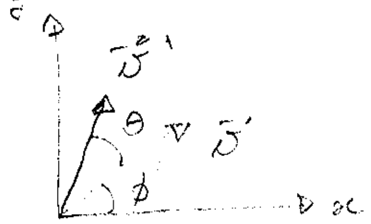
The relation between angular momentum and rotations

We began our study of angular momentum by saying that it was intimately related to rotations. However, this relation may still not be so clear because rotations are things which happen in $\mathbb{R}^3$, the space we live in. But angular momentum lives in the Hilbert space. So how are the two connected?

In order to understand that we first need to understand rotations in the real world.

Rotations in $\mathbb{R}^2$

Suppose you have a vector $\vec{v} = (v_x, v_y)$ in a plane and you want to rotate it by an angle θ.



Here is how you do it. First write it as

$$\begin{aligned} v_x &= v\cos\phi \\ v_y &= v\sin\phi \end{aligned} \qquad (1)$$

where

$$v = \sqrt{v_x^2 + v_y^2}$$

Rotations are always in the counter-clockwise direction.

when you rotate it by an angle θ you don't change its length v. Thus the components $\vec{v}' = (v_x', v_y')$ of the rotated vector will be

$$v_x' = v \cos(\phi + \theta) \qquad (2)$$
$$v_y' = v \sin(\phi + \theta)$$

Expanding the trigonometric quantities we get

$$v_x' = v \cos\phi \cos\theta - v \sin\phi \sin\theta \qquad (3)$$
$$v_y' = v \cos\phi \sin\theta + v \sin\phi \cos\theta$$

But $v\cos\phi = v_x$ and $v \sin\phi = v_y$. Thus we conclude that

$$\boxed{\begin{aligned} v_x' &= v_x \cos\theta - v_y \sin\theta \\ v_y' &= v_x \sin\theta + v_y \cos\theta \end{aligned}} \qquad (4)$$

We may also write this in matrix notation

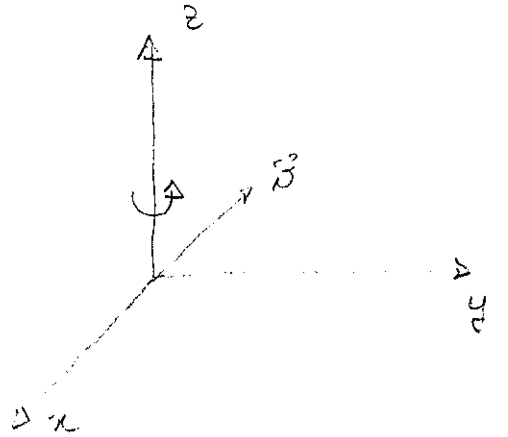
$$\boxed{\vec{v}' = \begin{bmatrix} v_x' \\ v_y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = R\,\vec{v}} \qquad (5)$$

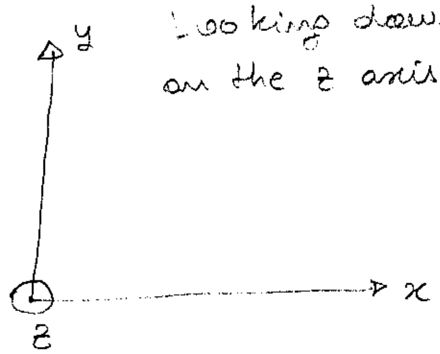
Rotation is a linear operation! To obtain the new $\vec{v}'$ from the old $\vec{v}$ simply multiply by the rotation matrix R.

Rotations in $\mathbb{R}^3$

Rotations in 3D space are more difficult because we need to specify the axis around which we are rotating. The trick is to try and reduce the problem to that of $\mathbb{R}^2$

Rotations around the z axis





Looking down on the z axis.

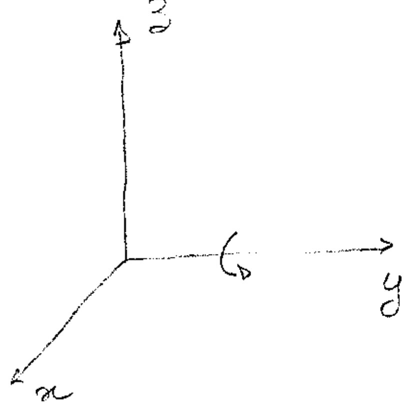
If we look down on the z axis, we see that this is just like a rotation in $\mathbb{R}^2$ for (x, y).

