Magnons

Review of spin S operators

A particle with spin S is described by 3 operators S_x, S_y and S_z, which satisfy the angular momentum algebra

$$[S_x, S_y] = i S_z \tag{1}$$

(plus cyclic permutations). We usually choose S_z to be diagonalized. For spin S it will have $2S+1$ eigenvalues of the form

$$S_z |m\rangle = m|m\rangle \tag{2}$$

$$m = S, S-1, \dots, -S+1, -S \tag{3}$$

If $S=1/2$ then $S_\alpha = \sigma_\alpha/2$, where σ_α are the Pauli matrices.

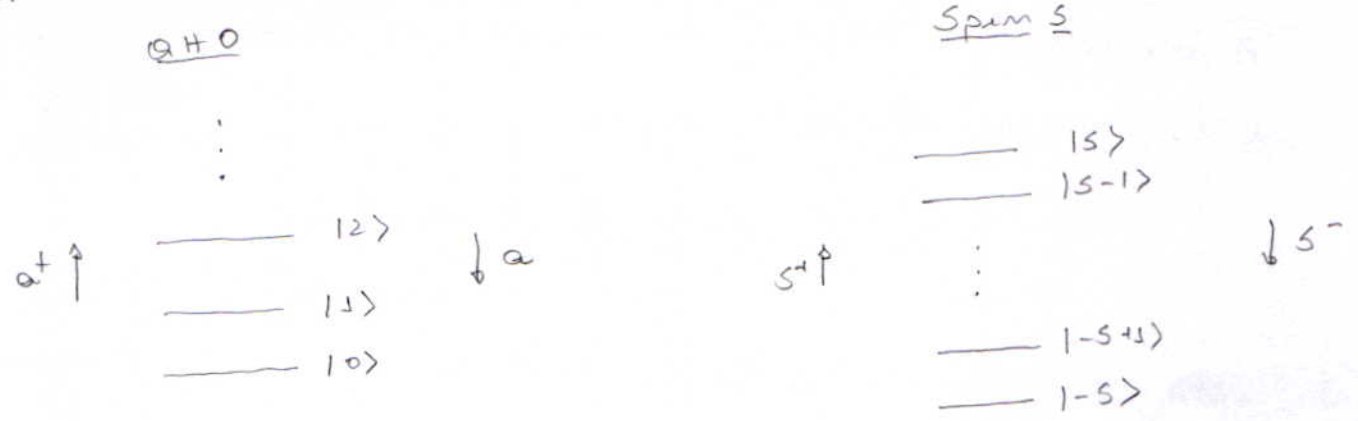
We also define lowering and raising operators $S_\pm$ as

$$S_\pm = S_x \pm i S_y \qquad \text{or} \qquad S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - iS_-}{2} \tag{4}$$

These guys operate on the basis $|m\rangle$ as

$$S_+ |m\rangle = \sqrt{(S-m)(S+m+1)}\;|m+1\rangle$$
$$S_- |m\rangle = \sqrt{(S+m)(S-m+1)}\;|m-1\rangle \tag{5}$$

The structure of the spin eigenstates is therefore quite similar to that of the harmonic oscillator. The difference is that for the QHO the ladder is infinite, whereas for spins it has $2S+1$ states



$a|0\rangle = 0$

$S^+|S\rangle = 0$

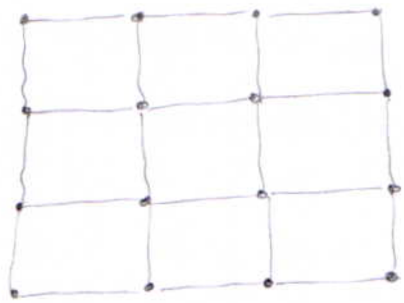
$S^-|-S\rangle = 0$

The algebra (1), in terms of the $S_\pm$ becomes

$$[S_z, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = 2 S_z \qquad (6)$$

The Heisenberg model

In these notes we shall investigate the properties of a system having N spin S particles displaced in a lattice. For instance, it could be a 2D lattice



We label each site by an index $i = 1, \dots, N$ and assume each site contains a spin operator $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$.

