r})^2 = \int d^3\mathbf{r}\ \frac{1}{V} \sum_{\mathbf{k}\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} \phi_{\mathbf{k}} \phi_{\mathbf{q}}$$

$$= \sum_{\mathbf{k},\mathbf{q}} \phi_{\mathbf{k}} \phi_{\mathbf{q}} \underbrace{\left[\frac{1}{V} \int d^3\mathbf{r}\ e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} \right]}_{\delta_{\mathbf{k},-\mathbf{q}}}$$

$$= \sum_{\mathbf{k}} \phi_{\mathbf{k}} \phi_{-\mathbf{k}}$$

and

$$\nabla\phi = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} (i\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} \phi_{\mathbf{k}}$$

so

$$\int d^3\mathbf{r}\ (\nabla\phi)^2 = \frac{1}{V} \int d^3\mathbf{r} \sum_{\mathbf{k},\mathbf{q}} (i\mathbf{k})(i\mathbf{q})\, e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}} \phi_{\mathbf{k}} \phi_{\mathbf{q}}$$

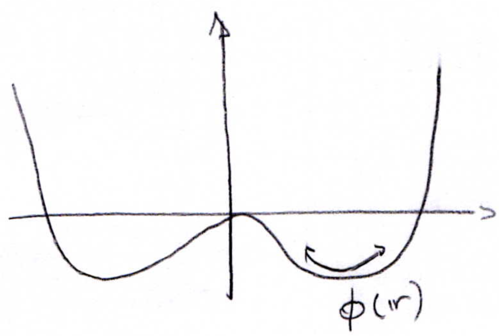
$$= \sum_{\mathbf{k}} (i\mathbf{k})(-i\mathbf{k})\, \phi_{\mathbf{k}} \phi_{-\mathbf{k}}$$

Thus we get

$$\boxed{F = \sum_{k} \left\{ \gamma k^2 + |a| \right\} \phi_k \phi_{-k}} \qquad (8)$$

This is a massive dispersion relation, where $|a|$ plays the role of the mass squared. Thus, magnetic excitations in this model are massive.

Just to clarify, we are talking here about excitations around the minimum of the free energy



Correlation functions

The correlation function is defined as

$$G(\mathbb{r}-\mathbb{r}') = \langle m(\mathbb{r})\, m(\mathbb{r}')\rangle - \langle m(\mathbb{r})\rangle\langle m(\mathbb{r}')\rangle \qquad (8)$$

Using (4) we get, since $\langle \phi \rangle = 0$

$$G(\mathbb{r}-\mathbb{r}') = \langle \phi(\mathbb{r})\,\phi(\mathbb{r}')\rangle \qquad (9)$$

But what are we averaging over anyway? I mean, what is the meaning of $\langle\,\rangle$?

The logic is that there can be many configurations $m(\mathbb{r})$ representing the different possible fluctuations above m^*. And we average over these different configurations.

But what are the probabilities? Well, motivated by the Gibbs formula $p = e^{-\beta E}$, we propose that the probabilities should be

$$P[m(\mathbb{r})] \propto e^{-\beta F[m]} \qquad (10)$$

It is easier to work in Fourier space using (7).

we then get

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{V} \sum_{\mathbf{k}\,\mathbf{k}'} e^{i(\mathbf{k}\cdot\mathbf{r} + \mathbf{k}'\cdot\mathbf{r}')} \langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \rangle \qquad (11)$$

The distribution of the ϕ_k is given by the F in Eq (8).

this distribution is quadratic in the ϕ_k. thus, we are talking here about a <u>Gaussian distribution</u>.

the only correlated variables are $\phi_{\mathbf{k}}$ and $\phi_{-\mathbf{k}}$, so

$\langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \rangle \propto \delta_{\mathbf{k}', -\mathbf{k}}$. Moreover, a Gaussian integral gives
(this comes from the theory of multi-dimensional Gaussians)

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle = \frac{1}{\gamma k^2 + |a|}} \qquad (12)$$