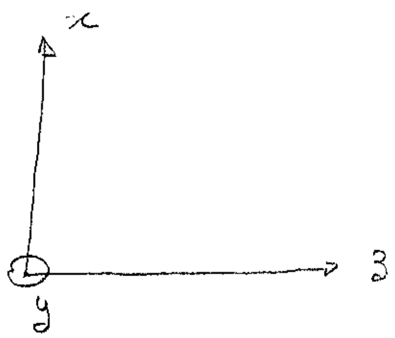
$$
\begin{aligned}
v_x' &= v_x \cos\theta - v_y \sin\theta \\
v_y' &= v_x \sin\theta + v_y \cos\theta \\
v_z' &= v_z
\end{aligned}
\qquad (6)
$$

or

$$
R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (7)
$$

Rotations around the y axis





This is the same as before, but with cyclic permutations

$$x \to z$$
$$y \to x$$
$$z \to y$$

Thus

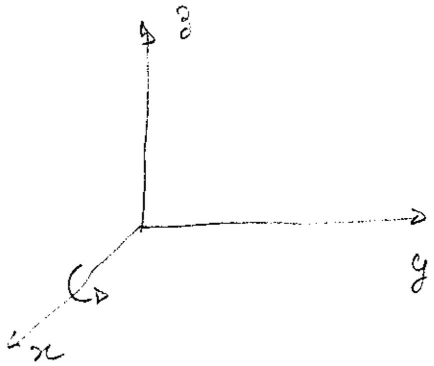
$$v_z' = v_z \cos\theta - v_x \sin\theta$$
$$v_x' = v_z \sin\theta + v_x \cos\theta \qquad (8)$$
$$v_y' = v_y$$

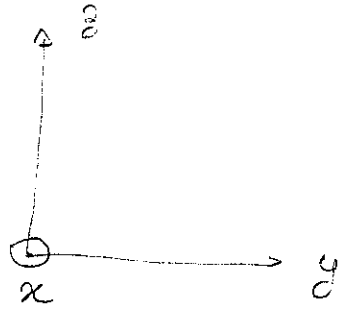
or

$$\hat{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \qquad (9)$$

Note how the $(-\sin\theta)$ is in a position different from Eq (7)

Rotations around the x axis





Now we have

$$
\begin{aligned}
v_y' &= v_y \cos\theta - v_z \sin\theta \\
v_z' &= v_y \sin\theta + v_z \cos\theta \qquad (10)\\
v_x' &= v_x
\end{aligned}
$$

or

$$
R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad (11)
$$

—//—

The rotation matrices R_x, R_y, R_z are orthogonal this is the same as unitary except that they are real):

$$
\boxed{R_\mu R_\mu^T = R_\mu^T R_\mu = 1} \qquad (12)
$$

$$\mu = x, y, z$$

[Since they are real, $R^+ = R^T$].

In $\mathbb{R}^3$ there is something you should know:

$$\boxed{\text{Rotations around different axes do not commute}}$$

I will leave to you as an exercise to show that

$$\boxed{R_z(\theta_1)\, R_z(\theta_2) = R_z(\theta_2)\, R_z(\theta_1) = R_z(\theta_1 + \theta_2)} \qquad (13$$

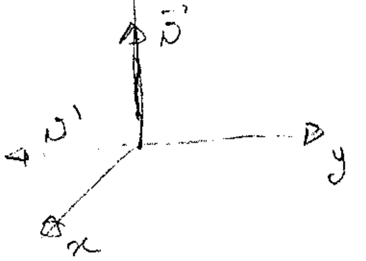
but

$$\boxed{R_z(\theta_1)\, R_x(\theta_2) \neq R_x(\theta_2)\, R_z(\theta_1)} \qquad (14)$$

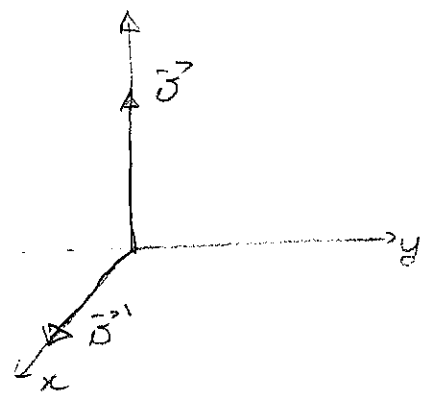
thus, two rotations around the same axis is the same as a simple rotation, but with a larger angle. However, when you make two rotations around different axis, the order is very important.

For instance, suppose you rotate the vector $\vec{v} = (0, 0, 1)$ around the z axis by $\pi/2$, then around the x axis by $\pi/2$

$R_x(\pi/2)\, R_z(\pi/2)\, \vec{v}$



$R_z(\pi/2)\, R_x(\pi/2)\, \vec{v}$



Generators of rotation

We know that unitary matrices can always be written as

$$U = e^{-iK}$$

Thus, since the R_p are orthogonal (unitary), I want to try and write them as

$$R_p = e^{-i\theta G_p} \qquad (15)$$

where G_p is a Hermitian matrix. What are the G_p? I will show you how to find G_z. The procedure for G_x and G_y is identical.