The system is then assumed to be described by the Heisenberg Hamiltonian

$$\boxed{H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle i,j \rangle} (S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z)} \qquad (7)$$

where $\langle i,j \rangle$ means a sum over all nearest-neighbors. This model is used to describe the physics of ferromagnets. An interaction like $-J\, \mathbf{S}_i \cdot \mathbf{S}_j$ favors the parallel alignment of the spins. A good way to understand that is to think about the $\mathbf{S}_i$ as classical vectors of length S. Thus to minimize $-J\, \mathbf{S}_i \cdot \mathbf{S}_j$ we need to put $\mathbf{S}_i$ and $\mathbf{S}_j$ parallel to each other.

The spin operators S_i satisfy the angular momentum algebra (1) for $i = j$, whereas they commute for $i \neq j$:

$$[S_i^x, S_j^y] = i \delta_{ij} S_i^z \tag{8}$$

This algebra is weird. It does not have the pretty structure of the bosonic and fermionic algebras. And what is worse, it is not invariant under unitary transformations (like Fourier transforms).

For this reason, it is unfortunately not possible to diagonalize (7) exactly. The only exception is for $S = 1/2$ in a 1D lattice, where there is a technique called Bethe ansatz.

Instead, what we will do here is to carry out an approximate analysis. We will show that finding the ground state is easy. Then we will study the low energy excitations above the ground state.

The ground - state

It turns out that finding the GS of (7) is quite easy. First we rewrite it in terms of $S_i^{\pm}$. I will leave for you to check that in this case it becomes

$$H = -J \sum_{\langle i,j \rangle} \left\{ \frac{S_i^+ S_j^- + S_i^- S_j^+}{2} + S_i^z S_j^z \right\} \tag{9}$$

A natural candidate for $|\psi_{gs}\rangle$ is the state where all spins are up

$$\boxed{|\psi_{gs}\rangle = |S, S, \dots, S\rangle} \tag{10}$$

which agrees with the classical intuition. To check that this works note that since $S_i^+ |S\rangle = 0$ we get

$$(S_i^+ S_j^- + S_i^- S_j^+) |\psi_{gs}\rangle = 0 \tag{11}$$

whereas

$$S_i^z S_j^z \, |\psi_{gs}\rangle = S^2 \, |\psi_{gs}\rangle \tag{12}$$

Thus

$$H |\psi_{gs}\rangle = -J \left(\sum_{\langle ij \rangle} S^2 \right) |\psi_{gs}\rangle \tag{13}$$

This sum is now the number of sites times the number of nearest neighbor bonds. In 2D this would be $N \times 2$. For a d-dimensional lattice it will be Nd. Thus

$$\mathcal{E}_{gs} = - J N d S^2 \tag{14}$$

Another question is why there cannot be a state with an energy smaller than this. The reason is that the norm of $\mathbf{S}_i \cdot \mathbf{S}_j$ is bounded by S^2, so this is indeed the lowest possible energy.

There is a catch however: the ground-state is degenerate. Again, it helps to think classically: the dot product $\mathbf{S}_i \cdot \mathbf{S}_j$ is maximized when $\mathbf{S}_i$ and $\mathbf{S}_j$ are parallel. But it does not matter in which direction they point to. In the quantum case this translates into a huge degeneracy. For instance $|-S, -S, \ldots, -S\rangle$ would also be a GS.

One way to fix this is to apply a magnetic field, so that (7) is replaced with

$$\boxed{H = - J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - h \sum_i S_i^z} \tag{15}$$

Now, for any $h > 0$, the GS (10) becomes the unique ground state.

The Holstein-Primakoff transformation

Now that we know the GS, let us study what are the low energy excitations above the GS. So starting at $|S, S, \ldots, S\rangle$, we can create an excitation by changing the state of one or a few spins away from S.

There is a neat way of implementing this idea directly on the Hamiltonian, which is to map spins to bosons. This is the idea of the Holstein-Primakoff transformation.

