Introduction to field theory

In mechanics the degrees of freedom are the positions $q_i(t)$ of point particles. The goal of mechanics is then to describe how these q_i evolve in time.

We will now turn to the description of a different kind of system, where the degrees of freedom are *fields* $\phi_i(t, \mathbf{r})$; ie, the quantities we want to describe are objects which depend on both time and space.

The simplest example is the temperature in a room, $T(t, \mathbf{r})$. It changes in time and also changes with $\mathbf{r}$. So it is a field. Another big example are the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$. Each is a vector so we have in total 6 fields. Each field changes both in time and space, $E_i(t, \mathbf{r})$, $B_i(t, \mathbf{r})$.

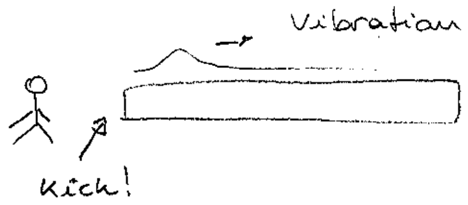
In principle the description of fields has nothing to do with mechanics. It is another area of physics. But the Lagrangian formulation of mechanics is so powerful that we may ask whether it is possible to generalize it to include fields as well.

The procedure to do this is actually straightforward. We just define the action as

$$S = \int d^4x \, \mathcal{L} \tag{1}$$

where $d^4x = dt\,dx\,dy\,dz$. We then cook up a Lagrangian $\mathcal{L}$ such that the condition $\delta S = 0$ gives the corresponding equations of motion (for instance Maxwell's equations in the electromagnetic case).

Let me give you an example. Suppose you kick a long iron bar. This will cause a displacement in the material which will propagate like a wave



this vibration may be described by a field $\phi(t, \vec{r})$ which measures how much the material was displaced at time t in the position r.

For simplicity I will suppose everything is one dimensional, so the field is $\phi(t, x)$. Engineers know that this vibration propagates through the material like a wave. So they know that ϕ satisfies the wave equation

$$\partial_t^2 \phi = v^2 \partial_x^2 \phi \tag{2}$$

where v is the velocity of sound in the material. In Iron $v = 5130\ m/s$.

Now we want to cook up a Lagrangian such that $\delta S = 0$ gives Eq (2). This is a bit difficult to do at first, but you get used to it with some practice. In our case the correct Lagrangian is

$$\mathcal{L} = (\partial_t \phi)^2 - v^2 (\partial_x \phi)^2 \tag{3}$$

Let us check that this works. Consider a variation of the field

$$\phi'(t,x) = \phi(t,x) + \eta(t,x) \qquad (4)$$

where η is a tiny field (this is exactly what we did with $q(t)$, except that now "q" depends on t and x).

The new Lagrangian becomes

$$\begin{aligned}
\mathcal{L}' &= (\partial_t \phi')^2 - v^2(\partial_x \phi')^2 \\
&= (\partial_t \phi + \partial_t \eta)^2 - v^2(\partial_x \phi + \partial_x \eta)^2 \\
&= \left[(\partial_t \phi)^2 - v^2(\partial_x \phi)^2\right] + \left[2(\partial_t \phi)(\partial_t \eta) - 2v^2(\partial_x \phi)(\partial_x \eta)\right] \\
&\qquad + \mathcal{O}(\eta)^2
\end{aligned}$$

where we consider only terms linear in η. We then get

$$\delta S = \int d^4x \; 2\left[(\partial_t \phi)(\partial_t \eta) - v^2(\partial_x \phi)(\partial_x \eta)\right] \qquad (5)$$

Integrating by parts and being careless about the cross term we obtain:

$$\int d^4x \, (\partial_t \phi)(\partial_t \eta) = -\int d^4x \, (\partial_t^2 \phi)\,\eta$$

$$\int d^4x \, (\partial_x \phi)(\partial_x \eta) = -\int d^4x \, (\partial_x^2 \phi)\,\eta$$

Thus

$$\delta S = -2\int d^4x \left[\partial_t^2 \phi - v^2 \partial_x^2 \phi\right]\eta$$

Since η is arbitrary, for $\delta S = 0$ we must have

$$\partial_t^2 \phi - v^2 \partial_x^2 \phi = 0$$

which is Eq (2).

We therefore started with a known Eq. of motion [Eq (2)] and constructed a corresponding Lagrangian theory out of it. This is exactly what we first did in mechanics, when we constructed a Lagrangian to describe Newton's law. So there is a close parallel.

But if you remember, even though we began by considering a Lagrangian simply to describe Newton's law, we quickly realized that there were several advantages in working with a Lagrangian. In particular, with a Lagrangian we were able to exploit deeply the symmetries of the problems.

Here it will be the same.

Relativistic fields

We will concentrate most (but not all) of our effort into fields which transform in a known way under Lorentz transformations.

Recall that a four-vector x^μ transforms under Lorentz transformations according to

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \qquad (6)$$

We will consider first the case of a scalar field; that is, a field which is invariant under Lorentz transformations. So we call it $\phi(x)$, where x is an abbreviation for x^μ.

$$\phi(x) = \phi(x^0, x^1, x^2, x^3)$$

We assume that under a Lorentz transformation ϕ transforms as

$$\phi(x) \to \phi'(x') = \phi(x) \qquad (7)$$

This is what defines a scalar field.