The idea is to expand Eq (15) and Eq (7) in a power series assuming an infinitesimal rotation angle θ. Eq (15) becomes

$$R_p \approx 1 - i\theta G_p \qquad (16)$$

and Eq (7) becomes

$$R_z \approx \begin{bmatrix} 1-\theta^2/2 & -\theta & 0 \\ \theta & 1-\theta^2/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 + \theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathcal{O}(\theta)^2 \qquad (17)$$

We need to compare terms of the same order in θ. Thus $\theta^2/2$ is not important here. We therefore conclude that

$$G_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{18}$$

Similarly, for G_x and G_y we find

$$G_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \tag{19}$$

$$G_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} \tag{20}$$

If you want, you can use the Mathematica notebook to check that Eq (15) indeed works

Next we verify the commutation relations of the G_s :

$$G_x G_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$G_y G_x = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad [G_x, G_y] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Oh my god! this is crazy. the G obey the algebra of angular momentum!

$$\boxed{[G_x, G_y] = i\, G_z} \qquad (21)$$

I'm being ironic. this is no coincidence.

Rotations around an arbitrary direction

To perform a more general rotation we need to specify, besides the angle θ, the direction around which we are rotating. This can be done by specifying a unit vector $\vec{n}$ ($|\vec{n}|=1$).

Now I claim that the rotation matrix which rotates a vector by an angle θ around a direction $\vec{n}$ is

$$\boxed{R(\vec{n},\theta) = e^{-i\theta\,\vec{n}\cdot\vec{G}}} \qquad \vec{G} = (G_x, G_y, G_z) \tag{22}$$

This of course makes sense because it reduces to R_x, R_y and R_z as particular cases.

To understand this better it is convenient to study the case of infinitesimal rotations $\delta\theta \ll 1$. In this case

$$R(\vec{n},\delta\theta) \simeq 1 - i\,\delta\theta\,\vec{n}\cdot\vec{G} \tag{23}$$

Well:

$$\vec{n}\cdot\vec{G} = n_x \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} + n_y \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix} + n_z \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\vec{n}\cdot\vec{G} = i \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}} \tag{24}$$

Now I want to show you something very interesting

Let us apply this infinitesimal rotation to a vector:

$$(-i\vec{n}\cdot\vec{G})\,\vec{v} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} n_y v_z - n_z v_y \\ n_z v_x - n_x v_z \\ n_x v_y - n_y v_x \end{bmatrix}$$

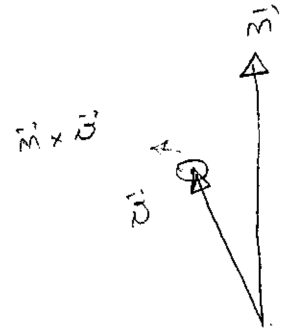
This is the vector product!

$$\boxed{(-i\vec{n}\cdot\vec{G})\,\vec{v} = \vec{n}\times\vec{v}} \qquad (25)$$

Therefore, in an infinitesimal rotation

$$\boxed{\vec{v}\,' = R(\vec{n}, \delta\theta)\,\vec{v} \simeq \vec{v} + \delta\theta\,(\vec{n}\times\vec{v})} \qquad (26$$

This is exactly what we expect from an infinitesimal rotation.



Since a finite rotation can be broken into a series of infinitesimal rotations, Eq (26) can be considered a demonstration of Eq (22).

An explicit formula for $R(\vec{m}, \theta)$

The calculation I am going to do now is a bit more difficult. I want to obtain an explicit formula for $R(\vec{m}, \theta)$ in Eq (22). To do this let

$$F = -i\,\vec{m}\cdot\vec{G} = \begin{bmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{bmatrix} \qquad (27)$$

Now we look at F^2.

$$F^2 = \begin{bmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -m_z^2 - m_y^2 & m_x m_y & m_x m_z \\ m_x m_y & -m_z^2 - m_x^2 & m_y m_z \\ m_x m_z & m_y m_z & -m_x^2 - m_y^2 \end{bmatrix}$$

Since $m_x^2 + m_y^2 + m_z^2 = 1$ we may write

$$F^2 = \begin{bmatrix} m_x^2 & m_x m_y & m_x m_z \\ m_y m_x & m_y^2 & m_y m_z \\ m_z m_y & m_z m_x & m_z^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first matrix may be written as an outer product

$$\vec{m}\,\vec{m}^T = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \begin{bmatrix} m_x & m_y & m_z \end{bmatrix} = \begin{bmatrix} m_x m_x & m_x m_y & m_x m_z \\ m_y m_x & m_y m_y & m_y m_z \\ m_z m_x & m_z m_y & m_z m_z \end{bmatrix}$$