Plugging this back into (11) we then get

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle$$

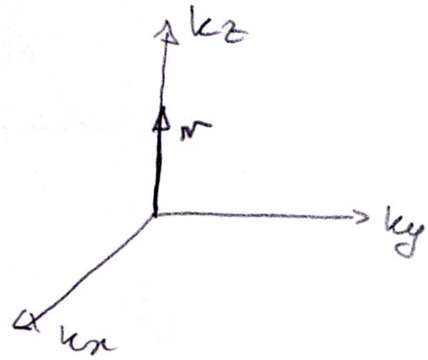
or, more simply

$$\boxed{G(\mathbf{r}) = \frac{1}{V} \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\gamma k^2 + |a|}} \qquad (13)$$

Let's compute this sum in $d=3$, by converting it to an integral. We get

$$G(\mathbf{r}) = \frac{1}{V}\frac{V}{(2\pi)^3}\int d^3\mathbf{k} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\gamma k^2 + |a|}$$

Define the k axes in such a way that k_z is parallel to $\mathbf{r}$. Then $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$ and we get



$$G(\mathbf{r}) = \frac{1}{(2\pi)^3}\int dk\, k^2 \frac{1}{\gamma k^2+|a|}\int_0^{2\pi} d\phi \int d(\cos\theta)\, e^{ikr\cos\theta}$$

$$= \frac{1}{(2\pi)^3}\, 2\pi \int dk \frac{k^2}{\gamma k^2+|a|}\,\frac{e^{ikr}-e^{-ikr}}{ikr}$$

$$= \frac{1}{2\pi^2 r}\int dk \frac{k\sin(kr)}{\gamma k^2+|a|}$$

To compute this integral we look instead at

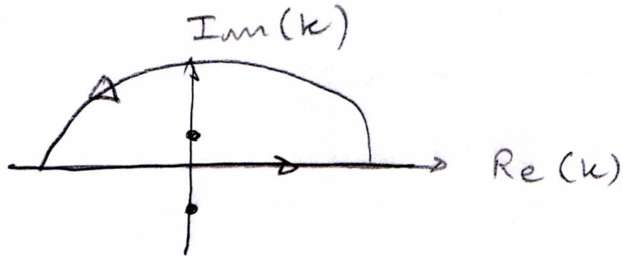
$$I = \int dk \frac{k\, e^{ikr}}{\gamma k^2+|a|}$$

Then in the end we take the imaginary part

This integral can be done using residues. From now on I will set $\chi = 1$. We write

$$k^2 + |a| = (k + i\sqrt{|a|})(k - i\sqrt{|a|})$$

this will have poles at $k = \pm i\sqrt{|a|}$ so we choose the path as



then we get

$$I = 2\pi i \frac{\sqrt{|a|}}{2\sqrt{|a|}} e^{-\sqrt{|a|}\, r} = \pi i\, e^{-r\sqrt{|a|}}$$

thus, taking the imaginary part we get

$$\boxed{G(r) = \frac{1}{4\pi r} e^{-r/\xi}} \qquad (14)$$

where

$$\boxed{\xi = \frac{1}{\sqrt{|a|}} \sim \frac{1}{|T - T_c|^{1/2}}} \qquad (15)$$

this is called the correlation length

We thus find that the correlation of $m(\mathbf{r})$ between two points a distance $\mathbf{r}$ apart decays exponentially as $e^{-r/\xi}$, with a typical correlation length ξ.

Moreover, we see that $\xi \to \infty$ as $T \to T_c$. Thus, at the critical point spins infinitely far apart are still correlated. This is the hallmark of criticality.

Let me summarize some results which appear often. Whenever the Ginzburg Landau free energy can be written as (8), we see from (15) that $|a| = 1/\xi^2$. Thus we may write a general quadratic free energy as

$$\boxed{\begin{aligned} F &= \int d^3\mathbf{r} \left\{ (\nabla\phi)^2 + |a|\,\phi^2 \right\} \\ &= \sum_{\mathbf{k}} \left(\mathbf{k}^2 + |a| \right) \phi_{\mathbf{k}}\, \phi_{-\mathbf{k}} \end{aligned}} \qquad (16)$$