A real scalar field ϕ does not describe any known physical system. But it is a very popular construct since it can be used to test many interesting techniques. Also, the complex scalar field describes the Higgs Boson and is thus very important.

the possible Lagrangians which may be constructed are severely constrained by the fact that $\mathcal{L}$ (and hence S) must be Lorentz invariant.

Typical Lorentz invariant terms include powers of ϕ, like

$$\phi,\ \phi^2,\ \phi^3,\ \ldots$$

But these terms have no dynamics. So we must also have derivatives of ϕ.

By definition, the infinitesimal form of ϕ is

$$d\phi = \frac{\partial \phi}{\partial x^\nu}\, dx^\nu \tag{8}$$

Since $d\phi$ is a scalar and dx^ν is a contravariant four-vector, $\frac{\partial \phi}{\partial x^\nu}$ must therefore be a covariant four-vector. We will therefore write it as

$$\frac{\partial \phi}{\partial x^\nu} = \partial_\nu \phi \tag{9}$$

with the index downstairs to emphasize that it is covariant.

In components we have

$$\partial_\nu \phi = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) \tag{10}$$

So the derivative is the opposite of the position: ∂_ν has all positives and ∂^ν has negative space components

$$\partial^\nu \phi = \left(\frac{1}{c}\frac{\partial \phi}{\partial t}, -\frac{\partial \phi}{\partial x}, -\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial z} \right) \tag{11}$$

I know this is a bit confusing. the best thing is to remember Eq (8).

Ok! Back to $\mathcal{L}$. We want to have some derivatives of ϕ in $\mathcal{L}$. But $\mathcal{L}$ is a scalar (Lorentz invariant) so these derivatives must appear in such a way as to produce a scalar. The only way to do this is to contract $\partial_\mu \phi$ with some other four-vector. But there are no other four-vectors because all we have is ϕ. Thus we conclude that the derivatives must appear in the Lagrangian as the combination

$$(\partial_\mu \phi)(\partial^\mu \phi)$$

because this is the only type of scalar that you can cook up with ∂_μ. Of course, we could also have something like $[(\partial_\mu\phi)(\partial^\mu\phi)]^2$, but this would produce things like $\partial_t^3 \phi$ in the equations of motion, which would be very weird.

Conclusion: the Lagrangian of a scalar field looks like

$$\mathcal{L} = \alpha (\partial_\mu \phi)(\partial^\mu \phi) + c_1 \phi + c_2 \phi^2 + c_3 \phi^3 + \dots \tag{12}$$

where $\alpha, c_1, c_2, \dots$ are constants.

The terms ϕ and ϕ^2 will produce equations of motion which are linear in the fields. This is what we call <u>free fields</u>. Terms like ϕ^3, ϕ^4, etc. produce non-linear equations which are much more difficult to treat.

The usual procedure is to consider first only the free field and then treat the higher order terms later as perturbations.

So as a first step we will consider only $c_1\phi$ and $c_2\phi^2$. Please note: I am not saying the other terms are not important. They are very important. But they are also very difficult to deal with, so to start we will simply neglect them.

As for $c_1\phi$ and $c_2\phi^2$, we may always complete squares and redefine ϕ so as to eliminate the linear part. We may also choose a value for α and thus leave c_2 as the only actual constant in the theory.

We will follow convention and therefore write

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2} \tag{13}$$

the minus sign is chosen because we will see later that m behaves like a mass.

I want to emphasize that the steps we just took to write down (13) are very solid. In fact, (13) is the most general Lagrangian for a scalar field satisfying

1) Lorentz invariance

2) No time derivatives in the Eqs of motion higher than 2

3) Linear in the field ϕ.

Natural system of units

Before we continue, I want to make an adjustment to our units. There exists a system of units which literally everyone uses called the natural system. It consists in setting

$$\boxed{c = \hbar = 1} \tag{14}$$

When you set $c=1$, time and space acquire the same dimensions. Since

$$[\hbar] = J.s = kg\,\frac{m^2}{s}$$

when we also set $\hbar = 1$ we obtain that mass must have units of 1/length:

$$\text{mass} = \frac{1}{\text{length.}} \tag{15}$$

Moreover, $J = kg\, m^2/s^2$ acquires units of mass. It is customary to measure everything in $GeV = 10^9\, eV$. We have the following conversion table:

$$\begin{aligned} \text{length}: &\quad GeV^{-1} = 1.97 \times 10^{-14}\, cm \\ \text{time}: &\quad GeV^{-1} = 6.58 \times 10^{-25}\, s \\ \text{mass}: &\quad GeV^{+1} = 1.78 \times 10^{-24}\, g. \end{aligned} \tag{16}$$

The action has units of $\hbar$ and is therefore dimensionless

$$\text{Action:} \quad GeV^{0} \tag{17}$$

Since d^4x has units of GeV^{-4} we must then have

$$\text{Lagrangian:} \quad GeV^{+4} \tag{18}$$

Looking at (13), $[x] = GeV^{-1}$ so $\left[\frac{\partial}{\partial x}\right] = GeV$. this fixes the dimension of ϕ

$$[\phi] = GeV$$

we then see that for the second term to have units of GeV^4, we must have

$$[m] = GeV$$

which is indeed the units of mass.