Whence

$$F^2 = \vec{m}\,\vec{m}^T - \mathbb{1} \qquad (28)$$

Next we look at F^3:

$$F^3 = F\,F^2 = F\,(\vec{m}\cdot\vec{m}^T) - \vec{F}$$

$$= (F\,\vec{m})\cdot\vec{m}^T - \vec{F}$$

But we already saw that $F\,\vec{v} = \vec{m} \times \vec{v}$ so that

$$\vec{F}\,\vec{m} = \vec{m} \times \vec{m} = 0$$

The cycle therefore resets:

$$F^3 = -F \qquad (29)$$

This is all we need to find $R(\vec{n}, \theta)$:

$$R(\vec{n}, \theta) = e^{-i\theta \vec{n}\cdot\vec{G}} = e^{\theta F}$$

$$= 1 + \theta F + \frac{\theta^2 F^2}{2!} + \frac{\theta^3 F^3}{3!} + \frac{\theta^4 F^4}{4!} + \dots$$

$$= 1 + \theta F + \frac{\theta^2 F^2}{2!} - \frac{\theta^3}{3!} F - \frac{\theta^4}{4!} F^2 + \dots$$

$$= 1 + F \underbrace{\left[\theta - \frac{\theta^3}{3!} + \dots\right]}_{\sin\theta} + F^2 \underbrace{\left[\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right]}_{1-\cos\theta}$$

whence

$$\boxed{R(\vec{n}, \theta) = 1 + F\sin\theta + F^2(1-\cos\theta)} \qquad (30)$$

This gives a general formula for R. Most of the times we want a formula for R applied to some vector $\vec{v}$.

We then have

$$F\vec{v} = \vec{n} \times \vec{v}$$

$$F^2\vec{v} = (\vec{n}\vec{n}^T - 1)\vec{v} = \vec{n}(\vec{n}\cdot\vec{v}) - \vec{v}$$

Thus

$$R(\vec{n}, \theta)\vec{v} = \vec{v} + (\vec{n} \times \vec{v})\sin\theta + [\vec{n}(\vec{n}\cdot\vec{v}) - \vec{v}](1-\cos\theta)$$

Or

$$\boxed{R(\vec{n},\theta)\,\vec{v} = \vec{v}\cos\theta + (\vec{n}\times\vec{v})\sin\theta + \vec{n}\,(\vec{n}\cdot\vec{v})(1-\cos\theta)} \quad (3$$

Alternatively we can recall that

$$F^2\vec{v} = F(\vec{n}\times\vec{v}) = \vec{n}\times(\vec{n}\times\vec{v}) \quad (32)$$

then

$$\boxed{R(\vec{n},\theta) = \vec{v} + (\vec{n}\times\vec{v})\sin\theta + \vec{n}\times(\vec{n}\times\vec{v})(1-\cos\theta)} \quad (3$$

Given 2 vectors $\vec{v}$ and $\vec{n}$, we may form a basis for $\mathbb{R}^3$ with $\vec{v}$, $\vec{n}\times\vec{v}$ and $\vec{n}\times(\vec{n}\times\vec{v})$. Eq (33) writes the rotated vector in this basis

Rotations from $\vec{n}_1$ to $\vec{n}_2$

What is the rotation matrix R which rotates a unit vector $\vec{n}_1$ to another unit vector $\vec{n}_2$? That is,

$$R\,\vec{n}_1 = \vec{n}_2$$

Answer: $R = R(\vec{n}_{12}, \theta_{12})$ where

$$\boxed{\begin{aligned} \vec{n}_{12} &= \frac{\vec{n}_1\times\vec{n}_2}{|\vec{n}_1\times\vec{n}_2|} \\ \theta_{12} &= \arccos(\vec{n}_1\cdot\vec{n}_2) \end{aligned}} \quad (34)$$

Introduction to group theory

What we have just done so far is a part of a more general theory called group theory. This is a very powerful theory and is intimately related to symmetries.

Definition of a group: a group g consists in a set of objects A such that

1) If $A \in g$ then there exists an inverse A^{-1} also in g such that $A A^{-1} = 1$ (the inverse is a member of the group)

2) If $A_1 \in g$ and $A_2 \in g$ then $A_1 A_2 \in g$ (products are also members of the group)

When I say "object" you may very well read "matrix" because we will mostly be interested in matrices. But some times people also use different objects and in these cases some additional properties may be necessary [for instance, associativity $A_1 (A_2 A_3) = (A_1 A_2) A_3$]

The SO(3) group

The matrices $R(\vec{n}, \theta)$ of rotations in $\mathbb{R}^3$ form a group called the SO(3) group. They clearly have an inverse because to go back you simply rotate by an angle $-\theta$:

$$[R(\vec{n}, \theta)]^{-1} = R(\vec{n}, -\theta) \qquad (35)$$

Thus, for any rotation, the inverse rotation is also a member of the group. Moreover, the product of two rotations is also a rotation.