and we see that

$$\boxed{\mu = \text{mass} = \sqrt{|a|} = \frac{1}{\xi}} \qquad (17)$$

Thus,

$$\boxed{\text{correlation length} = \frac{1}{\text{mass}}} \qquad (18)$$

At a phase transition the mass tends to zero and the correlation length diverges

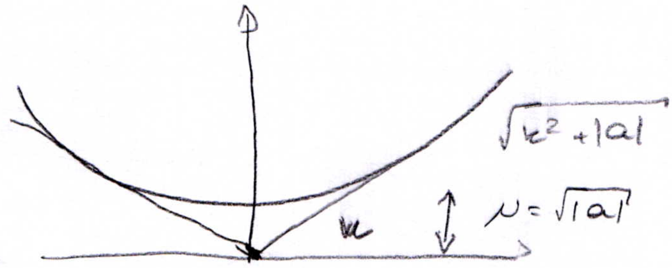
$$T = T_c \ : \quad \begin{cases} \text{mass} = \mu = 0 \\ \text{corr. length} = \xi = \infty \end{cases}$$

Another thing that is useful to remember is that, if F is quadratic like in (16), then

$$\boxed{\langle \phi_k \, \phi_{-k} \rangle = \frac{1}{k^2 + |a|}} \tag{19}$$

This is true in any dimension. The real space result $G(r)$, that we found in (13), on the other hand, holds only for $d=3$.

Oh, I forgot to say: we use the jargon "at a phase transition the gap closes and the excitations become massless." The relation between a gap closing and the mass is as follows



The gap is the mass

The Ginzburg criterion

The correlation function (14) was computed in $d=3$. It is also possible to find an approximate formula for arbitrary dimensions. It reads

$$G(r) \sim \frac{e^{-r/\xi}\,\xi^{(3-d)/2}}{r^{(d-1)/2}} \qquad (20)$$

where, again, $\xi = \frac{1}{\sqrt{|a|}}$.

The correlation function gives a measure of the fluctuations of the order parameter, as can be seen by the definitions (8) and (9).

The Ginzburg criterion is based on comparing the fluctuations with the average magnetization m'. For concreteness, we look at $G(r)$ evaluated at $r = \xi$. We then get

$$G(\xi) \sim \xi^{2-d} \sim |a|^{\frac{d-2}{2}} \qquad (21)$$

On the other hand

$$(m')^2 \sim |a| \qquad (22)$$

Thus

$$\frac{G(\mathcal{E})}{(m^*)^2} \sim |a|^{(d-4)/4} \tag{23}$$

As we approach the critical value, $|a| \to 0$, we get

$$\frac{G(\mathcal{E})}{(m^*)^2} \sim \begin{cases} 0 & d > 4 \\ \infty & d < 4 \end{cases} \tag{24}$$

Thus, we see that if $d \geqslant 4$ the relative fluctuations remain finite, but if $d < 4$ it diverges.

To understand the meaning of all this, recall that we started out with the mean-field approximation, which consisted in ignoring fluctuations. Now we see that this approximation is only reasonable at $d \geqslant 4$. Below this the fluctuations dominate.

This value $d = 4$ is called the upper critical dimension. It is the dimension above which the mean-field approximation becomes exact. The value $d = 4$ is specific for the Ising model. Each universality class has its own upper critical dimension

Higgs and Goldstone modes

Now let us consider a physical system whose order parameter $\psi(\mathbf{r})$ is complex. We have already seen that this is the case in a BEC. It is also the case in a superfluid or a superconductor, as we will learn later on. Another possibility is a magnetic system with an XY interaction

$$H = -J \sum_{\langle i,j \rangle} \left(S_i^x S_j^x + S_i^y S_j^y \right) \qquad (25)$$

This is a 2D analog of the Heisenberg interaction. In this system the order parameter will be a 2D magnetization vector $\vec{m}(\mathbf{r}) = (m_x(\mathbf{r}), m_y(\mathbf{r}))$. But since it has 2 components, we may also represent it by a complex number $\psi(\mathbf{r})$. So we are back to the previous case.