Back to the scalar field

The action corresponding to the Lagrangian (13) is

$$S = \int d^4x \left\{ \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \right\} \tag{19}$$

To obtain the equations of motion we perform a variation of the field

$$\phi \to \phi' = \phi + \eta$$

and set $\delta S = 0$. We have

$$(\partial_\mu \phi')(\partial^\mu \phi') = (\partial_\mu \phi + \partial_\mu \eta)(\partial^\mu \phi + \partial^\mu \eta)$$
$$= (\partial_\mu \phi)(\partial^\mu \phi) + (\partial_\mu \phi)(\partial^\mu \eta) + (\partial_\mu \eta)(\partial^\mu \phi) + \mathcal{O}(\eta)^2$$

and

$$\phi'^2 = (\phi + \eta)^2 = \phi^2 + 2\phi\eta + \mathcal{O}(\eta)^2$$

Thus

$$\delta S = \int d^4x \left\{ \frac{(\partial_\mu \phi)(\partial^\mu \eta) + (\partial_\mu \eta)(\partial^\mu \phi)}{2} - m^2 \phi \eta \right\}$$

Integrating by parts

$$(\partial_\mu \phi)(\partial^\mu \eta) \rightsquigarrow -(\partial^\mu \partial_\mu \phi)\eta$$
$$(\partial_\mu \eta)(\partial^\mu \phi) \rightsquigarrow -\eta(\partial_\mu \partial^\mu \phi)$$

Since $\partial_\nu \partial^\nu = \partial^\nu \partial_\nu$ we get

$$\delta S = \int d^4x \; 2\left[-\partial_\nu \partial^\nu \phi - m^2 \phi\right] \tag{20}$$

Hence, the equations of motion are

$$\boxed{\partial_\nu \partial^\nu \phi + m^2 \phi = 0} \tag{21}$$

This is called the <u>Klein-Gordon equation</u>. Looking back at (10) and (11) we see that ($c=1$ now)

$$\partial_\nu \partial^\nu \phi = \partial_t^2 \phi - \nabla^2 \phi$$

we also call this the D'Alembertian and use a funny symbol

$$\partial_\nu \partial^\nu \phi = \partial_t^2 \phi - \nabla^2 \phi = \Box \phi \tag{22}$$

Then (21) becomes

$$\boxed{(\Box + m^2)\phi = 0} \tag{23}$$

We will come back later to the solutions of this equation. For now let me just say that this is like a modified wave equation containing also a mass term.

Just like we did in mechanics, we may define the field conjugated to ϕ as

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

To compute it we write $\mathcal{L}$ explicitly

$$\mathcal{L} = \frac{1}{2}\left[(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2\right] - \frac{m^2}{2}\phi^2$$

Then we see that

$$\boxed{\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial_0 \phi} \tag{24}$$

Once we have π, we may define the Hamiltonian as we did in mechanics. Actually, now we have a Hamiltonian density but ok:

$$\mathcal{H} = \pi(\partial_0 \phi) - \mathcal{L}$$

or

$$\mathcal{H} = \pi^2 - \frac{1}{2}\left[(\partial_0 \phi)^2 - (\nabla \phi)^2\right] + \frac{m^2}{2}\phi^2$$

thus

$$\boxed{\mathcal{H} = \frac{1}{2}\left[\pi^2 + (\nabla \phi)^2 + m^2 \phi^2\right]} \tag{25}$$

The Lagrangian treats time and space in equal footing. and is therefore more suited for relativistic problems. The Hamiltonian, on the other hand, sepparates the time component.

In classical field theory most things are done using Lagrangians. Hamiltonians only becomes important when we quantize the theory. Quantization is always done using Hamiltonians.

The Euler - Lagrange equations

Now I want to take a break from the scalar field and develop more general results. We will therefore consider a system described by a series of fields ϕ_n, where $n = 1, 2, \dots$ Nothing will be said about the properties of this field. We will only assume that the Lagrangian depends on ϕ_n and $\partial_\mu \phi_n$. That is, it does not depend on higher order derivatives. So

$$\mathcal{L} = \mathcal{L}(\phi_n, \partial_\mu \phi_n) = \mathcal{L}(\phi_1, \phi_2, \dots, \partial_\mu \phi_1, \partial_\mu \phi_2, \dots)$$

Let us find the equations of motion, the Euler - Lagrange equations. The procedure is exactly the one used to derive (21), except that now we will consider a general Lagrangian.

We modify each field as

$$\phi_n \to \phi_n' = \phi_n + \eta_n$$

where the η_n are independent infinitesimal fields. The Lagrangian correspondingly changes to

$$\mathcal{L}' = \mathcal{L}(\phi_n', \partial_\mu \phi_n') = \mathcal{L}(\phi_n + \eta_n, \partial_\mu \phi_n + \partial_\mu \eta_n)$$

$$\simeq \mathcal{L}(\phi_n, \partial_\mu \phi_n) + \sum_n \left\{ \frac{\partial \mathcal{L}}{\partial \phi_n} \eta_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\mu \eta_n \right\}$$

Note how we are summing over μ in the last term.

the corresponding change in the action is

$$\delta S = \sum_m \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_m} \eta_m + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \eta_m \right\}$$

To put everything in terms of η_m we integrate the last term by parts, and assume that the fields vanish at the boundaries. So

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial_\mu \eta_m \rightsquigarrow -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \right) \eta_m$$

we then get

$$\delta S = \sum_m \int d^4x \, \eta_m \left\{ \frac{\partial \mathcal{L}}{\partial \phi_m} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \right) \right\} \tag{26}$$

Since the η_m are all independent, the stationarity condition $\delta S = 0$ gives

$$\boxed{\frac{\delta S}{\delta \phi_m(x)} = \frac{\partial \mathcal{L}}{\partial \phi_m} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \right) = 0} \tag{27}$$

which are the Euler-Lagrange equations, one for each field ϕ_m.
This result is very general and holds for any Lagrangian which depends at most in the first derivative of the fields. It turns out, however, that in many cases it is easier to find δS explicitly, like we did in (21).