The name SO(3) means the following :

- 3 is the dimension of the matrices
- "O" means orthogonal because the R are orthogonal matrices
- "S" means special because this corresponds to pure rotations.

There is also a group called O(3), which is more general. Besides rotations it also contains <u>reflections</u>. A rotation matrix has

$$\det(R) = 1$$

If reflections are involved then

$$\det(A) = -1$$

In our case we may check that the R correspond to rotations using the formula

$$\det(e^{\alpha A}) = e^{\alpha \, tr(A)}$$

A short dictionary to groups

- when the group depends continuously on certain parameters (like θ and $\vec{n}$ in our case) we say this is a Lie group.

- when we write the elements of the group as $R = e^{-i\theta \vec{n}\cdot\vec{G}}$ we say the matrices $\vec{G}$ are the generators of the group.

- these generators satisfy certain commutation relations which we call the Lie algebra of the group. In general these commutation relations may be written as

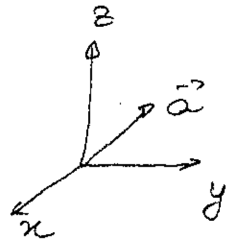
$$[G_i, G_j] = \sum_k f_{ijk} G_k \tag{36}$$

the f_{ijk} are coefficients called the structure constants of the group.

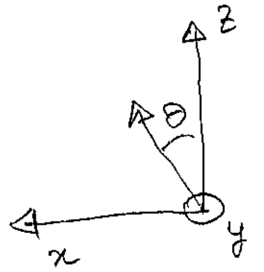
- If the f_{ijk} are all zero then the generators will commute: $[G_i, G_j] = 0$. In this case we say the group is Abelian. Abelian groups are easier to work with (example: the linear momenta are the generators of the group of translations) Otherwise the group is called non-Abelian (e.g. Angular momentum).

Suppose you start in $\mathbb{R}^3$ with the vector $(0,0,1)$, and you wish to rotate it to get an arbitrary vector $\vec{a}$ parametrized as

$$a_x = \sin\theta\cos\phi$$
$$a_y = \sin\theta\sin\phi \qquad (37)$$
$$a_z = \cos\theta$$



This can be done as follows. First you rotate around the y axis by an angle θ. This will produce a vector in the xz plane



Then you rotate around the z axis by an angle ϕ.

Thus, if $\vec{v} = (0,0,1)$

$$\vec{a} = R_z(\phi)\, R_y(\theta)\, \vec{v} \qquad (38)$$

Let's check this

$$R_y(\theta)\,\vec{v} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

$$R_z(\phi)\, R_y(\theta)\, \vec{v} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{bmatrix}$$

Yay! It works.

Angular momentum is the generator of rotations

Suppose we rotate our physical system in $\mathbb{R}^3$. Here I mean an actual rotation in the real world. Somehow this rotation must be represented by an operator in Hilbert space. The situation is similar to translations, where the operator was

$$\hat{T} = e^{-i\hat{p}a/\hbar}$$

Thus, given a certain rotation $R(\theta, \vec{n})$, we want to know what is the operator $\hat{D}(R)$ which <u>implements</u> this rotation.

If the system is in a state $|\alpha\rangle$ and we rotate it by R, the new state will be

$$|\alpha_R\rangle = \hat{D}(R)|\alpha\rangle \tag{39}$$

In order to conserve probability, $\hat{D}$ must be unitary

$$\hat{D}^\dagger(R)\hat{D}(R) = 1 \tag{40}$$

But the question is, how is this rotation actually implemented? The key is to look at expectation values. Let $\vec{v} = (\hat{v}_x, \hat{v}_y, \hat{v}_z)$ be a vector of vector operators. For instance $\vec{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$, etc. Before the operator. rotation the expectation values were

$$\langle \hat{v}_i \rangle = \langle \alpha | \hat{v}_i | \alpha \rangle$$

whereas after the rotation they become

$$\langle \alpha_R | \hat{v}_i | \alpha_R \rangle = \langle \alpha | D^\dagger \hat{v}_i D | \alpha \rangle = \langle \hat{v}_i \rangle'$$

We then impose that these expectation values transform like real vectors under a rotation. That is

$$\langle \alpha | \hat{D}^\dagger \hat{v}_i \hat{D} | \alpha \rangle = \sum_j R_{ij} \langle \alpha | \hat{v}_i | \alpha \rangle$$

Since this must be true for any $|\alpha\rangle$, we may as well stick to an equality involving operators only

$$\boxed{\hat{D}^\dagger(R)\, \hat{v}_i\, \hat{D}(R) = \sum_j R_{ij}\, \hat{v}_j} \qquad (41)$$

this is the fundamental relation between rotations in $\mathbb{R}^3$ and their implementation in Hilbert space. We expect that this Eq should hold for any vector operator. For instance, $\vec{J}$, $\vec{p}$, $\vec{r}$, etc.

