The BCS theory of superconductivity

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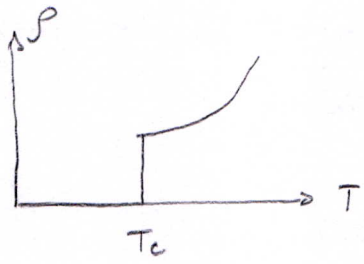
The BCS Theory of superconductivity

Introduction

The electrical resistivity of certain metals as a function of temperature looks somewhat like this

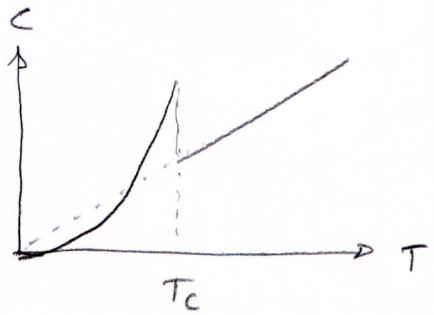


Below a certain T_c it drops abruptly to a value that is identically zero. Below T_c we say the system is in a superconducting state.

This superconducting state has many other remarkable properties. But we will not be discussing them here. You don't need lecture notes for that: you can just read about it on wikipedia.

Instead, what I want to discuss here is the more subtle point concerning the microscopic mechanisms responsible for superconductivity.

One hint about such mechanisms comes from measurements of the specific heat. They look somewhat like this



Above T_c the electronic specific heat is roughly linear, as expected from a metal. But at T_c it jumps and, below T_c it behaves exponentially like

$$C \sim e^{-\Delta/T} \qquad (1)$$

where Δ is some constant. This behavior is typical of systems that have an energy gap (to see why, compute the specific heat of a 2-state system with an energy separation Δ. In the limit $\Delta >> T$ you will find an exponential behavior like that of (1)).

This result therefore indicates that there must be a gap between the superconducting ground state and the first excited states. This gap plays a central role in the behavior of a superconductor, as we will see.

Perhaps the most startling evidence as to which is the mechanism behind superconductivity is the <u>isotope effect</u>: the value of T_c changes if we use a different isotopic composition.

This is absolutely remarkable. The mass of the nuclear has absolutely no effect on the chemistry of a system. That is determined solely by the electrons. In fact, the mass of the nuclei affects only one thing: <u>phonons</u>. It therefore follows that superconductivity <u>must</u> be related to the <u>electron-phonon interaction</u>.

In the previous set of notes we saw how the electron-phonon interaction gives rise to an effective electron-electron interaction of the form

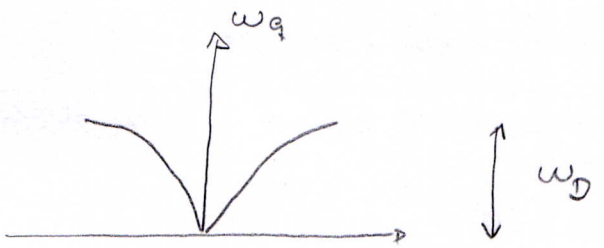
$$V = \sum_{\substack{kk'q \\ \sigma\sigma'}} V_{kq}\, c^{+}_{k+q,\sigma}\, c^{+}_{k'-q,\sigma'}\, c_{k'\sigma'}\, c_{k\sigma} \qquad (2)$$

where

$$V_{kq} = \frac{|g_q|^2 \omega_q}{(\epsilon_{k+q} - \epsilon_k)^2 - \omega_q^2} \qquad (3)$$

We see that for certain values of k and q, this potential may be negative, signalling an <u>attractive interaction</u>

The phonon spectra ω_q has a finite bandwidth



The bandwidth is of the order of the Debye frequency ω_D. For typical metals $\omega_D \sim 0.03\,eV$, which is a tiny number compared to typical Fermi energies, which are of the order of a few eV. Thus, the interaction is attractive only over a tiny window where $|\epsilon_{u-q} - \epsilon_u| < \omega_q$. For most other situations, it is repulsive as usual.

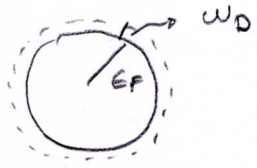
Notwithstanding, as we will learn next, an attractive interaction, as small as it may be, has a fundamental impact on the properties of the electrons

The Cooper pair idea

the fact that there is an attractive interaction means that the electrons may form bound states, where two or more electrons lower their energies by sticking together.

In Phys Rev. 104 (1956), L.N. Cooper showed that this will occur no matter how small the interaction is. And that this has profound consequences on the very stability of the Fermi sphere.

Here is what Cooper did. Suppose we have our good old metal, with the Fermi sphere entirely filled. And suppose we add two electrons to this metal, with momenta p and p'. we assume they cannot access any state below the Fermi sphere. Moreover, we assume that within a thin shell of width ω_D above ϵ_F, the interaction between the two electrons is attractive



we want to try to figure out if the two electrons can form a bound state and what is the configuration of this bound state and the binding energy.

We are going to do this using the variational principle: Let H be a Hamiltonian whose ground state is E_{gs}. Then, for any state $|\Psi\rangle$, it is always true that

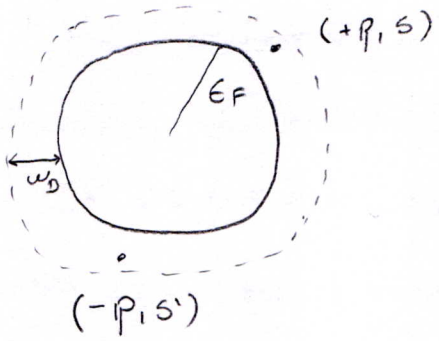
$$E_{gs} \leq \langle \Psi | H | \Psi \rangle \qquad (4)$$

thus, by giving good guesses about the wavefunction $|\Psi\rangle$, we can get closer and closer to E_{gs}.

If the electrons were non-interacting, their energies would be $\epsilon_p + \epsilon_{p'}$, where $\epsilon_p = p^2/2m$. What we want to show is that the presence of the electron electron interaction is to produce an energy lower than $\epsilon_p + \epsilon_{p'}$. If we can find any $|\Psi\rangle$ whose energy is lower than this, then by the variational principle we will know for sure that there exists a bound state. That is the magic of the variational principle.

Well, one thing we can say from the start: since V conserves momentum, if we want to minimize their energy we better put $p' = -p$. Otherwise we would have the extra energy of the center of mass

I therefore propose we analyze the following situation



we have two electrons with momentum p and $-p'$ same where in a shell between ϵ_F and $\epsilon_F + \omega_D$. Their spins are s, s'.