Consider a single spin and the states $|m\rangle$. We want excitations starting at $|S\rangle$. So what we try to do is map $|S\rangle$ into the vacuum $|0\rangle$ of a bosonic mode. The excitations $|S-1\rangle, |S-2\rangle, \ldots$ are generated by S^-. Thus we also try to map S^- into a creation operator $a^\dagger$, which takes $|0\rangle$ to $|1\rangle, |2\rangle, \ldots$

In this way $|S-1\rangle$ will be equivalent to a state with a single excitation, $|1\rangle$; similarly $|m = S - n\rangle$ will have $|n\rangle$ excitations

$$m = S - n \qquad (16)$$

m	S	$S-1$	$S-2$	$S-3$	$\ldots$
n	0	1	2	3	$\ldots$

This suggest we search for an operator mapping of the form

$$\boxed{S_z = S - a^\dagger a} \tag{17}$$

Now, you may protest and say that $a^\dagger a$ is infinite dimensional and S_z has dimension $2S+1$. And that's indeed true. But we can fix that by mapping S_- to $a^\dagger$ in such a way that we can never create more than $2S$ excitations.

The trick to that, which is the reason Holstein and Primakoff became known for, is to define

$$\boxed{S_- = a^\dagger \sqrt{2S - a^\dagger a}} \tag{18}$$

I know this looks weird. But let's take our time to convince ourselves that this works. First note that

$$S_- |m\rangle = a^\dagger \underbrace{\sqrt{2S - a^\dagger a}\,|m\rangle}_{\sqrt{2S-m}\,|m} = \sqrt{2S-m}\,a^\dagger\,|m\rangle$$

or

$$S_- |m\rangle = \sqrt{(2S-m)(m+1)}\,|m+1\rangle \tag{19}$$

If $m = 2S$ then we get 0. This is the truncation property I mentioned. Even though the bosonic space is infinite, with the term $\sqrt{2S - a^\dagger a}$ we can never create more than $2S$ excitations

If we also substitute $m = S - m$ in in (19) we get

$$\sqrt{(2S-m)(m+1)} = \sqrt{(S+m)(S-m+1)} \tag{20}$$

which is precisely the coefficient in Eq (5). Thus, the bosonic state $|m\rangle$ really corresponds to the spin state $|m\rangle$.

We should also check the algebra (6). Taking the adjoint of (18) we get

$$S_+ = \sqrt{2S - a^\dagger a}\; a \tag{21}$$

Then

$$\begin{aligned} S_+ S_- &= \sqrt{2S - a^\dagger a}\; a a^\dagger \sqrt{2S - a^\dagger a} \\ &= \sqrt{2S - a^\dagger a}\,(1 + a^\dagger a)\sqrt{2S - a^\dagger a} \\ &= (2S - a^\dagger a)(1 + a^\dagger a) \end{aligned}$$

← Now $a^\dagger a$ commutes with the square root!

$$\begin{aligned} S_- S_+ &= a^\dagger \sqrt{2S - a^\dagger a}\sqrt{2S - a^\dagger a}\; a \\ &= a^\dagger (2S - a^\dagger a)\, a \\ &= 2S\, a^\dagger a - a^\dagger a^\dagger a a \\ &= 2S\, a^\dagger a - a^\dagger a (a^\dagger a - 1) \\ &= (2S - a^\dagger a) a^\dagger a + a^\dagger a \end{aligned}$$

Thus

$$[S_+, S_-] = (2S - a^\dagger a)(1 + a^\dagger a) - (2S - a^\dagger a)a^\dagger a - a^\dagger a$$

$$= 2S - a^\dagger a - a^\dagger a$$

$$= 2(S - a^\dagger a)$$

$$= 2S_z$$

Voilà Eq (6)!

Similarly

$$[S_z, S_+] = [S - a^\dagger a, \sqrt{2S - a^\dagger a}\, a]$$

$$= -[a^\dagger a, \sqrt{2S - a^\dagger a}\, a]$$

$$= -\sqrt{2S - a^\dagger a}\, \underbrace{[a^\dagger a, a]}_{-a}$$

$$= \sqrt{2S - a^\dagger a}\, a$$

$$= S_+$$

and similarly for $[S_z, S_-] = -S_-$.