Example : the Schrödinger Lagrangian

Schrödinger's equation reads ($\hbar = 1$)

$$i\,\partial_t \psi = -\frac{1}{2m}\nabla^2\psi + V(x)\psi \qquad (28)$$

the wave-function $\psi(t,x,y,z)$ is a field since it depends on t and r. So we may cook up a Lagrangian whose Euler-Lagrange equations give exactly (28).

Since ψ is in general complex, we actually have two degrees of freedom. this can be treated by assuming that ψ and ψ^* are independent quantities. the correct Lagrangian to reproduce (28) is

$$\mathcal{L} = \frac{i}{2}\left[\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi\right] - \frac{1}{2m}(\nabla\psi^*)\cdot(\nabla\psi) - V\psi^*\psi \qquad (29)$$

To check that this works we apply (27) first to ψ^*:

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{i}{2}\partial_t\psi - V\psi$$

$$\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^*)} = \partial_t\left[-\frac{i}{2}\psi\right] = -\frac{i}{2}\partial_t\psi$$

$$\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\psi^*)} = \partial_i\left[-\frac{1}{2m}\partial_i\psi\right] = -\frac{1}{2m}\partial_i^2\psi$$

So (27) gives

$$\frac{i}{2}\partial_t\psi - V\psi + \frac{i}{2}\partial_t\psi + \frac{1}{2m}\nabla^2\psi = 0$$

which is exactly (28).

If you apply (27) to ψ you will find the equation for ψ^*, which is just the conjugate of (28).

The Lagrangian (29) is real. This is a property we always expect of a Lagrangian. But since what matters the most is the action, it is sometimes convenient to work with Lagrangians which are "real up to an integration by parts". What this means is this: if we integrate by parts the second term in (29) we get

$$-\frac{i}{2}\,(\partial_t \psi^*)\,\psi \;\rightsquigarrow\; \frac{i}{2}\,\psi^*\,\partial_t \psi$$

and similarly for the term with spatial derivatives. We could then work with

$$\mathcal{L} \;=\; i\,\psi^*\,\partial_t \psi \;+\; \frac{1}{2m}\,\psi^*\,(\nabla^2 \psi) \;-\; V\,\psi^*\,\psi \qquad (30)$$

This Lagrangian is not real, but it can be made real with some integrations by part. With this Lagrangian, finding the equations of motion is very easy because no derivatives of ψ^* are present:

$$\frac{\partial \mathcal{L}}{\partial \psi^*} \;=\; i\,\partial_t \psi \;+\; \frac{1}{2m}\,\nabla^2 \psi \;-\; V\,\psi \;=\; 0$$

which gives (28).

However, I have to warn you: when working with symmetries, using non-real Lagrangians can sometimes get you into trouble.

Nöether's theorem

We will now discuss Nöether's theorem, which associates to each symmetry of the system a corresponding conserved quantity.

For instance, we already know from mechanics that invariance under time translations imply the conservation of energy. We want to understand how this is implemented in the case of fields. Indeed, a field also carries energy and if the Lagrangian is invariant under time translations the energy of the field should be a constant of the motion.

Similarly, the quantity associated to invariance under space translations will be the momentum carried by the field. And invariance under rotations will allow us to define the angular momentum of the field.

As in classical mechanics, it suffices to consider infinitesimal transformations. We will therefore consider transformations of the form

$$x^{\nu} \to x'^{\nu} = x^{\nu} + \epsilon^{a} A^{\nu}_{a}(x) \qquad (31)$$

$$\phi_m(x) \to \phi'_m(x') = \phi_m(x) + \epsilon^{a} F_{ma}(\phi)$$

where ϵ^a are a set of small parameters. For instance, in a space-time translation we have

$$x^{\nu} \to x'^{\nu} = x^{\nu} + \epsilon^{\nu} \qquad (32)$$

$$\phi_m(x) \to \phi'_m(x') = \phi_m(x)$$

So in a spacetime translation the index a of ϵ^a is a Lorentz index $a = \nu = 0, 1, 2, 3$ and $A^\nu_a = \delta^\nu_a$. Moreover, in this case $F_{na} = 0$.

We will now demonstrate Nöether's theorem in all generality. This will be a very important step for us. We demonstrate this theorem once and then simply apply it as many times as we want.

But before we start let me tell you what to expect. We will consider the variation δS of the action due to the transformation (31). We will show that this transformation has the form

$$\delta S = -\int d^4x \; \epsilon^a \partial_\nu \theta^\nu_a \tag{33}$$

where ϵ^a are the infinitesimal parameters in (31) and θ^ν_a are a set of quantities which we will determine. If (31) is indeed a symmetry of the system then S should not be affected so we will then have

$$\partial_\nu \theta^\nu_a = 0 \tag{34}$$

The quantities θ^ν_a are called Nöether currents.