The Hamiltonian we want to analyze is

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + V \tag{5}$$

The most general wave function for two electrons may be written as

$$|\psi\rangle = \sum_{pss'} c(p, s, s') c^\dagger_{ps} c^\dagger_{-ps'} |F\rangle \tag{6}$$

where $|F\rangle$ is the Fermi state and c are arbitrary coefficients. The Hamiltonian (5) has no spin dependence, so the coefficients $c(p, s, s')$ will factor into a product like

$$c(p, s, s') = g(p) \chi(s, s') \tag{7}$$

The spin part $\chi(s,s')$ doesn't do much. But it affects the symmetry of the wave function: since we are talking about electrons, the total wave function must be anti-symmetric under the exchange of particles. Exchange means we need to compare $\chi(s,s')$ vs $\chi(s',s)$ and $g(p)$ vs $g(-p)$ (the electrons have opposite momenta). We therefore have two possibilities.

$$\text{Singlet}: \quad \chi(s,s') = -\chi(s',s) \rightsquigarrow g(-p) = g(p)$$
$$\text{Triplet}: \quad \chi(s,s') = \chi(s',s) \rightsquigarrow g(-p) = -g(p) \tag{8}$$

The distinction between these two possibilities will become important below.

After all this foreplay, we are finally ready for some action. Given the set of states (6), we construct the eigenvalue eigenvector equation

$$H|\Psi\rangle = E|\Psi\rangle \tag{9}$$

The state $|\Psi\rangle$ which represents the ground-state of this equation will be our estimate of the true ground-state.

Now

$$\left(\sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma}\right)|\Psi\rangle = \epsilon_F + \sum_{pss'} 2\epsilon_p \, g(p) \chi(s,s') c^\dagger_{ps} c^\dagger_{-p,s'} |F\rangle \qquad (10)$$

Next we look at the effect of V. we do it slowly, one term at a time.

$$c_{k\sigma}\left[c^\dagger_{ps} c^\dagger_{-ps'}|F\rangle\right] = (\delta_{kp}\delta_{\sigma s} - c^\dagger_{ps} c_{k\sigma}) c^\dagger_{-ps'}|F\rangle$$

$$= \delta_{kp}\delta_{\sigma s}\, c^\dagger_{-ps'}|F\rangle +$$

$$- c^\dagger_{ps}(\delta_{k,-p}\delta_{\sigma,s'} - c^\dagger_{-ps'} c_{k\sigma})|F\rangle$$

we only want configurations where the Fermi sphere is completely filled. So we may set $c_{k\sigma}|F\rangle = 0$. Even though for some values of k that will not be true (when $k < k_F$), these situations will produce states with electrons that are not in the energy shell and thus are of no interest to us (at least right now!). We then get

$$c_{k\sigma}\left[c^\dagger_{ps} c^\dagger_{-ps'}|F\rangle\right] = \delta_{kp}\delta_{\sigma s}\, c^\dagger_{-ps'}|F\rangle - \delta_{k,-p}\delta_{\sigma s'} c^\dagger_{ps}|F\rangle \qquad (11)$$

Next:

$$c_{k'\sigma'} c_{k\sigma} \left[c^{\dagger}_{ps} c^{\dagger}_{-p s'} |F\rangle \right] = \left[\delta_{kp} \delta_{\sigma s} \, \delta_{k',-p} \delta_{\sigma',s'} + \right.$$

$$\left. - \delta_{k,-p} \delta_{\sigma,s'} \delta_{k',p} \delta_{\sigma',s} \right] |F\rangle \qquad (12)$$

If we now apply the full V on this state, we will get

$$V \left[c^{\dagger}_{ps} c^{\dagger}_{-p s'} |F\rangle \right] = \sum_{q} \left[V_{pq} \, c^{\dagger}_{p+q,s} \, c^{\dagger}_{-p-q,s'} + \right.$$

$$\left. - V_{-pq} \, c^{\dagger}_{-p+q,s'} \, c^{\dagger}_{p-q,s} \right] |F\rangle \qquad (13)$$

Thus, the action of V on the $|\Psi\rangle$ of (6) will be

$$V|\Psi\rangle = \sum_{pqss'} \left\{ V_{pq} \, g(p) \, \chi(s,s') \, c^{\dagger}_{p+q,s} \, c^{\dagger}_{-p-q,s'} |F\rangle \right.$$

$$\left. + V_{-p,q} \, g(p) \, \chi(s,s') \, c^{\dagger}_{p-q,s} \, c^{\dagger}_{-p+q,s'} |F\rangle \right\} \qquad (14)$$

where, in the last term, I exchange the two operators to get rid of the minus sign.

In the first term of (14) we change variables as $p' = p + q$. And in the 2nd we change as $p' = p - q$. We then get

$$V|\psi\rangle = \sum_{p'qss'} \left\{ V_{p'-q,q}\, g(p'-q)\, \chi(s,s') + V_{-p'-q,q}\, g(p'+q)\, \chi(s,s') \right\} c^{\dagger}_{p's}\, c^{\dagger}_{-p's'} |F\rangle$$

Now we can change back to p. Moreover, note from (3) that $V_{-p-q,q} = V_{p+q,q}$. Thus we get

$$V|\psi\rangle = \sum_{pqss'} \left\{ V_{p-q,q}\, g(p-q) + V_{p+q,q}\, g(p+q) \right\} \chi(s,s') \cdot c^{\dagger}_{ps}\, c^{\dagger}_{-p,s'} |F\rangle$$

Another thing we notice is that $V_{k,-q} = V_{k,q}$, so in the first term we can change $q \to -q$. We then get something which is identical to the 2nd term. Thus we finally arrive at

$$V|\psi\rangle = \sum_{pqss'} 2 V_{p+q,q}\, g(p+q)\, \chi(s,s')\, c^{\dagger}_{p,s}\, c^{\dagger}_{-p,s'} |F\rangle \qquad (15)$$

We are now ready to write down our eigenstuff equation $H|\Psi\rangle = E|\Psi\rangle$. To get $H|\Psi\rangle$ we combine (10) and (15), which gives

$$H|\Psi\rangle = \sum_{pss'} \left\{ 2\epsilon_p g(p) + \sum_q 2V_{p+q,q}\, g(p+q) \right\} \chi(s,s')\, c^\dagger_{ps} c^\dagger_{-ps'} |F\rangle$$

Moreover

$$E|\Psi\rangle = \sum_{pss'} E\, g(p)\, \chi(s,s')\, c^\dagger_{ps} c^\dagger_{-ps'} |F\rangle$$

Thus, comparing the two term by term, we get the equation

$$\boxed{2\epsilon_p\, g(p) + \sum_q 2V_{p+q,q}\, g(p+q) = E\, g(p)} \qquad (16)$$

This is our eigenstuff equation. The eigenfunctions are $g(p)$ and the eigenenergies are E

As discussed previously, we are going to assume the electrons can only have momentum k in a thin shell between ϵ_F and $\epsilon_F + \omega_D$. This shell is indeed very thin so we may suppose the potential is roughly constant within this shell. We therefore set

$$V_{kq} = \begin{cases} -\frac{V_0}{N} & \epsilon_F < \epsilon_k < \epsilon_F + \omega_D \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where $V_0 > 0$ is a constant. Then Eq (16) becomes

$$(2\epsilon_p - E)\, g(p) = \frac{2V_0}{N} \sum_k g(k) \tag{18}$$

where I changed variables in the q sum to $k = p + q$.