What is common to all these systems is that they are invariant under U(1) symmetries

$$\psi(\mathbf{r}) \rightarrow e^{i\theta} \psi(\mathbf{r}) \qquad (26)$$

This is also called a global gauge transformation. For the XY model, it represents a rotation by an angle θ

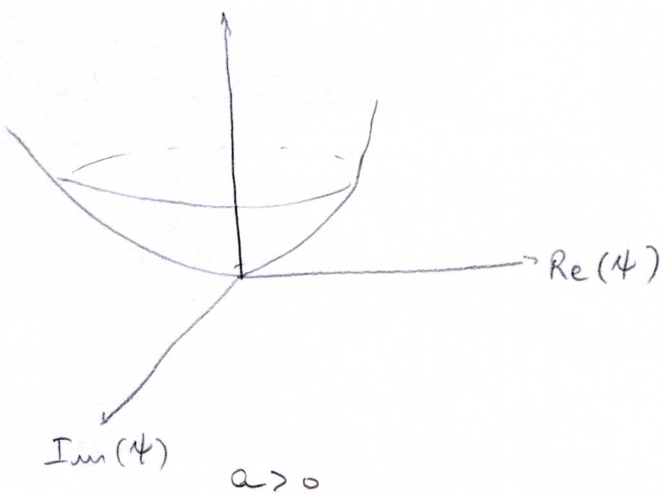
The natural generalization of the Ginzburg Landau free energy (2) for a complex order parameter is

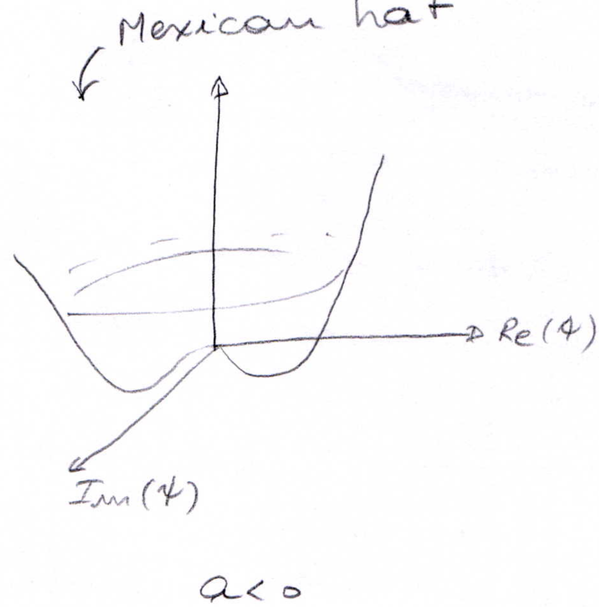
$$\boxed{F = \int d^3 r \left\{ |\nabla \psi|^2 + \frac{a}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \right\}} \tag{27}$$

this may at first look similar to what we had before. But now there is one fundamental difference: since ψ is complex, the potential

$$V(\psi) = \frac{a}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \tag{28}$$

will have an additional degree of freedom. Thus it will look like this





In magnitude, the minimum of $V(\psi)$ continues to occur at

$$|\psi_0| = \begin{cases} 0 & a > 0 \\ \sqrt{-\frac{a}{b}} & a < 0 \end{cases} \qquad (29)$$

However, the phase is arbitrary. Any configuration $\psi_0 = |\psi_0| \, e^{i\theta_0}$ will lie at the bottom of the mexican hat and thus will be a valid minimum.

As before, let us now consider small fluctuations around the minimum. That is, we write

$$\psi(\mathbf{r}) = [\psi_0 + \rho(\mathbf{r})] \, e^{i(\theta_0 + \theta(\mathbf{r}))}$$

where we assume $\rho(\mathbf{r})$ and $\theta(\mathbf{r})$ will be small. Since the phase is arbitrary, we choose $\theta_0 = 0$. We now get

$$\nabla\psi = (\nabla\rho) \, e^{i\theta(\mathbf{r})} + (\psi_0 + \rho)(i\nabla\theta) \, e^{i\theta}$$

$$\simeq \left[(\nabla\rho) + \psi_0 (i\nabla\theta)\right] e^{i\theta}$$

thus

$$|\nabla\psi|^2 = (\nabla\rho)^2 + \psi_0^2 (\nabla\theta)^2$$

$$= (\nabla\rho)^2 + \frac{|a|}{b} (\nabla\theta)^2$$

Moreover

$$|\psi|^2 = \psi_0^2 + 2\psi_0\rho + \rho^2$$

$$|\psi|^4 = \psi_0^4 + 4\psi_0^3\rho + 6\psi_0^2\rho^2 + O(\rho)^3$$

Thus, exactly like in Eq (5), we find

$$\frac{a}{2}|\psi|^2 + \frac{b}{4}|\psi|^4 = -\frac{a^2}{4b} - a\rho^2$$

Thus, neglecting constants, the free energy (27) becomes

$$\boxed{F = \int d^3r \left\{ \left[(\nabla\rho)^2 + |a|\rho^2 \right] + \frac{|a|}{b}(\nabla\theta)^2 \right\}} \qquad (30)$$