The physical meaning of (34) will be discussed later. Now let us first find out what the θ^ν_a are.

Demonstration of Nöether's theorem

The variation of the action due to the infinitesimal transformation (31) is

$$\delta S = \int d^4x' \, \mathcal{L}(\phi'_m(x'), \partial'_\nu \phi'_m(x')) - \int d^4x \, \mathcal{L}(\phi_m(x), \partial_\nu \phi_m(x)) \qquad (35)$$

So we have two things to deal with: d^4x' and $\mathcal{L}'$. The coordinat change according to (31) so

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x \qquad (36)$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the determinant of the Jacobian of the transformation

From (31) we have

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \epsilon^a \partial_\nu A^\mu_a \qquad (37)$$

The determinant is

$$\left|\frac{\partial x'}{\partial x}\right| = \begin{vmatrix} 1 + \epsilon^a \partial_0 A^0_a & \epsilon^a \partial_0 A^1_a & \epsilon^a \partial_0 A^2_a & \cdots \\ \epsilon^a \partial_1 A^0_a & 1 + \epsilon^a \partial_1 A^1_a & \epsilon^a \partial_1 A^2_a & \cdots \\ \vdots & \vdots & & \ddots \end{vmatrix}$$

We need to compute the determinant of this big matrix. But lucky for us, we are only interested in terms which are of first order in ϵ^a. Of the many products in this determinant, most will be of order $(\epsilon^a)^2$ or higher. In fact, the only term which will be linear in ϵ^a will come from the product of the diagonal elements

It will be:

$$(1+\epsilon^a \partial_0 A^0_a)(1+\epsilon^b \partial_1 A^1_b)(1+\epsilon^c \partial_2 A^2_c)(1+\epsilon^d \partial_3 A^3_d)$$

$$= 1 + \epsilon^a(\partial_0 A^0_a + \partial_1 A^1_a + \partial_2 A^2_a + \partial_3 A^3_a)$$

Hence, we conclude that, to first order in ϵ^a,

$$d^4x' = (1+\epsilon^a \partial_\mu A^\mu_a)\, d^4x \qquad (38)$$

Good. This is the first part of the expansion. Now we look at $\mathcal{L}'$. It will have terms like $\partial'_\mu \phi'_m(x')$ which we need to deal with:

$$\partial'_\mu \phi'_m(x') = \frac{\partial}{\partial x'^\mu}\left[\phi_m(x) + \epsilon^a F_{ma}\right]$$
$$= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}\left[\phi_m + \epsilon^a F_{ma}\right] \qquad (39)$$

The first term is the inverse of (37). We can compute it as follows.
First we write (31) as

$$x^\nu = x'^\nu - \epsilon^a A^\nu_a(x)$$

Then

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \epsilon^a \frac{\partial A^\nu_a}{\partial x'^\mu}$$

$$= \delta^\nu_\mu - \epsilon^a \frac{\partial A^\nu_a}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\mu}$$

We now use the equation in itself:

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu_\mu - \epsilon^a \frac{\partial A^\nu_a}{\partial x^\alpha}\left[\delta^\alpha_\mu - \epsilon^b \frac{\partial A^\alpha_b}{\partial x'^\mu}\right]$$

when we expand this and keep only those terms which are linear in ϵ^a, we get

$$\frac{\partial x^\nu}{\partial x'^\mu} = \delta_\mu^\nu - \epsilon^a \partial_\mu A_a^\nu \qquad (40)$$

So to first order, we may invert (37) by simply changing a minus sign. Eq (39) then becomes

$$\partial'_\mu \phi'_m(x') = \left[\delta_\mu^\nu - \epsilon^a \partial_\mu A_a^\nu\right]\left[\partial_\nu \phi_m + \epsilon^a \partial_\nu F_{ma}\right]$$

$$= \partial_\mu \phi_m + \epsilon^a \left(\partial_\mu F_{ma} - (\partial_\nu \phi_m)(\partial_\mu A_a^\nu)\right)$$

Thus

$$\mathcal{L}' = \mathcal{L}\left(\phi_m + \epsilon^a F_{ma},\ \partial_\mu \phi_m + \epsilon^a\left(\partial_\mu F_{ma} - (\partial_\nu \phi_m)(\partial_\mu A_a^\nu)\right)\right)$$

This now has the form $f(x+\Delta x, y+\Delta y)$ so we may expand it in a Taylor series:

$$\mathcal{L}' \simeq \mathcal{L} + \sum_m \left\{ \frac{\partial \mathcal{L}}{\partial \phi_m} \epsilon^a F_{ma} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \epsilon^a \left[\partial_\mu F_{ma} - \partial_\nu \phi_m \partial_\mu A_a^\nu\right]\right\} \qquad (41)$$

$$= \mathcal{L} + \delta \mathcal{L}.$$

We may use the Euler-Lagrange equations to bundle together the first two terms.

$$\frac{\partial \mathcal{L}}{\partial \phi_m} \epsilon^a F_{ma} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \epsilon^a \partial_\mu F_{ma} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)}\right) \epsilon^a F_{ma} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \epsilon^a \partial_\mu F_{ma}$$

$$= \epsilon^a \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} F_{ma}\right]$$

Thus we get

$$\delta \mathcal{L} = \sum_m \left\{ \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \epsilon^a F_{ma} \right] - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \epsilon^a \partial_\nu \phi_m \partial_\nu A^\nu_a \right\} \quad (42)$$