Thus, to conclude, we see that the mapping in Eqs (17) and (18) are really genuine in all senses: the labelling of the states, the matrix elements and the algebras

Holstein - Primakoff in the Heisenberg model

Let's now go back to the full Heisenberg model in Eq (15)

$$\boxed{H = -h \sum_{i=1}^{N} S_i^z - J \sum_{\langle ij \rangle} \left\{ \frac{S_i^+ S_j^- + S_i^- S_j^+}{2} + S_i^z S_j^z \right\}} \tag{22}$$

we introduce a HP transf. for each spin, as

$$S_i^z = S - a_i^\dagger a_i \tag{23a}$$

$$S_i^- = a_i^\dagger \sqrt{2S - a_i^\dagger a_i} \tag{23b}$$

$$S_i^+ = \sqrt{2S - a_i^\dagger a_i}\, a_i \tag{23c}$$

this transformation is exact. However, it would lead to a very ugly Hamiltonian having weird square roots. Thus, the real use of the HP transformation is as an <u>approximation</u> <u>method</u>.

Recall that we are interested in <u>low</u> <u>energy</u> <u>excitations</u>. This means essentially values of m_i which are far away from $2S$. that is, such that we are far from the end of the ladder. Alternatively, you may think we are assuming S is very large. Then we can work under a "large S" approximation.

We therefore expand

$$\sqrt{2S - a_i^\dagger a_i} = \sqrt{2S}\sqrt{1 - \frac{a_i^\dagger a_i}{2S}}$$

$$\simeq \sqrt{2S}\left(1 - \frac{1}{2}\frac{a_i^\dagger a_i}{2S}\right) \qquad (24)$$

The lowering operators then become

$$S_i^- \simeq \sqrt{2S}\, a_i^\dagger - \frac{1}{2}\frac{1}{\sqrt{2S}}\, a_i^\dagger a_i^\dagger a_i \qquad (25)$$

whereas the raising operator becomes

$$S_i^+ \simeq \sqrt{2S}\, a_i - \frac{1}{2}\frac{1}{\sqrt{2S}}\, a_i^\dagger a_i a_i \qquad (26)$$

When we multiply them, as in (22), we will get

$$S_i^+ S_j^- = \left[\sqrt{2S}\, a_i - \frac{1}{2}\frac{1}{\sqrt{2S}}\, a_i^\dagger a_i a_i\right]\left[\sqrt{2S}\, a_j^\dagger - \frac{1}{2}\frac{1}{\sqrt{2S}}\, a_j^\dagger a_j^\dagger a_j\right]$$

$$= 2S\, a_i a_j^\dagger - \frac{1}{2}\, a_i a_j^\dagger a_j^\dagger a_j - \frac{1}{2}\, a_i^\dagger a_i a_i a_j^\dagger + \dots$$

Now comes the important point: the first term is of order S, whereas the others are of order 1. Thus, for large S we may keep only the first term

Thus,

$$S_i^+ S_j^- \simeq 2S\, a_j^\dagger a_i \qquad (27)$$

($a_i a_j^\dagger = a_j^\dagger a_i$ because $j \neq i$ in (22)).

Similarly, the term $S_i^z S_j^z$ in (22) becomes, using (23a),

$$\begin{aligned} S_i^z S_j^z &= (S - a_i^\dagger a_i)(S - a_j^\dagger a_j) \\ &= S^2 - S(a_i^\dagger a_i + a_j^\dagger a_j) + a_i^\dagger a_i a_j^\dagger a_j \qquad (28) \end{aligned}$$

Again, you see that the quartic term is of order 1 and thus much smaller than the others.