Now I want to show that the correct expression for $\hat{D}(R)$ is

$$\boxed{\hat{D}(R) = e^{-i\theta\vec{n}\cdot\vec{J}}} \qquad (42)$$

where $\vec{J}$ is the angular momentum operator. This formula has a structure similar to $R(\theta,\vec{n}) = e^{-i\theta\vec{n}\cdot\vec{G}}$. But note that R is a 3×3 matrix whereas $\hat{D}$ is an operator acting on Hilbert space.

Before I demonstrate $\hat{D}(R)$ in the general case, let us analyze it for the case of spin 1/2. In this case

$$\vec{J} = \frac{\vec{\sigma}}{2} \qquad (43)$$

For a rotation around the z axis

$$\hat{D}_z = e^{-i\frac{\theta}{2}\sigma_z} \qquad (44)$$

Since σ_z is diagonal

$$\hat{D}_z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \qquad (45)$$

We now apply Eq (41) to $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. We have

$$\boxed{e^{i\frac{\theta}{2}\sigma_z}\,\sigma_z\,e^{-i\frac{\theta}{2}\sigma_z} = \sigma_z} \tag{46a}$$

and

$$e^{i\frac{\theta}{2}\sigma_z}\,\sigma_x\,e^{-i\frac{\theta}{2}\sigma_z} = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & e^{i\theta/2} \\ e^{-i\theta/2} & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} = \begin{bmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cos\theta + i\sin\theta \\ \cos\theta - i\sin\theta & 0 \end{bmatrix}$$

$$\therefore \quad \boxed{D_z^{+}\,\sigma_x\,D_z = \sigma_x\cos\theta - \sigma_y\sin\theta} \tag{46b}$$

This is precisely what we expect from a rotation around z. Similarly we get

$$\boxed{D_z^{+}\,\sigma_y\,D_z = \sigma_x\sin\theta + \sigma_y\cos\theta} \tag{46c}$$

Thus, at least for spin 1/2, Eq (42) gives the correct answer.

Since we are talking about spin 1/2, let me show you a neat application of Eq (42). In a previous occasion we have shown that if A is an operator such that $\hat{A}^2 = 1$, then

$$e^{i\lambda A} = \cos\lambda + iA\sin\lambda \tag{47}$$

this is true for Pauli matrices: $\sigma_i^2 = 1$. Thus

$$e^{-i\frac{\theta}{2}\hat{\sigma}_j} = \cos\frac{\theta}{2} - i\hat{\sigma}_j\sin\frac{\theta}{2} \tag{48}$$

Now let's use this to construct the eigenvectors of the matrix

$$\hat{\sigma}_a = \vec{a}\cdot\vec{\sigma_a} \tag{49}$$

where $\vec{a}$ is a direction in space parametrized as in Eq (37). We learned that

$$\vec{a} = R_z(\phi)R_y(\theta)\hat{z}$$

where $\hat{z} = (0,0,1)$. Thus, we may relate the eigenvectors $|a\pm\rangle$ to the eigenvectors $|z\pm\rangle$ by the same procedure

$$|a\pm\rangle = D_z(\phi)D_y(\theta)|z\pm\rangle \tag{50}$$

Well

$$\sigma_y |z+\rangle = i|z-\rangle$$
$$\sigma_y |z-\rangle = -i|z+\rangle$$

Thus

$$e^{-i\frac{\theta}{2}\sigma_y}|z+\rangle = \left[\cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2}\right]|z+\rangle$$
$$= \cos\frac{\theta}{2}|z+\rangle + \sin\frac{\theta}{2}|z-\rangle$$

Moreover

$$e^{-i\frac{\phi}{2}\sigma_z}|z+\rangle = e^{-i\phi/2}$$
$$e^{-i\frac{\phi}{2}\sigma_z}|z-\rangle = e^{i\phi/2}$$

so that

$$|a+\rangle = e^{-i\frac{\phi}{2}\sigma_z}\left[\cos\frac{\theta}{2}|z+\rangle + \sin\frac{\theta}{2}|z-\rangle\right]$$

or

$$\boxed{|a+\rangle = e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|z+\rangle + e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}|z-\rangle} \quad (51a$$

By a similar calculation we arrive at

$$\boxed{|a-\rangle = -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}|z+\rangle + e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}|z-\rangle} \quad (51b$$

These are precisely the results we have derived a while ago by exactly diagonalizing σ_a.