The variation of the action (35) may now be written as

$$\delta S = \int d^4x' \, \mathcal{L}' - \int d^4x \, \mathcal{L}$$

$$= \int d^4x \, (1 + \epsilon^a \partial_\nu A^\nu_a)(\mathcal{L} + \delta \mathcal{L}) - \int d^4x \, \mathcal{L}$$

$$= \int d^4x \left[\epsilon^a (\partial_\nu A^\nu_a) \mathcal{L} + \delta \mathcal{L} \right]$$

where I used the fact that $\delta \mathcal{L} \propto \epsilon$ to discard one term. Let us write everything together:

$$\delta S = \epsilon^a \int d^4x \left\{ (\partial_\nu A^\nu_a) \mathcal{L} + \sum_m \left[\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} F_{ma} \right) - \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\nu \phi_m \partial_\nu A^\nu_a \right] \right\} \quad (43)$$

The second term is already in the form $\partial_\nu(\cdots)$ which is what we want (see (33)). So now we have to massage the first and third terms. Let us first write

$$(\partial_\nu A^\nu_a) \mathcal{L} = \partial_\nu (A^\nu_a \mathcal{L}) - A^\nu_a (\partial_\nu \mathcal{L}) \quad (44)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\nu \phi_m \partial_\nu A^\nu_a = \partial_\nu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\nu \phi_m A^\nu_a \right\} + - \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\nu \phi_m \right] A^\nu_a \quad (44')$$

The first term in each formula has the shape $\partial_\mu(\cdots)$, which is what we want. Lucky for us, the remaining terms will cancel out exactly. To see this we expand:

$$\partial_\mu \mathcal{L} = \sum_m \frac{\partial \mathcal{L}}{\partial \phi_m} \partial_\mu \phi_m + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\mu \partial_\nu \phi_m$$

and use the Euler-Lagrange equations again to write

$$\partial_\mu \mathcal{L} = \sum_m \left\{ \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \right) \partial_\mu \phi_m + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\mu \partial_\nu \phi_m \right\}$$

$$= \partial_\nu \sum_m \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\mu \phi_m \right\}$$

Combining (44) and (44') we then get

$$(\partial_\mu A^\mu_a)\mathcal{L} - \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial_\nu \phi_m \partial_\mu A^\nu_a = \partial_\mu \left\{ \mathcal{L} A^\mu_a - \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial_\nu \phi_m A^\nu_a \right\}$$

$$- A^\mu_a \partial_\mu \mathcal{L} + \sum_m \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial_\nu \phi_m \right] A^\nu_a$$

$$= \partial_\mu \left\{ \mathcal{L} A^\mu_a - \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial_\nu \phi_m A^\nu_a \right\}$$

$$- A^\mu_a \partial_\nu \sum_m \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_m)} \partial_\mu \phi_m \right\} + A^\nu_a \partial_\mu \sum_m \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial_\nu \phi_m \right\}$$

The last two terms look different but they are not. Since μ and ν are both being summed, we may exchange their orders in one of the terms, which will then become equal to the other. Thus, the last two terms indeed cancel.

Eq (43) then becomes

$$\delta S = \epsilon^a \int d^4x\, \partial_\mu \left\{ \mathcal{L} A^\mu_a + \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \left[F_{ma} - (\partial_\nu \phi_m) A^\nu_a \right] \right\}$$

we now define

$$\boxed{\Theta^\mu_a = \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \left[(\partial_\nu \phi_m) A^\nu_a - F_{ma} \right] - \mathcal{L} A^\mu_a} \qquad (45)$$

with which we get

$$\delta S = -\epsilon^a \int d^4x\, \partial_\mu \Theta^\mu_a \qquad (46)$$

If our transformation is indeed a symmetry of the system, then

$$\partial_\mu \Theta^\mu_a = 0 \quad \text{for each } a. \qquad (47)$$

So the Θ^μ_a are the Nöether currents.

Congratulations. You survived. Now you can go drink that coffee you wanted so badly. Don't worry. Go ahead. You deserve it.

Example: Schrödinger's probability current

In order to understand the meaning of (47), it is useful to look at an example. The Schrödinger Lagrangian (29)

$$\mathcal{L} = \frac{i}{2}\left[\psi^* \partial_t \psi - (\partial_t \psi^*)\psi\right] - \frac{1}{2m}(\nabla\psi^*)(\nabla\psi) - V\psi^*\psi$$

has the property of being invariant under the phase transformation

$$\psi \to e^{i\theta}\psi \qquad \psi^* \to e^{-i\theta}\psi^* \tag{48}$$

we may use Nöether's theorem to find the corresponding conserved quantity. First we change to an infinitesimal transformation:

$$\psi \to \psi + i\theta\psi \qquad \psi^* \to \psi - i\theta\psi^* \tag{49}$$

So there is only one parameter, $\epsilon^a = \theta$ $(a=1)$, and $A^\nu_a = 0$ and

$$F_1 = i\psi \qquad F_2 = -i\psi^*$$

Here I am using $\phi_1 = \psi$ and $\phi_2 = \psi^*$

The Nöether current (45) then becomes

$$\Theta^\nu = -\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)}(i\psi) - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi^*)}(-i\psi^*)$$

$$= -i\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi)}\psi - \frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi^*)}\psi^*\right\}$$