The contribution $\rho(\mathbf{r})$ describes the radial fluctuations in the mexican hat. It is called the Higgs mode. Conversely, θ describes the radial fluctuations which loop around the mexican hat. It is called the Goldstone mode. When the order parameter was m, we had no Goldstone mode. Just a Higgs mode.

The Goldstone mode appears whenever we have a continuous symmetry. This is the content of Goldstone's theorem.

If we move to Fourier space,

$$\rho(\mathbb{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbb{k}} e^{i\mathbb{k}\cdot\mathbb{r}} \rho_{\mathbb{k}} \tag{31}$$

$$\theta(\mathbb{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbb{k}} e^{i\mathbb{k}\cdot\mathbb{r}} \theta_{\mathbb{k}}$$

we get, as before

$$\boxed{F = \sum_{\mathbb{k}} \left\{ (k^2 + |a|) |\rho_{\mathbb{k}}|^2 + k^2 |\theta_k|^2 \right\}} \tag{32}$$

Thus, we see that the Higgs mode is massif, as before, with mass $\mu = \sqrt{|a|}$. The Goldstone mode, on the other hand, is massless. Thus, exciting a Goldstone mode is very easy (since it can just slide around the mexican hat potential).

Recall now the result (19). We can apply it to the Goldstone mode, but without the mass term. Thus

$$\langle |\theta_{\mathbb{k}}|^2 \rangle = \frac{1}{k^2} \tag{33}$$

Now we can go back to real space and compute

$$\langle \theta(\mathbb{r})\, \theta(\mathbb{r}') \rangle = \frac{1}{V} \sum_{\mathbb{k} \mathbb{q}} e^{i(\mathbb{k}\cdot\mathbb{r} + \mathbb{q}\cdot\mathbb{r}')} \underbrace{\langle \theta_{\mathbb{k}}\, \theta_{\mathbb{q}} \rangle}_{\frac{1}{k^2} \delta_{\mathbb{q}, -\mathbb{k}}}$$

Thus

$$\langle \Theta(\mathbb{r})\, \Theta(\mathbb{r}') \rangle = \frac{1}{V} \sum_{\mathbb{k}} \frac{e^{i\mathbb{k}\cdot(\mathbb{r}-\mathbb{r}')}}{k^2}$$

Let us convert this to an integral, but now assuming an arbitrary dimension d. Then

$$\langle \Theta(\mathbb{r})\, \Theta(\mathbb{r}') \rangle = \frac{1}{(2\pi)^d} \int d^d\mathbb{k}\, \frac{e^{i\mathbb{k}\cdot(\mathbb{r}-\mathbb{r}')}}{k^2}$$

This integral is the Coulomb potential due to a charge at the origin. Thus, borrowing results from electromagnetism textbooks, we get

$$\langle \Theta(\mathbb{r})\, \Theta(\mathbb{r}') \rangle = - \frac{|\mathbb{r}-\mathbb{r}'|^{2-d}}{(2-d)\, S_d} \qquad (34)$$

where $S_d = \dfrac{2\pi^{d/2}}{(\frac{d}{2}-1)!}$.