It is nice to know that we may obtain these eigenvectors by this alternative procedure

Back to the general formula

Now I want to show that if $\hat{D}(R)$ is defined as in (42), then (41) will be satisfied. I will do this for the case of a rotation around z. Since there is nothing special about this direction, if it is true for z, it is true for any $\vec{n}$. Moreover, I will do the calculation for $\vec{V} = \vec{J}$. The more general case will be discussed later. Thus I want to analyze

$$e^{i\theta \hat{J}_z} \hat{J}_i \, e^{-i\theta \hat{J}_z}$$

of course

$$e^{i\theta \hat{J}_z} \hat{J}_z \, e^{-i\theta \hat{J}_z} = \hat{J}_z$$

so we are off to a good start. Next, using BCH:

$$e^{i\theta J_z} J_x e^{-i\theta J_z} = J_x + i\theta [J_z, J_x] + \frac{(i\theta)^2}{2} [J_z, [J_z, J_x]] + \cdots$$

$$= J_x + i\theta (i J_y) + \frac{(i\theta)^2}{2!} [J_z, i J_y] + \frac{(i\theta)^3}{3!} [J_z, [J_z, i J_y]]$$

$$= J_x - \theta J_y - \frac{\theta^2}{2} J_x - \frac{i\theta^3}{3!} [J_z, J_x] + \cdots$$

$$= J_x \left[1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots\right] - J_y \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right]$$

$$= J_x \cos\theta - J_y \sin\theta$$

YAY! Success! This is exactly the rotation we wanted. Only this time we used just the commutation relations of the angular momentum. The calculation for J_y is identical.

Vector operators

We have shown that Eqs (41) and (42) work when we rotate the angular momentum operators. But what about another vector operator $\vec{v}$?

Well, first we need to answer something else. Given 3 arbitrary crazy operators $\hat{A}$, $\hat{B}$, $\hat{C}$, is $\vec{v} = (\hat{A}, \hat{B}, \hat{C})$ a vector operator? I hope not!

In other words, Eq (41) should not be true for any triplet of operators. So now we need to determine for which operators it should work.

To do this we may consider infinitesimal rotations. I will also start with a rotation around z. The right side of (41) is then,

$$R_z \vec{v} \approx \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} + \theta \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{bmatrix}$$

$$= \begin{bmatrix} v_x - \theta v_y \\ v_y + \theta v_x \\ v_z \end{bmatrix}$$

On the left side we have

$$D^{+} v_i D \approx (1 - i\theta J_z)\, \hat{v}_i\, (1 + i\theta J_z) = \hat{v}_i - i\theta [J_z, v_i] + \Theta(\theta)^2$$

Comparing terms to the same order in θ we conclude that

$$[\hat{J}_z, \hat{v}_x] = i\, v_y$$

$$[J_z, \hat{v}_y] = -i\, \hat{v}_x \qquad (52)$$

$$[J_z, v_z] = 0$$

These are the commutation relations which must be satisfied by a triplet of operators $(\hat{v}_x, \hat{v}_y, \hat{v}_z)$ if they are to be considered as <u>vector operators</u>.

If we do the same thing for rotations around the different axes we get similar relations with J_y and J_z. Note how they have a structure similar to the angular momentum commutation relations. In general we may write

$$\boxed{[\hat{J}_i, \hat{v}_j] = i\, \epsilon_{ijk}\, \hat{v}_k} \qquad (53)$$

This is certainly true for $\vec{v} = \vec{J}$ so angular momentum is obviously a vector operator.

However, we may also verify that Eq (53) is also true for $\vec{p}$ and $\vec{r}$, as we expected. For this case, instead of $\vec{J}$ we need to use

$$\vec{L} = \frac{\vec{r} \times \vec{p}}{\hbar} \qquad (54)$$

Now let $\vec{v}_1$ and $\vec{v}_2$ be two vector operators (ie, they satisfy (53)). We may then show that

$$[\vec{J}_i, \vec{v}_1 \cdot \vec{v}_2] = 0 \tag{55}$$

This is the quantum mechanical statement that "scalar products produce scalars" [Recall that a scalar is an object which is invariant under rotations].

Thus, for instance,

$$\vec{p}^{\,2} = \vec{p} \cdot \vec{p}$$

is a scalar operator and therefore is invariant under rotations. The same is true for $\vec{r}^{\,2}$ and $\vec{L}^2$ or $\vec{J}^2$.