For $\nu = 0$:

$$\theta^0 = -i\left\{ \left(\frac{i}{2}\psi^*\right)\psi - \psi^*\left(-\frac{i}{2}\psi\right) \right\}$$

or

$$\theta^0 = |\psi|^2 \tag{50}$$

For $\nu = i$:

$$\theta^i = -i\left\{ -\frac{1}{2m}(\partial_i \psi^*)\psi + \frac{1}{2m}\psi^*(\partial_i \psi) \right\}$$

Eq (47) then means that

$$\partial_0 \theta^0 + \partial_i \theta^i = 0 \tag{51}$$

Let

$$\vec{J} = \frac{i}{2m}\left[(\nabla\psi^*)\psi - \psi^*(\nabla\psi) \right] \tag{52}$$

then we get

$$\frac{d}{dt}|\psi|^2 + \nabla\cdot\vec{J} = 0 \tag{53}$$

this is a <u>continuity equation</u>. To understand what it means, integrate it over a certain volume V in space

$$\frac{d}{dt}\int_V d^3x\, |\psi|^2 = -\int_V d^3x\, \nabla\cdot\vec{J}$$

According to the divergence theorem

$$\int_V d^3x\, \nabla\cdot J = \int_{\partial V} d\vec{\sigma}\cdot\vec{J}$$

where the integral is to be taken over the surface bounding the volume V

We then obtain

$$\boxed{\frac{d}{dt}\int_V d^3x\, |\psi|^2 = -\int_{\partial V} d\vec{\sigma}\cdot\vec{J}} \tag{54}$$

This equation has a beautiful physical interpretation. The left-hand side is the rate of change of the probability of finding the system in the volume V. And the right-hand side is the flow of probability through the surface bounding V.

We therefore conclude that the symmetry of the Schrödinger Lagrangian under Global phase transformations, $\psi \to e^{i\theta}\psi$, implies the conservation of the probability density. This is Nöether's theorem at it's finest!

If we extend the volume V to encompass all space, and if we assume that $\psi \to 0$ at infinity, then the right-hand side of (54) vanishes and we are left with

$$\frac{dP}{dt} = 0 \quad , \quad P = \int_{\substack{\text{all}\\ \text{space}}} d^3x\, |\psi|^2 \tag{55}$$

Hence, the probability of finding the electron somewhere is a conserved quantity: the electron never disappears.

This reasoning is also valid for any conserved quantity obtained from Nöether's theorem. Eq (47) reads

$$\frac{d\theta^0_a}{dt} + \nabla\cdot\vec{\theta}_a = 0 \tag{56}$$

So from the four-vector θ^ν_a (for each a), we define the <u>conserved</u> <u>charge</u>

$$q_a = \int d^3x\, \theta^0_a \tag{57}$$

This quantity satisfies

$$\frac{dq_a}{dt} = 0 \tag{58}$$

ie, it is independent of time / a constant of the motion.

The quantity θ^0_a is therefore a charge density, whereas θ^i_a is a <u>flux</u> <u>density</u>. It describes the flux of the conserved quantity.

For all these reasons, q_a plays a very special role.

<u>Summary</u>:

$$\boxed{\begin{aligned} x^\nu &\to x^\nu + \epsilon^a A^\nu_a \\ \phi_m &\to \phi_m + \epsilon^a F_{ma} \\ \theta^\nu_a &= \sum_m \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_m)}\left[(\partial_\nu\phi_m)A^\nu_a - F_{ma}\right] - \mathcal{L}A^\nu_a \\ \partial_\nu\theta^\nu_a &= 0 \\ q_a &= \int d^3x\,\theta^0_a \qquad (\dot{q}_a = 0) \end{aligned}} \tag{59}$$

The energy-momentum tensor

Let us now consider spacetime translations:

$$
\begin{aligned}
x^\nu &\to x^\nu + \epsilon^\nu \\
\phi_m(x) &\to \phi_m(x)
\end{aligned}
\qquad (60)
$$

Looking at (59) we see that $a = \nu$, $F_{ma} = 0$ and $A^\nu{}_a = \delta^\nu{}_a$.
Using $a = \nu$ we obtain the energy-momentum tensor:

$$
\boxed{T^\mu{}_\nu = \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} (\partial_\nu \phi_m) - \delta^\mu{}_\nu \mathcal{L}} \qquad (61)
$$

It is customary to write it with both indices up:

$$
\boxed{T^{\mu\nu} = \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_m)} \partial^\nu \phi_m - \eta^{\mu\nu} \mathcal{L}} \qquad (62)
$$

where I used the fact that $\delta^{\mu\nu} = \eta^{\mu\nu}$ is the Minkowski metric.

The indices μ and ν have very different meanings. The index μ is such that $\mu = 0$ means a charge and $\mu = i$ means a current. As for ν, $\nu = 0$ means a time translation and $\nu = i$ is a space translation.

So the energy density is

$$
T^{00} = \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_m)} \partial^0 \phi_m - \mathcal{L} \qquad (63)
$$

This makes sense because

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_m)} = \pi_m \tag{64}$$

is the field conjugated to ϕ_m. We then get

$$T^{00} = \sum_m \pi_m \partial^0 \phi_m - \mathcal{L} \tag{65}$$

which is the formula for the energy density / Hamiltonian that we already know from classical mechanics.