We see that the behavior of this correlation function changes dramatically at $d=2$. To understand how this affects the system, consider the correlation function

$$\langle \psi(\mathbb{r})\, \psi^*(0) \rangle$$

Assume $\rho = 0$ so $\psi = \psi_0 e^{i\theta(\mathbf{r})}$ and

$$\langle \psi(\mathbf{r})\psi(0)\rangle = |\psi_0|^2 \langle e^{i[\theta(\mathbf{r}) - \theta(0)]}\rangle \tag{35}$$

Now I will borrow a result from Gaussian variables. Sorry about that! The result is

$$\langle e^{\alpha\theta}\rangle = e^{\frac{\alpha^2}{2}\langle\theta^2\rangle} \tag{36}$$

Thus

$$\langle \psi(\mathbf{r})\psi(0)\rangle = |\psi_0|^2 \exp\left\{-\frac{1}{2}\langle[\theta(\mathbf{r}) - \theta(0)]^2\rangle\right\}$$

$$= |\psi_0|^2 \exp\left\{-\frac{1}{2}\left[\langle\theta(\mathbf{r})^2\rangle + \langle\theta(0)^2\rangle + -2\langle\theta(\mathbf{r})\theta(0)\rangle\right]\right\}$$

Using Eq (34) we therefore get

$$\boxed{\langle \psi(\mathbf{r})\psi(0)\rangle = |\psi_0|^2 \exp\left\{-\frac{r^{2-d}}{(2-d)S_d}\right\}} \tag{37}$$

Now recall that the system will have long range order when the correlation function remains finite as $r \to \infty$. Here we see that

$$\boxed{\lim_{r \to \infty} \langle \psi(r) \psi^*(0) \rangle = \begin{cases} 0 & d \leqslant 2 \\ |\psi_0|^2 & d > 2 \end{cases}} \qquad (38)$$

Thus, for $d \leqslant 2$ the Goldstone mode completely destroys the order in the system. This is the <u>Mermin - Wagner theorem</u>. Since the Goldstone mode appears due to the existence of a continuous symmetry, we conclude that

> "A system with a continuous symmetry can have no long-range order for $d \leqslant 2$"

Hence, for instance, a <u>Heisenberg model will have no magnetic order at $d \leqslant 2$</u>. This is what we found when we discussed magnons.

However, it was found that the system may nonetheless undergo a type of transition, known as the <u>Kosterlitz Touless transition</u>.

The Anderson - Higgs mechanism

Now suppose we have a system with a complex order parameter ψ, but such that the system is electrically charged. This means that the system will also couple to the electromagnetic field. This is approximately what happens in a superconductor:

$$\boxed{\text{superconductor} \simeq \text{charged superfluid}}$$

As before, we now want to modify the Ginzburg - Landau equations to include the effect of the EM field. This is done through something called minimal coupling. That is, the simplest coupling to preserve gauge invariance. It consists in replacing

$$\nabla \to \nabla - \frac{2ie}{\hbar} \mathbb{A} \tag{39}$$

where $\mathbb{A}$ is the vector potential. This is the same coupling that appears in quantum mechanics when we replace the momentum p by $p - 2eA$. The factor of 2 is because we know the charges in a superconductor are Cooper pairs, which have charge $2e$

Our free energy then becomes

$$\boxed{F = \int d^3 r \left\{ \left| \nabla \psi - \frac{2ie}{\hbar} \mathbb{A} \psi \right|^2 + \frac{a}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \right\}} \tag{40}$$

To be more general, we should also introduce the electromagnetic energy

$$\int d^3 r \, \frac{\mathbb{B}^2}{2\mu} \tag{41}$$

where I will assume there is only a magnetic field and no $\mathbb{E}$ field. Thus, the complete free energy will be

$$\boxed{F = \int d^3 r \left\{ \frac{\mathbb{B}^2}{2\mu} + \left| \nabla \psi - \frac{2ie}{\hbar} \mathbb{A} \psi \right|^2 + \frac{a}{2} |\psi|^2 + \frac{b}{4} |\psi|^4 \right\}} \tag{42}$$

this Ginzburg-Landau energy is invariant under the <u>local gauge transformation</u>

$$\psi(\mathbb{r}) \rightarrow \psi(\mathbb{r})\, e^{i\Lambda(\mathbb{r})}$$

$$\mathbb{A}(\mathbb{r}) \rightarrow \mathbb{A}(\mathbb{r}) + \frac{\hbar}{2e}\nabla\Lambda(\mathbb{r}) \tag{43}$$

where $\Lambda(\mathbb{r})$ is an arbitrary function

Now let us consider again small fluctuations

$$\psi(\mathbb{r}) = [\psi_0 + \rho(\mathbb{r})]\, e^{i\theta(\mathbb{r})} \tag{44}$$

