Magnons

Review of spin S operators

A particle with spin S is described by 3 operators S_x, S_y and S_z, which satisfy the angular momentum algebra

$$[S_x, S_y] = i S_z \tag{1}$$

(plus cyclic permutations). We usually choose S_z to be diagonalized. For spin S it will have $2S+1$ eigenvalues of the form

$$S_z |m\rangle = m |m\rangle \tag{2}$$

$$m = S, S-1, \ldots, -S+1, -S \tag{3}$$

If $S=1/2$ then $S_\alpha = \sigma_\alpha/2$, where σ_α are the Pauli matrices.

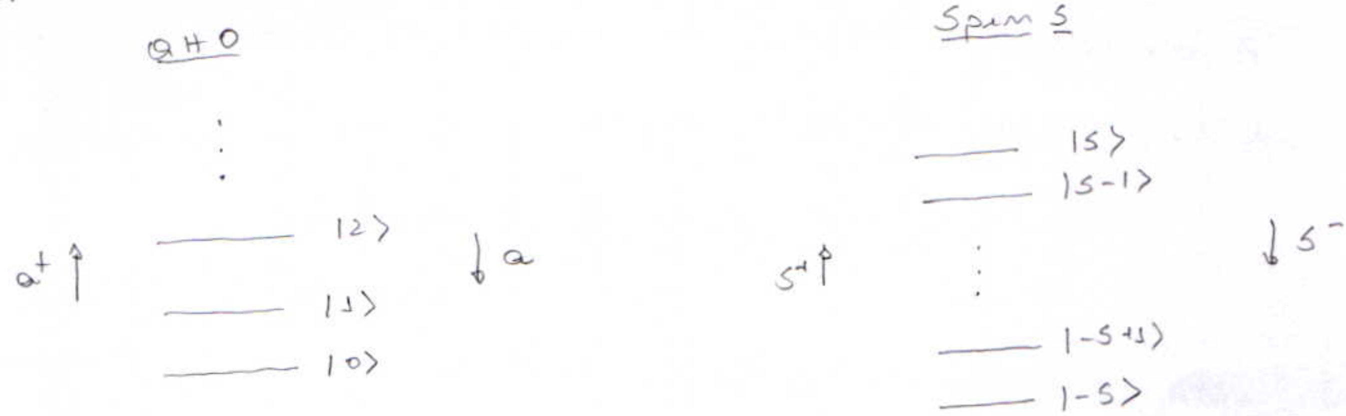
We also define lowering and raising operators $S_\pm$ as

$$S_\pm = S_x \pm i S_y \qquad \text{or} \qquad S_x = \frac{S_+ + S_-}{2}, \quad S_y = \frac{S_+ - iS_-}{2} \tag{4}$$

These guys operate on the basis $|m\rangle$ as

$$S_+ |m\rangle = \sqrt{(S-m)(S+m+1)}\; |m+1\rangle$$
$$S_- |m\rangle = \sqrt{(S+m)(S-m+1)}\; |m-1\rangle \tag{5}$$

The structure of the spin eigenstates is therefore quite similar to that of the harmonic oscillator. The difference is that for the QHO the ladder is infinite, whereas for spins it has $2S+1$ states



$$a|0\rangle = 0$$

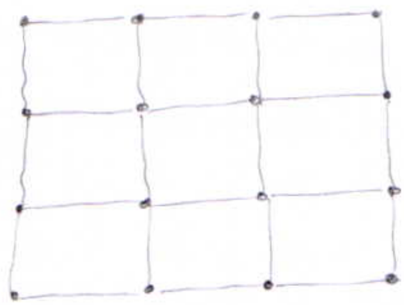
$$S^{+}|S\rangle = 0$$
$$S^{-}|-S\rangle = 0$$

The algebra (1), in terms of the $S_{\pm}$ becomes

$$[S_z, S_{\pm}] = \pm S_{\pm} \qquad [S_+, S_-] = 2 S_z \qquad (6)$$

The Heisenberg model

In these notes we shall investigate the properties of a system having N spin S particles displaced in a lattice. For instance, it could be a 2D lattice



We label each site by an index $i = 1, \dots, N$ and assume each site contains a spin operator $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$.

The system is then assumed to be described by the Heisenberg Hamiltonian

$$\boxed{H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle i,j \rangle} \left(S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z \right)} \qquad (7)$$

where $\langle i,j \rangle$ means a sum over all nearest-neighbors. This model is used to describe the physics of ferromagnets. An interaction like $-J\, \mathbf{S}_i \cdot \mathbf{S}_j$ favors the parallel alignment of the spins. A good way to understand that is to think about the $\mathbf{S}_i$ as classical vectors of length S. Then to minimize $-J\, \mathbf{S}_i \cdot \mathbf{S}_j$ we need to put $\mathbf{S}_i$ and $\mathbf{S}_j$ parallel to each other.