The total energy of the system is

$$E = \int d^3x \, T^{00} \tag{66}$$

and it is a constant of the motion: $dE/dt = 0$. The quantities T^{i0} represent the energy flux of the system. But T^{0i} are the momentum densities

$$p^i = T^{0i} = \sum_m \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_m)} \partial^i \phi_m \tag{67}$$

the total momentum of the field is

$$\underline{P}^i = \int d^3x \, p^i \tag{68}$$

and it is also a constant of the motion, $\dot{\underline{P}}^i = 0$.

Example 1 : scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi)(\partial^\nu \phi) - \frac{m^2}{2}\phi^2 \tag{69}$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\partial^0 \phi - \mathcal{L} = (\partial^0 \phi)^2 - \mathcal{L}$$

$$T^{00} = \frac{1}{2}(\partial^0 \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{m^2}{2}\phi^2 \tag{70}$$

the total energy is

$$E = \int d^3x \left[\frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{m^2}{2}\phi^2 \right] \tag{71}$$

If we write it in terms of $\pi = \partial_0 \phi$ we may call it a Hamiltonian

$$H = \int d^3x \left[\frac{\pi^2}{2} + \frac{(\nabla \phi)^2}{2} + \frac{m^2}{2}\phi^2 \right] \tag{72}$$

The momentum density is

$$p^i = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\partial^i \phi = \pi \partial^i \phi \tag{73}$$

since $\partial^i = (-\partial x, -\partial y, -\partial z)$ we obtain

$$P^i = \int d^3x \; \pi \partial^i \phi \tag{74}$$

or

$$\vec{P} = \int d^3x \; \pi \nabla \phi \tag{74'}$$

or the momentum of the field.

Example: Schrödinger Lagrangian

$$\mathcal{L} = \frac{i}{2}\left[\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi\right] - \frac{1}{2m}(\nabla\psi^*)(\nabla\psi) - V\psi^*\psi$$

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\partial^0\psi + \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^*)}\partial^0\psi^* - \mathcal{L}$$

$$= \frac{1}{2m}(\nabla\psi)\cdot(\nabla\psi) + V\psi^*\psi$$

the energy is then

$$E = \int d^3x \left[\frac{1}{2m}(\nabla\psi^*)\cdot(\nabla\psi) + V\psi^*\psi\right] \tag{75}$$

the field conjugated to ψ is

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \frac{i}{2}\psi^*$$

so (75) is already the Hamiltonian because it is written in terms of ψ and its conjugated field. To write (75) in a way which may be more familiar, integrate the first term by parts

$$H = \int d^3x\, \psi^*\left\{-\frac{1}{2m}\nabla^2 + V(x)\right\}\psi \tag{76}$$

the energy is only conserved if V is independent of time

the momentum is

$$p^i = \frac{i}{2}\psi^*(\partial^i\psi) - \frac{i}{2}\psi(\partial^i\psi^*)$$

so

$$\vec{P} = \int d^3x\, \psi^*(-i\nabla)\psi \tag{77}$$

where I bundled the two terms integrating by parts. The momentum is only conserved if V is independent of r.

Angular momentum

Now let us consider the conserved quantity associated to rotations in $\mathbb{R}^3$. It is simpler to consider a specific rotation around the z axis and through an infinitesimal angle ϵ:

$$\begin{aligned} x^0 &\to x^0 \\ x^1 &\to x^1 \cos\epsilon - x^2 \sin\epsilon \simeq x^1 - x^2\epsilon \\ x^2 &\to x^1 \sin\epsilon + x^2\cos\epsilon \simeq x^1\epsilon + x^2 \\ x^3 &\to x^3 \end{aligned} \tag{67}$$

Comparing with (53) we see that $a=3$, there is only one parameter, and

$$A^0_a = 0 \quad A^1_a = -x^2 \quad A^2_a = x^1 \quad A^3_a = 0. \tag{68}$$

But now we must know how the field transforms under rotations. This will determine the F_{ia} in (53). For the scalar field, since it is a scalar, it does not change under rotations so $F=0$.

Eq (54) then gives

$$\begin{aligned} m^\nu_a &= \frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi)} (\partial_\gamma\phi) A^\gamma_a - \mathcal{L} A^\nu_a \\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi)} \left[(\partial_2\phi) x^1 - (\partial_1\phi) x^2 \right] - \mathcal{L} A^\nu_a \end{aligned}$$

As before we want only $\nu = 0$, which will give the conserved charge. Then the last term is zero and we get

$$m^0_a = \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} (\partial_2 \phi) \right] x^1 - \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} (\partial_1 \phi) \right] x^2$$

Looking at (64) we identify the momentum density P_i.

Thus

$$m^0_3 = P_2 x^1 - P_1 x^2 = p^1 x^2 - p^2 x^1$$

We define the angular momentum as

$$M^3 = \int d^3x \; m^{03} = \int d^3x \, (x^1 p^2 - x^2 p^1)$$

(since $m^{03} = -m^0_3$). Thus we see that in general we will have

$$\boxed{\vec{M} = \int d^3x \, (\vec{x} \times \vec{p})} \qquad (69)$$

This is the <u>orbital angular momentum</u> since it depends on the particular choice of coordinate axis. We therefore see that the scalar field has no spin. This is due to the fact that $F_{ia} = 0$. When we consider other fields which are not invariant under rotations we will have the angular part (69) plus another part which we will be able to identify as the spin component. The spin will therefore appear as a consequence of the internal structure of the fields.