Now comes the first important result to get out of this. In Eq (8) we argued that the spin configuration could be either a singlet or a triplet, which would affect the parity of $g(p)$.

Suppose the configuration is a triplet. Then $g(-p) = -g(p)$. But in the RHS of (18) we have a sum over all k. Thus, if $g(k) = -g(-k)$, this sum will be zero. Thus, we conclude that if the two electrons are in the triplet state, the electron-electron attraction V_0 will have no effect whatsoever, and they will not form a bound state.

As a consequence, if the electrons are to form a bound state, they must be in the singlet configuration. Thus, Cooper pairs turn up as singlets. This is what is called s-wave superconductivity. Normal superconductors like Al, Hg, Nb, etc. are all s-wave superconductors. The more sophisticated superconductors, like the cuprates, are in general not singlets.

Now return to Eq (18) and divide both sides by $(2\epsilon_p - E)$. We get

$$g(p) = \frac{2V_0}{N} \sum_k \frac{g(k)}{2\epsilon_p - E}$$

Next sum over p:

$$\sum_p g(p) = \frac{2V_0}{N} \sum_p \frac{1}{2\epsilon_p - E} \sum_k g(k)$$

The two sums over g cancel out and we are left with

$$1 = \frac{2V_0}{N} \sum_p \frac{1}{2\epsilon_p - E} \qquad (19)$$

We convert the sum to an integral. In doing so, we don't use the density of states $D(\epsilon)$ but, for convenience, we define $D(\epsilon) = N n_0(\epsilon)$. Then we write

$$\sum_p f(\epsilon_p) = N \int d\epsilon \, n_0(\epsilon) f(\epsilon) \qquad (20)$$

Eq (19) then becomes

$$I = 2V_0 \int_{\epsilon_F}^{\epsilon_F + \omega_D} \frac{d\epsilon \; n_0(\epsilon)}{2\epsilon - E} \tag{21}$$

The integral is only over the thin shell of thickness ω_D. Within this shell we may take $n_0(\epsilon) \approx n_0(\epsilon_F)$, a constant. We then get

$$I = 2V_0 n_0(\epsilon_F) \int_{\epsilon_F}^{\epsilon_F + \omega_D} \frac{d\epsilon}{2\epsilon - E}$$

$$= 2V_0 \frac{n_0(\epsilon_F)}{2} \ln\left[\frac{2\epsilon_F + 2\omega_D - E}{2\epsilon_F - E}\right]$$

We define

$$\Delta = 2\epsilon_F - E \tag{22}$$

This is the binding energy: it is how much the energy of the two electrons is lowered, with respect to putting them on the surface of the Fermi sphere (if there were no attraction, the ground-state would be to simply sit them down on the Fermi surface). Alternatively, Δ represents the energy gap between the ground state (the two electrons bound together) and the excited state (the two electrons flying off apart)

We then get

$$\frac{1}{V_0 m_0(\epsilon_F)} = \ln\left(\frac{\Delta + 2\omega_D}{\Delta}\right)$$

or

$$\frac{\Delta + 2\omega_D}{\Delta} = e^{-1/V_0 m_0}$$

Solving for Δ:

$$\Delta\left(1 - e^{-1/V_0 m_0}\right) = -2\omega_D e^{-1/V_0 m_0}$$

or

$$\Delta = 2\omega_D \frac{e^{-1/V_0 m_0}}{e^{-1/V_0 m_0} - 1} \tag{23}$$

In the <u>weak coupling limit</u> we have

$$V_0 m_0 \ll 1 \tag{24}$$

then we may approximate

$$\boxed{\Delta \simeq 2\omega_D e^{-1/V_0 m_0(\epsilon_F)}} \tag{25}$$

This is a beautiful result: note that the gap will in general be small when $V_0 n_0 \ll 1$. However, it always exists, no matter how small V_0 is is.

Thus, we see that two electrons don't want to be piled up to form a Fermi sphere. they want to live in an energy shell and form Cooper pairs. the usual "electron gas" we are used to is therefore unstable. Any tiny attractive interaction will cause dramatic changes on the ground-state properties of the electron gas.

Another thing we notice from (25) is that Δ is not an analytic function of V_0 in the limit $V_0 \to 0$. thus, we cannot hope to find Cooper pairs using any type of perturbation theory. For in perturbation theory we always start with an unperturbed state and find small corrections to it.

Another interesting consequence of (25) is that the gap will be larger when V_0 is larger. But V_0 represents the electron-phonon interaction, which is also responsible for the resistivity of metals. Thus superconductivity should be more likely to occur on bad conductors! For instance, lead, which is a bad conductor, becomes a superconductor at $T_c = 7.19\,K$. Conversely, the good conductors like Cu, Au and Ag don't become superconductors at all.

The BCS theory using Bogoliubov's method

The calculation we just did concerned only two electrons. But superconductivity is a many-body phenomena. So now we must try to tackle the full many body problem. Of course, we can't solve it exactly, so we will use a mean-field approximation.

This is the Bardeen, Cooper, Schrieffer (BCS) theory of superconductivity, Phys Rev. 106 (1957) (one year after Cooper's paper).

We will not reproduce the original BCS calculations. Instead, we are going to consider a simpler method developed by Bogoliubov (Nuovo Cimento, 7, 1958).

The starting point is again the full Hamiltonian (5). But, motivated by the Cooper pair idea, we are only going to consider those interactions in V where $k' = -k$. That is from

$$V = \sum_{\substack{kk'q \\ \sigma\sigma'}} V_{k,q}\, c^{+}_{k-q,\sigma}\, c^{+}_{k'-q,\sigma'}\, c_{k'\sigma'}\, c_{k\sigma}$$

we are only going to consider the terms with $k' = -k$:

$$V \simeq \sum_{\substack{kq \\ \sigma\sigma'}} V_{kq}\, c^{+}_{k-q,\sigma}\, c^{+}_{-k-q,\sigma'}\, c_{-k,\sigma'}\, c_{k\sigma}$$

We change variables to $k' = k - q$ and write $V_{k,q}$ as $V_{kk'}$:

$$V = \sum_{\substack{kk' \\ \sigma\sigma'}} V_{kk'} \, c^{+}_{k',\sigma} \, c^{+}_{-k',\sigma'} \, c_{-k,\sigma'} \, c_{k\sigma}$$

Finally, again motivated by the Cooper pair idea, we assume the pairs have opposite momentum, so $\sigma' = -\sigma$. We therefore consider

$$V = \sum_{kk'\sigma} V_{kk'} \, c^{+}_{k'\sigma} \, c^{+}_{-k',-\sigma} \, c_{-k,-\sigma} \, c_{k\sigma}$$