The spin operators S_i satisfy the angular momentum algebra (1) for $i=j$, whereas they commute for $i \neq j$:

$$[S_i^x, S_j^y] = i\,\delta_{ij}\, S_i^z \tag{8}$$

This algebra is weird. It does not have the pretty structure of the bosonic and fermionic algebras. And what is worse, it is not invariant under unitary transformations (like Fourier transforms).

For this reason, it is unfortunately not possible to diagonalize (7) exactly. The only exception is for $S=1/2$ in a 1D lattice, where there is a technique called Bethe ansatz.

Instead, what we will do here is to carry out an approximate analysis. We will show that finding the ground state is easy. Then we will study the low energy excitations above the ground state.

The ground-state

It turns out that finding the GS of (7) is quite easy. First we rewrite it in terms of $S_i^{\pm}$. I will leave for you to check that in this case it becomes

$$H = -J \sum_{\langle ij \rangle} \left\{ \frac{S_i^+ S_j^- + S_i^- S_j^+}{2} + S_i^z S_j^z \right\} \tag{9}$$

A natural candidate for $|\psi_{gs}\rangle$ is the state where all spins are up

$$\boxed{|\psi_{gs}\rangle = |S, S, \ldots, S\rangle} \tag{10}$$

which agrees with the classical intuition. To check that this works note that since $S_i^+ |S\rangle = 0$ we get

$$(S_i^+ S_j^- + S_i^- S_j^+)\, |\psi_{gs}\rangle = 0 \tag{11}$$

whereas

$$S_i^z S_j^z\, |\psi_{gs}\rangle = S^2\, |\psi_{gs}\rangle \tag{12}$$

Thus

$$H |\psi_{gs}\rangle = -J \left(\sum_{\langle ij \rangle} S^2 \right) |\psi_{gs}\rangle \tag{13}$$

This sum is now the number of sites times the number of nearest neighbor bonds. In 2D this would be $N \times 2$. For a d-dimensional lattice it will be Nd. Thus

$$\mathcal{E}_{gs} = -JNdS^2 \qquad (14)$$

Another question is why there cannot be a state with an energy smaller than this. The reason is that the norm of $\mathbf{S}_i \cdot \mathbf{S}_j$ is bounded by S^2, so this is indeed the lowest possible energy.

There is a catch however: the ground-state is degenerate. Again, it helps to think classically: the dot product $\mathbf{S}_i \cdot \mathbf{S}_j$ is maximized when $\mathbf{S}_i$ and $\mathbf{S}_j$ are parallel. But it does not matter in which direction they point to. In the quantum case this translates into a huge degeneracy. For instance $|-S, -S, \ldots, -S\rangle$ would also be a GS.

One way to fix this is to apply a magnetic field, so that (7) is replaced with

$$\boxed{H = -J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - h \sum_i S_i^z} \qquad (15)$$

Now, for any $h > 0$, the GS (10) becomes the unique ground state.

The Holstein - Primakoff transformation

Now that we know the GS, let us study what are the low energy excitations above the GS. So starting at $|S, S, \dots, S\rangle$, we can create an excitation by changing the state of one or a few spins away from S.

There is a neat way of implementing this idea directly on the Hamiltonian, which is to map spins to bosons. This is the idea of the Holstein - Primakoff transformation.

Consider a single spin and the states $|m\rangle$. We want excitations starting at $|S\rangle$. So what we try to do is map $|S\rangle$ into the vacuum $|0\rangle$ of a bosonic mode. The excitations $|S-1\rangle$, $|S-2\rangle, \dots$ are generated by S^-. Thus we also try to map S^- into a creation operator $a^\dagger$, which takes $|0\rangle$ to $|1\rangle, |2\rangle, \dots$

In this way $|S-1\rangle$ will be equivalent to a state with a single excitation, $|1\rangle$; similarly $|m = S - n\rangle$ will have $|n\rangle$ excitations

$$m = S - n \tag{16}$$

m	S	$S-1$	$S-2$	$S-3$	$\dots$
n	0	1	2	3	$\dots$

This suggest we search for an operator mapping of the form

$$\boxed{S_z = S - a^\dagger a} \tag{17}$$

Now, you may protest and say that $a^\dagger a$ is infinite dimensional and S_z has dimension $2S+1$. And that's indeed true. But we can fix that by mapping S_- to $a^\dagger$ in such a way that we can never create more than $2S$ excitations.