The potential energy remains as before

$$\frac{a}{2}|\psi|^2 + \frac{b}{4}|\psi|^4 = -\frac{a^2}{4b} + |a|\rho^2 \tag{45}$$

As for the kinetic part, using the result in the end of page 17,

$$\nabla\psi = [(\nabla\rho) + \psi_0(i\nabla\theta)]\, e^{i\theta}$$

so

$$\nabla\psi - \frac{2ie}{\hbar c}\mathbb{A}\psi = \left\{\nabla\rho + \psi_0(i\nabla\theta) - \frac{2ie}{\hbar c}\mathbb{A}(\psi_0+\rho)\right\} e^{i\theta}$$

$$\simeq \left\{\nabla\rho + i\psi_0\left[\nabla\theta - \frac{2e}{\hbar c}\mathbb{A}\right]\right\} e^{i\theta}$$

Thus

$$|\nabla\psi - \frac{2ie}{\hbar}\mathbb{A}\psi|^2 \simeq (\nabla\rho)^2 + \psi_0^2\left[\nabla\theta - \frac{2e}{\hbar}\mathbb{A}\right]^2$$

Eq (42) then becomes

$$\boxed{F \simeq \int d^3\mathbb{r}\left\{\frac{B^2}{2\mu} + \left[(\nabla\rho)^2 + |a|\rho^2\right] + \frac{|a|}{b}\left(\nabla\theta - \frac{2e}{\hbar}\mathbb{A}\right)^2\right\}} \quad (46)$$

But now comes something quite remarkable. In fact, so remarkable that it is behind two Nobel prizes, one for Anderson and one for Higgs. We can make a Gauge transformation

$$\mathbb{A} \to \mathbb{A} + \frac{\hbar}{2e}\nabla\theta$$

then the last term becomes

$$\boxed{F = \int d^3\mathbb{r}\left\{\frac{B^2}{8\pi} + \left[(\nabla\rho) + |a|\rho^2\right] + \frac{|a|}{b}\left(\frac{2e}{\hbar}\right)^2\mathbb{A}^2\right\}} \quad (47)$$

The Goldstone mode has disapeared. And what we find is a term proportional to $\mathbb{A}^2$. In analogy with the term $|a|\rho^2$, which we interpret as a mass term, this term proportional to $\mathbb{A}^2$ is a <u>mass term for photons</u>. This is the Anderson mechanism: inside a superconductor "the photons become massive"

Higgs used this idea to describe how gluons and the W and Z Boson can acquire mass: they couple to a Higgs field (which is complex, like ours) whose symmetry is spontaneously broken.

Returning to (46), we can find the state which minimizes F. For ρ it is easy. Set $\rho = 0$. For θ and $\mathbb{A}$, on the other hand, we set

$$\mathbb{A} = \frac{\hbar c}{2e} \nabla \theta$$

But then

$$B = \nabla \times \mathbb{A} = \frac{\hbar c}{2e} \nabla \times (\nabla \theta)$$

Thus

$$\boxed{\mathbb{B} = 0}$$

This is the <u>Meissner effect</u>: the field is expelled inside a superconductor

The magnetic flux across a surface is

$$\Phi = \int \mathbb{B} \cdot d\mathbb{S} = \oint \mathbb{A} \cdot d\mathbb{r}$$

But $\mathbb{A} = \frac{\hbar c}{2e} \nabla \theta$, so

$$\Phi = \frac{\hbar}{2e} \oint \nabla \theta \cdot d\mathbb{r}$$

The order parameter $\psi = (\psi_0 + \rho) e^{i\theta}$ must be single-valued, so if we loop θ around, it can change at most by $2\pi n$, where n is an integer. Thus

$$\boxed{\Phi = 2\pi n \frac{\hbar}{2e}}$$

the magnetic flux inside a superconductor is therefore quantized in units of $(2\pi\hbar = h)$

$$\boxed{\Phi_0 = \frac{h}{2e}}$$

The quantization appears which charge $2e$, not e. Thus, by measuring the flux quantum we can have an experimental confirmation for the existence of cooper pairs.