We can spell this out explicitly

$$V = \sum_{kk'} V_{kk'} \Big[c^{+}_{k'\uparrow} \, c^{+}_{-k',\downarrow} \, c_{-k\downarrow} \, c_{k\uparrow} + \\ + c^{+}_{k'\downarrow} \, c^{+}_{-k'\uparrow} \, c_{-k\uparrow} \, c_{k\downarrow} \Big]$$

But now we can use the following trick. In the second term we change $k \to -k$ and $k' \to -k'$. This does not affect V, which is even in the k's. The second term then becomes identical with the first and we get

$$V = \sum_{kk'} 2V_{kk'} \, c^{+}_{k'\uparrow} \, c^{+}_{-k'\downarrow} \, c_{-k\downarrow} \, c_{k\uparrow}$$

Notice now how spin up always appears with $+k$ and spin down with $-k$. Thus, we may simplify a bit our notation by writing

$$c_{k\uparrow} \equiv c_k$$

$$c_{-k,\downarrow} \equiv c_{-k}$$

Rescaling also $2V_{kk'} \to V_{kk'}$, we then get

$$\mathcal{V} = \sum_{kk'} V_{kk'}\, c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'} \tag{26}$$

As for the kinetic energy, we write

$$\sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} = \sum_k \left(\epsilon_k c_{k\uparrow}^\dagger c_{k\uparrow} + \epsilon_k c_{k\downarrow}^\dagger c_{k\downarrow}\right)$$

$$= \sum_k \left(\epsilon_k c_{k\uparrow}^\dagger c_{k\uparrow} + \epsilon_k c_{-k\downarrow}^\dagger c_{-k\downarrow}\right)$$

where I changed $k \to -k$ in the last term and used the fact that $\epsilon_{-k} = \epsilon_k$. We then get

$$\sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} = \epsilon_k \left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right)$$

There is one final twist: we now include explicitly the chemical potential μ. Thus we replace ϵ_k with

$$\epsilon_k = \frac{k^2}{2m} - \mu \tag{27}$$

Hence, we finally arrive at the BCS Hamiltonian

$$H = \sum_k \epsilon_k (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + \sum_{kk'} V_{kk'} c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'} \tag{28}$$

This is the Hamiltonian we will explore. For simplicity, we will choose a specific form of $V_{kk'}$ as BCS did

$$V_{kk'} = \begin{cases} -\frac{V_0}{N} & \text{if } -\omega_D < \varepsilon_k, \varepsilon_{k'} < \omega_D \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

Recall that $\varepsilon_k = \epsilon_k - \mu$ and $\mu \simeq \epsilon_F$. Thus, this means we choose $V_{kk'}$ to be non-zero in a shell from $\epsilon_F - \omega_D$ to $\epsilon_F + \omega_D$.

To proceed we use a mean-field approximation in the Hamiltonian (28). The idea of Bogoliubov was to do the mean-field for the pair $c_k c_{-k}$. We therefore write

$$c_k^\dagger c_{-k}^\dagger = a_k + (c_k^\dagger c_{-k}^\dagger - a_k) \tag{30}$$

where

$$a_k = \langle c_k^\dagger c_{-k}^\dagger \rangle \tag{31}$$

Without loss of generality we choose the a_k as real. We then get

$$c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'} = [a_k + (c_k^\dagger c_{-k}^\dagger - a_k)][a_{k'} + (c_{-k'} c_{k'} - a_{k'})]$$

$$= a_k a_{k'} + a_k (c_{-k'} c_{k'} - a_{k'}) +$$

$$+ a_{k'} (c_k^\dagger c_{-k}^\dagger - a_k) +$$

$$+ \underbrace{(c_k^\dagger c_{-k}^\dagger - a_k)(c_{-k'} c_{k'} - a_{k'})}_{\text{neglect.}}$$

$$\simeq - a_k a_{k'} + a_k c_{-k'} c_{k'} + a_{k'} c_k^\dagger c_{-k}^\dagger \tag{32}$$

The Hamiltonian (28) then becomes

$$H = \sum_k \epsilon_k (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + \sum_{kk'} \{ -V_{kk'} a_k a_{k'} + \\ + V_{kk'} (a_k c_{-k'} c_{k'} + a_{k'} c_k^\dagger c_{-k}^\dagger) \}$$

Now define

$$\Delta_k = -\sum_{k'} V_{kk'} a_{k'} \tag{32'}$$

Then we get

$$H = \sum_k \epsilon_k (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + \sum_k \Delta_k a_k + \\ - \sum_k \Delta_k (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k)$$

where I used the fact that we are always summing over k, k' to write everything as a sum over k, with k' being placed on Δ_k. Organizing a bit better

$$\boxed{H = \sum_k \Delta_k a_k + \sum_k \{ \epsilon_k (c_k^\dagger c_k + c_{-k}^\dagger c_{-k}) + \\ - \Delta_k (c_k^\dagger c_{-k}^\dagger + c_{-k} c_k) \}} \tag{33}$$

Note how we have now factored the full Hamiltonian H into a sum of independent terms for each k. We therefore only need to diagonalize each term in the sum independently. This is the Bogoliubov method to diagonalize a Hamiltonian of the form

$$\epsilon(a^\dagger a + b^\dagger b) - \Delta(a^\dagger b^\dagger + ba)$$

we introduce two new Fermionic operators γ_k and γ_{-k} according to

$$\begin{aligned} c_k &= u_k \gamma_k + v_k \gamma^\dagger_{-k} \\ c^\dagger_{-k} &= -v_k \gamma_k + u_k \gamma^\dagger_{-k} \end{aligned} \tag{34}$$

where u_k, v_k are taken to be real numbers. We want γ_k and γ_{-k} to satisfy the Fermionic algebra, which imposes the constraint

$$u_k^2 + v_k^2 = 1 \tag{35}$$

thus, they may be parametrized as

$$\begin{aligned} u_k &= \cos\theta_k \\ v_k &= \sin\theta_k \end{aligned} \tag{36}$$

Then (34) is found to be nothing but a simple rotation by θ_u:

$$\begin{pmatrix} c_u \\ c_{-u}^+ \end{pmatrix} = \begin{pmatrix} \cos\theta_u & \sin\theta_u \\ -\sin\theta_u & \cos\theta_u \end{pmatrix} \begin{pmatrix} \gamma_u \\ \gamma_{-u}^+ \end{pmatrix} \tag{37}$$

the inverse is then a rotation by $-\theta_u$

$$\begin{pmatrix} \gamma_u \\ \gamma_{-u}^+ \end{pmatrix} = \begin{pmatrix} \cos\theta_u & -\sin\theta_u \\ \sin\theta_u & \cos\theta_u \end{pmatrix} \begin{pmatrix} c_u \\ c_{-u}^+ \end{pmatrix} \tag{38}$$

Next we substitute (34) in ~~to~~ the Hamiltonian (33). We get

$$c_u^+ c_u = (\sin\theta_u \gamma_u^+ + \cos\theta_u \gamma_{-u})(\sin\theta_u \gamma_u + \cos\theta_u \gamma_{-u}^+)$$

$$= \sin^2\theta_u \gamma_u^+ \gamma_u + \cos^2\theta_u \gamma_{-u} \gamma_{-u}^+ +$$

$$+ \sin\theta_u \cos\theta_u (\gamma_u^+ \gamma_{-u}^+ + \gamma_{-u} \gamma_u)$$

$$c_{-u}^+ c_{-u} = (-\cos\theta_u \gamma_u + \sin\theta_u \gamma_{-u}^+)(-\cos\theta_u \gamma_u^+ + \sin\theta_u \gamma_{-u})$$

$$= \cos^2\theta_u \gamma_u \gamma_u^+ + \sin^2\theta_u \gamma_{-u}^+ \gamma_{-u} +$$

$$- \sin\theta_u \cos\theta_u (\gamma_u \gamma_{-u} + \gamma_{-u}^+ \gamma_u^+)$$

To add the two, we use the algebra of the operators.