The trick to that, which is the reason Holstein and Primakoff became known for, is to define

$$\boxed{S_- = a^\dagger \sqrt{2S - a^\dagger a}} \tag{18}$$

I know this looks weird. But let's take our time to convince ourselves that this works. First note that

$$S_- |m\rangle = a^\dagger \underbrace{\sqrt{2S - a^\dagger a}\, |m\rangle}_{\sqrt{2S-m}\,|m} = \sqrt{2S-m}\, a^\dagger\, |m\rangle$$

or

$$S_- |m\rangle = \sqrt{(2S-m)(m+1)}\, |m+1\rangle \tag{19}$$

If $m = 2S$ then we get 0. This is the truncation property I mentioned. Even though the bosonic space is infinite, with the term $\sqrt{2S - a^\dagger a}$ we can never create more than $2S$ excitations

If we also substitute $m = S - m$ in in (19) we get

$$\sqrt{(2S-m)(m+1)} = \sqrt{(S+m)(S-m+1)} \tag{20}$$

which is precisely the coefficient in Eq (5). Thus, the bosonic state $|m\rangle$ really corresponds to the spin state $|m\rangle$.

We should also check the algebra (6). Taking the adjoint of (18) we get

$$S_+ = \sqrt{2S - a^\dagger a}\; a \tag{21}$$

Then

$$\begin{aligned} S_+ S_- &= \sqrt{2S - a^\dagger a}\; a a^\dagger \sqrt{2S - a^\dagger a} \\ &= \sqrt{2S - a^\dagger a}\,(1 + a^\dagger a)\sqrt{2S - a^\dagger a} \\ &= (2S - a^\dagger a)(1 + a^\dagger a) \end{aligned}$$

← Now $a^\dagger a$ commutes with the square root!

$$\begin{aligned} S_- S_+ &= a^\dagger \sqrt{2S - a^\dagger a}\sqrt{2S - a^\dagger a}\; a \\ &= a^\dagger (2S - a^\dagger a)\, a \\ &= 2S\, a^\dagger a - a^\dagger a^\dagger a a \\ &= 2S\, a^\dagger a - a^\dagger a (a^\dagger a - 1) \\ &= (2S - a^\dagger a) a^\dagger a + a^\dagger a \end{aligned}$$

Thus

$$[S_+, S_-] = (2S - a^\dagger a)(1 + a^\dagger a) - (2S - a^\dagger a)a^\dagger a - a^\dagger a$$

$$= 2S - a^\dagger a - a^\dagger a$$

$$= 2(S - a^\dagger a)$$

$$= 2S_z$$

Voilà Eq (6)!

Similarly

$$[S_z, S_+] = [S - a^\dagger a, \sqrt{2S - a^\dagger a}\, a]$$

$$= -[a^\dagger a, \sqrt{2S - a^\dagger a}\, a]$$

$$= -\sqrt{2S - a^\dagger a}\, \underbrace{[a^\dagger a, a]}_{-a}$$

$$= \sqrt{2S - a^\dagger a}\, a$$

$$= S_+$$

and similarly for $[S_z, S_-] = -S_-$.

Thus, to conclude, we see that the mapping in Eqs (17) and (18) are really genuine in all senses: the labelling of the states, the matrix elements and the algebras

Holstein - Primakoff in the Heisenberg model

Let's now go back to the full Heisenberg model in Eq (15)

$$\boxed{H = -h \sum_{i=1}^{N} S_i^z - J \sum_{\langle ij \rangle} \left\{ \frac{S_i^+ S_j^- + S_i^- S_j^+}{2} + S_i^z S_j^z \right\}} \qquad (22)$$

we introduce a HP transf. for each spin, as

$$S_i^z = S - a_i^\dagger a_i \qquad (23a)$$

$$S_i^- = a_i^\dagger \sqrt{2S - a_i^\dagger a_i} \qquad (23b)$$

$$S_i^+ = \sqrt{2S - a_i^\dagger a_i}\, a_i \qquad (23c)$$

this transformation is exact. However, it would lead to a very ugly Hamiltonian having weird square roots. Thus, the real use of the HP transformation is as an approximation method.

Recall that we are interested in low energy excitations. This means essentially values of n_i which are far away from 2S. that is, such that we are far from the end of the ladder. Alternatively, you may think we are assuming S is very large. then we can work under a "large S" approximation.