To leading order in S, the Hamiltonian (22) then becomes

$$\begin{aligned} H = -h \sum_{i=1}^{N} (S - a_i^\dagger a_i) - J \sum_{\langle ij \rangle} \Big\{ \frac{2S}{2}(a_j^\dagger a_i + a_i^\dagger a_j) + S^2 + \\ - S(a_i^\dagger a_i + a_j^\dagger a_j) \Big\} \qquad (29) \\ + \mathcal{O}(S^0) \end{aligned}$$

Let's now tidy things up a bit. First

$$-h \sum_{i=1}^{N} (S - a_i^\dagger a_i) = -hNS + h \sum_{i=1}^{N} a_i^\dagger a_i \qquad (30)$$

Second

$$-J \sum_{\langle i,j\rangle} S^2 = -JS^2 dN$$

And third

$$JS \sum_{\langle i,j\rangle} a_i^\dagger a_i = JSd \sum_i a_i^\dagger a_i$$

$$JS \sum_{\langle i,j\rangle} a_j^\dagger a_j = JSd \sum_i a_i^\dagger a_i$$

thus we get

$$\boxed{H = \mathcal{E}_{gs} + \sum_{i=1}^{N} (h + 2JSd)\, a_i^\dagger a_i - JS \sum_{\langle i,j\rangle} (a_i^\dagger a_j + a_j^\dagger a_i)} \tag{31}$$

where

$$\mathcal{E}_{gs} = -JS^2 Nd - hNS \tag{32}$$

is the ground state energy (14) (plus a new field term)

Eq (31) is really really cool. It allows us to write the Heisenberg Hamiltonian as a ground state energy plus excitations. Moreover, we see that these excitations are bosonic and, moreover, they are mobile: they can hop through the lattice. These bosonic excitations are called magnons, they are the fundamental quanta of spin waves.

To finish our problem, we simply note that (31) is nothing but the tight-binding model. Thus, we can diagonalize it by moving to Fourier space

$$a_i = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot x_i} b_{\mathbf{k}} \tag{33}$$

As a result we get, using the results from the previous lecture notes

$$\boxed{H = \mathcal{E}_{gs} + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}} \tag{34}$$

where

$$\boxed{\varepsilon_{\mathbf{k}} = h + 2JSd - 2JS(\cos k_1 + \dots + \cos k_d)} \tag{35}$$

(I'm assuming a general d-dimensional lattice).

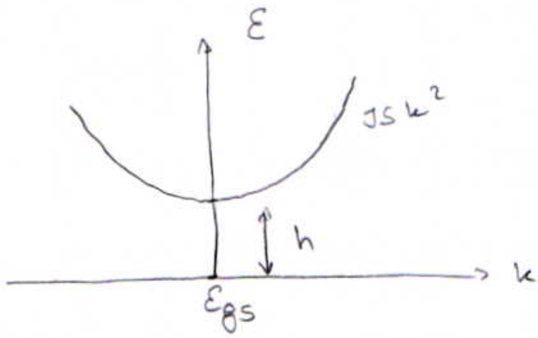
In the long wavelength limit (low k) we expand the cosines to get

$$\mathcal{E}_{\mathbf{k}} \approx h + 2JSd - 2JSd + JS\,|\mathbf{k}|^2$$

or

$$\boxed{\mathcal{E}_{\mathbf{k}} \approx h + JSk^2} \qquad (36)$$

We therefore see that magnons behave like non-relativistic particles ($E \sim p^2$) but the magnetic field playing the role of an energy gap



The field h measures the minimum energy required to create an excitation. That's why it gives the gap.

Magnons are like phonons or photons, in that the number of particles is not conserved. Thus the GS is still $|\psi_{gs}\rangle$ with zero magnons present. But we can then externally excite the system and produce some magnons. These magnons will then propagate through lattice with different momenta k.

In our leading order approximation, the magnons propagate freely as plane waves (because our Hamiltonian is quadratic). But we could now reintroduce the higher order terms in (26) and (28). This will lead to quartic terms representing <u>magnon-magnon interactions</u>.

Another thing which is quite important in practice are the interactions between magnons and phonons. This interaction causes the phonons to behave as a bath for the magnons, leading to decoherence and damping. The precise mechanisms through which this occurs are still the subject of debate and an active area of research.