In particular, $\gamma_k \gamma_{-k} = -\gamma_{-k}\gamma_k$ and $\gamma_k \gamma_k^\dagger = 1 - \gamma_k^\dagger \gamma_k$.

We then get

$$c_k^\dagger c_k + c_{-k}^\dagger c_{-k} = (u_k^2 - v_k^2)(\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k}) + 2v_k^2 + 2u_k v_k (\gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_{-k}\gamma_k) \tag{39}$$

Now we do the same for the other term in (33).

$$c_k^\dagger c_{-k}^\dagger = (u_k \gamma_k^\dagger + v_k \gamma_{-k})(-v_k \gamma_k + u_k \gamma_{-k}^\dagger)$$

$$= u_k v_k (-\gamma_k^\dagger \gamma_k + \gamma_{-k}\gamma_{-k}^\dagger) + u_k^2 \gamma_k^\dagger \gamma_{-k}^\dagger - v_k^2 \gamma_{-k}\gamma_k$$

$$c_{-k} c_k = (c_k^\dagger c_{-k}^\dagger)^\dagger$$

$$= u_k v_k (-\gamma_k^\dagger \gamma_k + \gamma_{-k}\gamma_{-k}^\dagger) + u_k^2 \gamma_{-k}\gamma_k - v_k^2 \gamma_k^\dagger \gamma_{-k}^\dagger$$

Thus

$$c_k^\dagger c_{-k}^\dagger + c_{-k} c_k = -2u_k v_k (\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k}) + 2u_k v_k + (u_k^2 - v_k^2)(\gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_{-k}\gamma_k) \tag{40}$$

Combining (39) and (40) into (33) we then get

$$H = \sum_k \Big\{ \Delta_k a_k + 2\epsilon_k \vartheta_k^2 - 2\Delta_k u_k v_k \vartheta_k + \tag{41}$$

$$+ \left[\epsilon_k (u_k^2 - v_k^2) + 2\Delta_k u_k v_k \right] (\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k})$$

$$+ \left[2\epsilon_k u_k v_k - \Delta_k (u_k^2 - v_k^2) \right] (\gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_{-k} \gamma_k) \Big\}$$

To diagonalize H we now choose u_k, v_k such that the last term vanishes. That is

$$\boxed{2\epsilon_k u_k v_k - \Delta_k (u_k^2 - v_k^2) = 0} \tag{42}$$

Using the trigonometric parametrization (36) we get

$$2 u_k v_k = 2 \cos\theta_k \sin\theta_k = \sin 2\theta_k$$

$$u_k^2 - v_k^2 = \cos^2\theta_k - \sin^2\theta_k = \cos 2\theta_k$$

Thus (42) becomes

$$\epsilon_k \sin 2\theta_k = \Delta_k \cos 2\theta_k$$

or

$$\boxed{\tan 2\theta_k = \frac{\Delta_k}{\epsilon_k}} \tag{43}$$

Now we use this to recover u_u and v_u:

$$u_u^2 - v_u^2 = 1$$

$$u_u^2 - v_u^2 = \cos 2\theta_u = \frac{\epsilon_u}{\sqrt{\epsilon_u^2 + \Delta u^2}} \tag{44}$$

Thus

$$u_u^2 = \frac{1}{2}\left[1 + \frac{\epsilon_u}{\sqrt{\epsilon_u^2 + \Delta u^2}}\right] \tag{45}$$

$$v_u^2 = \frac{1}{2}\left[1 - \frac{\epsilon_u}{\sqrt{\epsilon_u^2 + \Delta u^2}}\right]$$

we also have

$$u_u v_u = \frac{\Delta u}{2\epsilon_u}\left(u_u^2 - v_u^2\right)$$

Thus

$$u_u v_u = \frac{1}{2}\frac{\Delta u}{\sqrt{\epsilon_u^2 + \Delta u^2}} \tag{46}$$

Eqs (43) - (46) completely solve for u_u, v_u.

Now we return to the Hamiltonian (41). The constant term in the first line becomes

$$W_u := \Delta_u a_u + 2\epsilon_u \vartheta_u^2 - 2\Delta_u u_u \vartheta_u$$

$$= \Delta_u a_u + \epsilon_u \left[1 - \frac{\epsilon_u}{\sqrt{\epsilon_u^2 + \Delta_u^2}} \right] - \frac{\Delta_u^2}{\sqrt{\epsilon_u^2 + \Delta_u^2}}$$

$$= \Delta_u a_u + \epsilon_u - \frac{(\epsilon_u^2 + \Delta_u^2)}{\sqrt{\epsilon_u^2 + \Delta_u^2}}$$

or

$$\boxed{W_u = \Delta_u a_u + \epsilon_u - \sqrt{\epsilon_u^2 + \Delta_u^2}} \qquad (47)$$

the constant in the 2nd line of (41) becomes

$$\epsilon_u (u_u^2 - \vartheta_u^2) + 2\Delta_u u_u \vartheta_u = \frac{\epsilon_u^2}{\sqrt{\epsilon_u^2 + \Delta_u^2}} + \frac{\Delta_u^2}{\sqrt{\epsilon_u^2 + \Delta_u^2}}$$

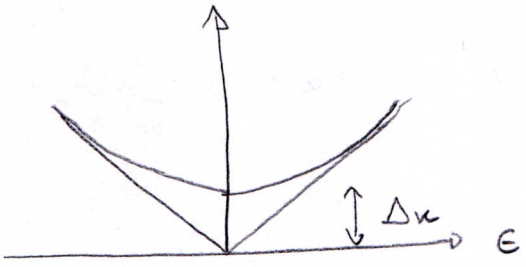
$$= \sqrt{\epsilon_u^2 + \Delta_u^2} \qquad (48)$$

Thus we finally obtain the Hamiltonian

$$H = W + \sum_k \sqrt{\epsilon_k^2 + \Delta_k^2}\,(\gamma_k^\dagger \gamma_k + \gamma_{-k}^\dagger \gamma_{-k}) \tag{49}$$

where $W = \sum_k W_k$ is a constant.