The SU(2) group

Now let us go back to spin 1/2. The most general rotation matrix is

$$\hat{D} = e^{-i\frac{\theta}{2}\vec{n}\cdot\vec{\sigma}} \tag{56}$$

Since $(\vec{n}\cdot\vec{\sigma})^2 = 1$ we may write

$$\hat{D} = \cos\frac{\theta}{2} - i(\vec{n}\cdot\vec{\sigma})\sin\frac{\theta}{2} \tag{57}$$

Explicitly we have

$$(\vec{n}\cdot\vec{\sigma}) = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} \tag{58}$$

so that

$$\hat{D} = \begin{bmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & -(in_x + n_y)\sin\frac{\theta}{2} \\ -(in_x - n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{bmatrix} \tag{59}$$

These matrices form a group called the SU(2) group, where "U" stands for unitary. It is customary to write (59) as

$$\boxed{\hat{D}(a,b) = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}} \tag{60}$$

where

$$\boxed{\begin{aligned} a &= \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} \\ b &= -(in_x + n_y)\sin\frac{\theta}{2} \end{aligned}} \tag{61}$$

these are called the Cayley-Klein parameters. The SU(2) group and the Cayley-Klein parameters are much older than quantum mechanics. In fact, they date from 1875!

The reason why people were interested in this at that time is because you may use the SU(2) group to rotate vectors in $\mathbb{R}^3$! I know it sounds strange since the matrices of the SU(2) are 2x2. But they are complex so we have extra room for to fit all parameters.

A rotation in $\mathbb{R}^3$ has 4 parameters: n_x, n_y, n_z and θ.
A rotation in the SU(2) has only 2: a and b. But they are complex numbers so in fact we have 4, just like $\mathbb{R}^3$.

The formula for rotation in the SU(2) is defined as follows. Given a vector $\vec{v} \in \mathbb{R}^3$ define

$$\sigma_v = \vec{v} \cdot \vec{\sigma} = \begin{bmatrix} v_z & v_x - i v_y \\ v_x + i v_y & -v_z \end{bmatrix}$$

then the rotation

$$\vec{v}' = R\vec{v}$$

may be written as

$$\boxed{\sigma_{v'} = D(R)\, \sigma_v\, D^{+}(R)} \qquad (62)$$

Note how the order of D and D^{+} is different from what we had before.

I will verify this formula for the case of a rotation around z; In this case

$$e^{-i\frac{\theta}{2}\sigma_z}(\vec{v}\cdot\vec{\sigma})e^{i\frac{\theta}{2}\sigma_z} = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2}\end{bmatrix}\begin{bmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z\end{bmatrix}\begin{bmatrix} e^{+i\theta/2} & 0 \\ 0 & e^{-i\theta/2}\end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2}\end{bmatrix}\begin{bmatrix} e^{i\theta/2} v_z & (v_x - iv_y)e^{-i\theta/2} \\ (v_x + iv_y)e^{i\theta/2} & -v_z e^{-i\theta/2}\end{bmatrix}$$

$$= \begin{bmatrix} v_z & (v_x - iv_y)e^{-i\theta} \\ (v_x + iv_y)e^{i\theta} & -v_z\end{bmatrix}$$

we must compare this with

$$\sigma v' = \begin{bmatrix} v_z' & v_x' - iv_y' \\ v_x' + iv_y' & -v_z'\end{bmatrix}$$

We see that $v_z' = v_z$ (phew!) and

$$v_x' - iv_y' = (v_x - iv_y)e^{-i\theta}$$

$$= (v_x - iv_y)(\cos\theta - i\sin\theta)$$

$$= (v_x\cos\theta - v_y\sin\theta) + -i(v_x\sin\theta + v_y\cos\theta)$$

so

$$v_x' = v_x\cos\theta - v_y\sin\theta$$

$$v_y' = v_x\sin\theta + v_y\cos\theta$$

It works!

The main reason why rotating in the SU(2) is useful is because now it is easier to compose rotations. That is, suppose we first rotate by R and then by R'. The composite rotation is

$$R'' = R' R$$

But what are the values of θ and $\vec{n}$ of R'', as a function of the values for R and R'? The answer is not obvious (try it out and you will see what I mean).

In the SU(2) the answer is easier. If the Cayley-Klein parameters for R are a and b, and those of R' are a' and b', then

$$D(R'') = D(R')D(R) = \begin{bmatrix} a' & b' \\ -(b')^* & (a')^* \end{bmatrix} \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

$$= \begin{bmatrix} a'a - b'b^* & a'b + b'a^* \\ \cdots & \cdots \end{bmatrix}$$

I don't even need to know what the other elements are. The result, since it is a member of the group, must have the form (60) for parameters

$$\boxed{\begin{aligned} a'' &= a'a - b'b^* \\ b'' &= a'b + b'a^* \end{aligned}} \qquad (63)$$

This gives the parameters of the composite rotation in terms of those of the two original rotations.