We therefore conclude that the BCS Hamiltonian has a ground-state energy W and a spectrum of fermionic excitations $\gamma_k^\dagger \gamma_k$ with dispersion relation $\sqrt{\epsilon_k^2 + \Delta_k^2}$. We still don't know what Δ_k is, but we see that it represents the <u>energy gap</u> of the excitations



the value of Δ is determined by the recursiveness of the mean-field approximation. Recall that a_u is $\langle c_u^\dagger c_{-u}^\dagger \rangle$. But in (40) we found that

$$2a_u = \langle c_u^\dagger c_{-u}^\dagger \rangle + \langle c_{-u} c_u \rangle$$

$$= -2\mu_u \vartheta_u \langle \gamma_u^\dagger \gamma_u + \gamma_{-u}^\dagger \gamma_{-u} \rangle + 2\mu_u \vartheta_u + (\mu_u^2 - \vartheta_u^2) \langle \gamma_u^\dagger \gamma_{-u}^\dagger + \gamma_{-u} \gamma_u \rangle$$

In the thermal state of the Hamiltonian (49) we always have $\langle \gamma_u^\dagger \gamma_{-u}^\dagger \rangle = 0$. Moreover

$$\langle \gamma_u^\dagger \gamma_u \rangle = \langle \gamma_{-u}^\dagger \gamma_{-u} \rangle := f_u = \frac{1}{e^{\beta\sqrt{\epsilon_u^2 + \Delta_u^2}} + 1}$$

is the Fermi-Dirac distribution. Thus

$$\boxed{a_u = \mu_u \vartheta_u (1 - 2f_u)} \qquad (50)$$

Let us first assume that we are at zero temperature. Then $f_k = 0$ and we get $a_k = u_k v_k$. Combining (46) with the definition of Δ_k in (32'), we then get

$$\Delta_k = -\sum_{k'} \frac{V_{kk'}}{2} \frac{\Delta_{k'}}{\sqrt{\epsilon_{k'}^2 + \Delta_{k'}^2}} \tag{51}$$

This is a self-consistent equation to be solved for Δ_k. Since we are assuming the constant potential (29) for $V_{kk'}$, it follows that

$$\Delta_k = \begin{cases} \Delta & -\omega_D < \epsilon_k < \omega_D \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

Thus, converting the sum in (51) to an integral we get

$$\Delta = \frac{V_0}{2} \int_{-\omega_D}^{\omega_D} d\epsilon \, \frac{n_0(\epsilon)\Delta}{\sqrt{\epsilon^2 + \Delta^2}}$$

Approximating again $n_0(\epsilon) \approx n_0(\epsilon_F)$ we get

$$1 = \frac{V_0 n_0}{2} \operatorname{arcsinh}\left(\frac{\epsilon}{\Delta}\right) \Big|_{-\omega_D}^{+\omega_D}$$

$$= V_0 n_0 \operatorname{arcsinh}\left(\frac{\omega_D}{\Delta}\right)$$

Thus

$$\frac{\omega_D}{\Delta} = \sinh(1/V_0 n_0)$$

or

$$\Delta = \frac{\omega_D}{\sinh(1/V_0 n_0)} \tag{53}$$

In the weak-coupling limit we get again

$$\boxed{\Delta \simeq 2\omega_D e^{-1/V_0 n_0(\epsilon_F)}} \tag{54}$$

Thus we see that the electrons which are in the shell $\epsilon_F - \omega_D < \epsilon_p \leq \epsilon_F + \omega_D$ have an energy gap Δ. Those deep inside the Fermi sphere have no gap, as in a normal metal. But those in the shell are gap.

The conclusion is that electrons in the ω_D shell form cooper pairs and it requires an energy gap Δ to break them apart.

Momentum distribution and coherence length

Let us now look at $\langle c_k^\dagger c_k \rangle$, still at zero temperature. To do this we use the result at the end of page 26

$$c_k^\dagger c_k = u_k^2 \gamma_k^\dagger \gamma_k + v_k^2 + \gamma_{-k}^\dagger \gamma_{-k} + u_k v_k (\gamma_k^\dagger \gamma_{-k}^\dagger + \gamma_{-k} \gamma_k) \quad (55)$$

Again, at $T=0$ all expectation values are zero so we get

$$\langle c_k^\dagger c_k \rangle = v_k^2 = \frac{1}{2}\left[1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta_k^2}}\right] \quad (56)$$

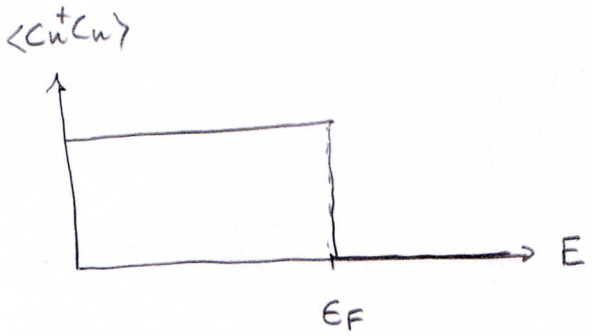
where I used Eq (45)

In a normal metal we would have $\Delta_k = 0$ so

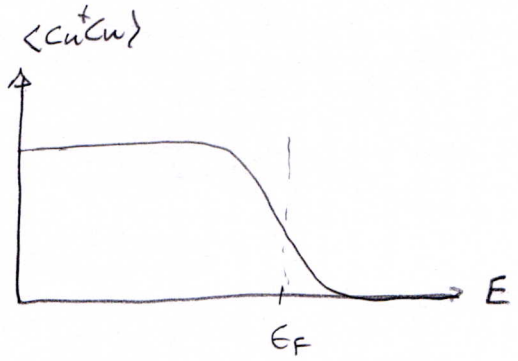
$$\langle c_k^\dagger c_k \rangle = \frac{1}{2}\left[1 - \frac{\epsilon_k}{|\epsilon_k|}\right] = \begin{cases} 1 & \epsilon_k < 0 \\ 0 & \epsilon_k > 0 \end{cases} \quad (57)$$

(Recall that we are measuring the zero of energy at the Fermi level).

Thus, we have the following picture



Normal metal



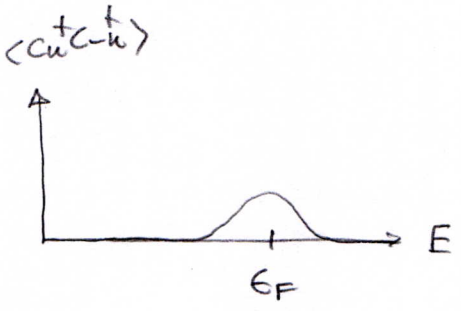
Superconductor

The quantity $\langle c_k^\dagger c_k \rangle$ is the number of electrons with momentum k. As we approach the Fermi momentum k_F the electrons start to condense into Cooper pairs, so the density of free electrons goes down.

We can also compare this with $\langle c_k^\dagger c_{-k}^\dagger \rangle$. Recalling that this is simply $u_k v_k$, we get that at $T=0$

$$\langle c_k^\dagger c_{-k}^\dagger \rangle = u_k v_k = \frac{1}{2} \frac{\Delta_k}{\sqrt{\epsilon_k^2 + \Delta_k^2}} \tag{58}$$

this quantity is always zero for a normal metal. But for a superconductor it will be non-zero in a small region of width Δ around ϵ_F



This is the region where Cooper pairs exist

From this last analysis we find that Cooper pairs exist in a region of width Δ around ϵ_F. This establishes the typical variance of the momentum of Cooper pairs.

$$E = \frac{k^2}{2m} \quad \rightsquigarrow \quad \delta E \sim \Delta \sim \frac{k}{m}\delta k$$

Using $k = k_F$ we then get

$$\delta k \sim \frac{m\Delta}{k_F} \sim \frac{\Delta}{v_F} \tag{59}$$

this gives the typical magnitude of the fluctuations of the momentum.

We may now use the uncertainty principle to relate this with the fluctuations in position, called the <u>coherence length</u> ξ_0. they are related by

$$\xi_0 \, \delta k \sim \hbar$$

thus

$$\boxed{\xi_0 \sim \frac{\hbar v_F}{\Delta}} \tag{60}$$

Using $\Delta \sim 1$ meV and $\hbar v_F \sim 10^{-9}$ m.eV we get

$$\xi_0 \sim 10^4 \text{ Å}$$

this is the typical length scale of the correlations between electrons in a Cooper pair.

Thus, we see that Cooper pairs are not bound close to each other in real space, but may be quite far apart.

The gap at finite temperatures

Let us return now to the gap self-consistency equation:

$$\Delta_k = -\sum_{k'} V_{kk'} a_{k'}$$

For finite temperatures a_k is given in (50)

$$a_k = u_k v_k (1 - 2 f_k)$$

Recall also that f_k is the Fermi-Dirac distribution. In general if $f = (e^{\beta E} + 1)^{-1}$ we may write

$$1 - 2f = 1 - \frac{2}{e^{\beta E} + 1} = \frac{e^{\beta E} + 1 - 2}{e^{\beta E} + 1}$$

$$= \frac{e^{\beta E} - 1}{e^{\beta E} + 1}$$

or

$$1 - 2f = \frac{e^{\beta E/2}}{e^{\beta E/2}} \left(\frac{e^{\beta E/2} - e^{-\beta E/2}}{e^{\beta E/2} + e^{-\beta E/2}} \right)$$

Thus

$$1 - 2f = \tanh\left(\frac{\beta E}{2}\right) \qquad (61)$$

we then get

$$a_k = \frac{1}{2} \frac{\Delta_k}{\sqrt{\epsilon_k^2 + \Delta_k^2}} \tanh\left(\frac{\beta}{2} \sqrt{\epsilon_k^2 + \Delta_k^2} \right) \tag{62}$$

Hence the self-consistency Eq for Δ becomes

$$\Delta_k = - \sum_{k'} V_{kk'} \frac{1}{2} \frac{\Delta_{k'}}{\sqrt{\epsilon_{k'}^2 + \Delta_{k'}^2}} \tanh\left(\frac{\beta}{2} \sqrt{\epsilon_{k'}^2 + \Delta_{k'}^2} \right)$$

or, using the same procedure as before

$$\Delta = V_0 \int_{-\omega_D}^{\omega_D} d\epsilon \, \frac{n_0(\epsilon)}{2} \frac{\Delta}{\sqrt{\epsilon^2 + \Delta^2}} \tanh\left(\frac{\beta}{2} \sqrt{\epsilon^2 + \Delta^2} \right)$$

Approximating $n_0(\epsilon) \simeq n_0(\epsilon_F)$ we then finally get

$$\boxed{\int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2 + \Delta^2}} \tanh\left(\frac{\beta}{2} \sqrt{\epsilon^2 + \Delta^2} \right) = \frac{2}{V_0 n_0}} \tag{63}$$

this is an inverse equation we must solve for Δ. In the limit $T \to 0$ we get $\tanh(\cdots) \to 1$ and we recover our zero temperature analysis in page 33.

In general Eq (63) cannot be solved analytically. But there is notwithstanding, a lot of information we can extract from it. First, we can get the critical temperature T_c as the value of T where the gap vanishes. Setting $\Delta = 0$ we then get

$$\int_{-\omega_D}^{\omega_D} \frac{d\epsilon}{|\epsilon|} \tanh\left(\frac{|\epsilon|}{2T_c}\right) = \frac{2}{V_0 \nu_0}$$

the integrand is even. So we may integrate from 0 to ω_D and multiply by 2:

$$\int_0^{\omega_D} \frac{d\epsilon}{\epsilon} \tanh\left(\frac{\epsilon}{2T_c}\right) = \frac{1}{V_0 \nu_0}$$

Now let $x = \epsilon/T_c$. Then $\frac{d\epsilon}{\epsilon} = \frac{dx}{x}$ and we get

$$\int_0^{\omega_D/T_c} \frac{1}{x} \tanh\frac{x}{2}\, dx = \frac{1}{V_0 \nu_0}$$

when ω_D / T_c is large this integral becomes approximately

$$\int_0^{\omega_D/T_c} \frac{1}{x} \tanh\frac{x}{2}\, dx \cong \ln(1.13\, \omega_D/T_c)$$

Thus, we get

$$1.13\ \frac{\omega_D}{T_c} = \exp(1/V_0 n_0)$$

or, finally

$$\boxed{T_c = 1.13\,\omega_D\, e^{-1/V_0 n_0}} \qquad (64)$$

We can also compare this with Eq (54)

$$\Delta(0) = 2\,\omega_D\, e^{-1/V_0 n_0} \qquad (65)$$

where I write $\Delta(0)$ to emphasize that this is the gap at $T=0$.

We therefore see that

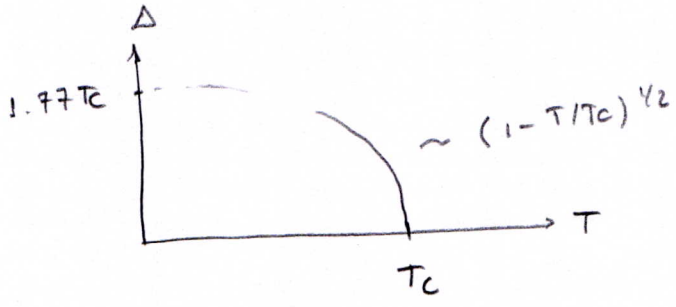
$$\boxed{\Delta(0) = 1.77\ T_c} \qquad (66)$$

The gap at zero temperature is therefore <u>easily</u> estimated from the value of T_c.

we can also estimate the integral close to $T = T_c$. One then finds

$$\Delta(T) \simeq 3.06\, T_c \left(1 - \frac{T}{T_c}\right)^{1/2} \qquad (T \lesssim T_c) \tag{67}$$

the gap therefore goes down with T_c with the typical mean-field exponent of 1/2.



(don't try to apply (67) to $T = 0$. It is only valid close to T_c ☺)

We therefore emerge from these calculations with the idea that the gap is maximum at $T = 0$ and goes down to zero as T approaches T_c.

Electronic heat capacity

To compute the heat capacity of the superconductor I will borrow a formula from statistical mechanics. Namely, the entropy of a Fermi gas with Fermi distribution f_k is given by

$$S = -2 \sum_k \{ f_k \ln f_k + (1-f_k) \ln (1-f_k) \} \tag{68}$$

we can then find the heat capacity from

$$C = T \frac{\partial S}{\partial T} \tag{69}$$

Now:

$$\frac{\partial S}{\partial T} = \sum_k \frac{\partial S}{\partial f_k} \frac{\partial f_k}{\partial T} \tag{70}$$

we have

$$\frac{\partial S}{\partial f_k} = -2 \ln \frac{f_k}{1-f_k}$$

using $f_k = \left(e^{\beta \sqrt{\epsilon_k^2 + \Delta_k^2}} + 1 \right)^{-1}$ we then get

$$\frac{\partial S}{\partial f_k} = \frac{2}{T} \sqrt{\epsilon_k^2 + \Delta_k^2} \tag{71}$$

Moreover

$$\frac{\partial f_k}{\partial T} = \frac{(-1)\, e^{\beta \sqrt{\epsilon_k^2 + \Delta_k^2}}}{\left(e^{\beta \sqrt{\epsilon_k^2 + \Delta_k^2}} + 1 \right)^2} \frac{\partial}{\partial T} \left[\beta \sqrt{\epsilon_k^2 + \Delta_k^2} \right]$$

or $\left(\omega_k = \sqrt{\epsilon_k^2 + \Delta_k^2}\right)$

$$\frac{\partial f_k}{\partial T} = \frac{e^{\beta\omega_k}}{(e^{\beta\omega_k}+1)^2}\left\{\frac{\sqrt{\epsilon_k^2+\Delta_k^2}}{T^2} - \frac{1}{2T}\frac{1}{\sqrt{\epsilon_k^2+\Delta_k^2}}\frac{\partial}{\partial T}\Delta_k^2\right\}$$

Thus we get

$$C = T\sum_k \frac{\partial S}{\partial f_k}\frac{\partial f_k}{\partial T}$$

$$= T\frac{2}{T}\sum_k \sqrt{\epsilon_k^2+\Delta_k^2}\,\frac{e^{\beta\omega_k}}{(e^{\beta\omega_k}+1)}\left\{\frac{\sqrt{\epsilon_k^2+\Delta_k^2}}{T^2} - \frac{1}{2T}\frac{1}{\sqrt{\epsilon_k^2+\Delta_k^2}}\frac{\partial}{\partial T}\Delta_k^2\right\}$$

$$\therefore \quad C = \frac{2}{T^2}\sum_k \frac{e^{\beta\omega_k}}{(e^{\beta\omega_k}+1)^2}\left\{\epsilon_k^2+\Delta_k^2 - \frac{T}{2}\frac{\partial}{\partial T}\Delta_k^2\right\} \qquad (72)$$

Now we convert the sum to an integral. There is a contribution from all values of the energy, even though the gap Δ_k is non-zero only in a small energy range. Let us then write $c = C/N$, and

$$c \simeq \frac{2N_0(\epsilon_F)}{T^2}\int_{-\infty}^{\infty} d\epsilon\, \frac{e^{\frac{\beta}{2}\sqrt{\epsilon^2+\Delta_\epsilon^2}}}{\left(e^{\frac{\beta}{2}\sqrt{\epsilon^2+\Delta_\epsilon^2}}+1\right)^2}\left\{\epsilon^2+\Delta_\epsilon^2 - \frac{T}{2}\frac{\partial}{\partial T}\Delta_\epsilon^2\right\} \qquad (73)$$

where $\Delta\epsilon$ is such that $\Delta\epsilon = \Delta$ if $-\omega_D < \epsilon < \omega_D$ and zero otherwise.

If $\Delta = 0$ we are in the normal metal phase. We then get

$$C_m \simeq \frac{2 n_0(\epsilon_F)}{T^2} \underbrace{\int_{-\infty}^{\infty} \frac{e^{\beta|\epsilon|}}{(e^{\beta|\epsilon|}+1)^2} \epsilon^2 \, d\epsilon}_{\frac{\pi^2 T^3}{3}}$$

Thus

$$C_m = \frac{2\pi^2}{3} n_0(\epsilon_F) \, T \tag{74}$$

which is the usual Sommerfeld result giving $C \sim T$.

The important aspect of Eq (73) is that it depends on $\frac{\partial}{\partial T}\Delta^2$. The gap as a function of temperature is continuous. But its derivative is not. In fact, at T_c, $\frac{\partial}{\partial T}\Delta^2$ has a discontinuity.

From Eq (67) we have

$$\Delta(T)^2 = (3.06)^2 T_c^2 \left(1 - \frac{T}{T_c}\right)$$

$$\frac{\partial}{\partial T}\Delta(T)^2 = -(3.06)^2 \frac{T_c^2}{T_c} = -(3.06)^2 T_c$$

Plugging this back into (73) we see that the discontinuity in the specific heat will be, at $T = T_c$

$$\Delta C = \frac{2 n_0(\epsilon_F)}{T_c^2} \int_{-\infty}^{\infty} d\epsilon \, \frac{e^{\beta_c|\epsilon|}}{(e^{\beta_c|\epsilon|}+1)^2} \, \frac{T_c}{2} (3.06)^2 T_c$$

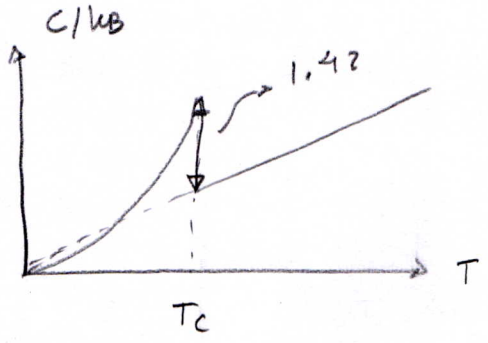
$$\Delta C = 9.36\, n_0(\epsilon_F) \underbrace{\int_{-\infty}^{\infty} \frac{e^{\beta_c|\epsilon|}}{(e^{\beta_c|\epsilon|}+1)^2}\, d\epsilon}_{T_c}$$

$$\Delta C = 9.36\, n_0(\epsilon_F)\, T_c \tag{75}$$

we then get, at T_c

$$\boxed{\frac{\Delta C}{C_n} = \frac{9.36}{2\pi^2/3} = 1.42} \tag{76}$$

This looks somewhat like this



this jump of 1.42 is the first signature that your superconductor is of "BCS type": if the jump is around 1.4, then it certainly is BCS.

This is the last paragraph of the last set of notes of my course. If you've read this far you are a champion. Congratulations. If you want, send me an e-mail and I'll give you a chocolate.