Quantum spin chains

In these notes I want to talk about a topic I think is really cool: a family of exactly solvable 1D spin 1/2 chains. The basic Hamiltonian has the form

$$\boxed{H = -\frac{h}{2}\sum_{i=1}^{L} \sigma_z^i - \frac{1}{2}\sum_{i=1}^{L-1} \left(J_x \sigma_x^i \sigma_x^{i+1} + J_y \sigma_y^i \sigma_y^{i+1} \right)} \qquad (1)$$

this goes by the name of XY model. If $J_x = J_y$ it is called the XX model:

$$H = -\frac{h}{2}\sum_{i=1}^{L} \sigma_z^i - \frac{J}{2}\sum_{i=1}^{L-1} \left(\sigma_x^i \sigma_x^{i+1} + \sigma_y^i \sigma_y^{i+1} \right) \qquad (2)$$

this situation is special as the system has rotation invariance around the z axis (think about the last term as the dot product of 2 vectors in the XY plane).

Instead, if $J_y = 0$ in Eq (1) we obtain the transverse field Ising model (TFIM)

$$H = -\frac{h}{2}\sum_{i=1}^{L} \sigma_z^i - \frac{J}{2}\sum_{i=1}^{L-1} \sigma_x^i \sigma_x^{i+1} \qquad (3)$$

this is perhaps one of the most famous models in statistical physics. We will also see that this is equivalent to a model by Kitaev for topological superconductors.

Oh, and I forgot to say, in (1) - (3) I am assuming open boundary conditions (OBC). But in these notes we will consider both OBC and periodic boundary conditions (PBC).

The Jordan - Wigner transformation

Let us change from σ_x^i, σ_y^i to $\sigma_\pm^i$:

$$\sigma_x^i = \sigma_+^i + \sigma_-^i \qquad \sigma_\pm^i = \frac{\sigma_x^i \pm i\,\sigma_y^i}{2} \qquad (4)$$

$$\sigma_y^i = \frac{\sigma_+^i - \sigma_-^i}{i}$$

then

$$\sigma_x^i \sigma_x^{i+1} = (\sigma_+^i + \sigma_-^i)(\sigma_+^{i+1} + \sigma_-^{i+1})$$

$$= \sigma_+^i \sigma_+^{i+1} + \sigma_-^i \sigma_-^{i+1} + \sigma_+^i \sigma_-^{i+1} + \sigma_-^i \sigma_+^{i+1} \qquad (5a)$$

$$\sigma_y^i \sigma_y^{i+1} = -(\sigma_+^i - \sigma_-^i)(\sigma_+^{i+1} - \sigma_-^{i+1})$$

$$= -\sigma_+^i \sigma_+^{i+1} - \sigma_-^i \sigma_-^{i+1} + \sigma_+^i \sigma_-^{i+1} + \sigma_-^i \sigma_+^{i+1} \qquad (5b)$$

Thus

$$J_x \sigma_x^i \sigma_x^{i+1} + J_y \sigma_y^i \sigma_y^{i+1} = (J_x - J_y)(\sigma_+^i \sigma_+^{i+1} + \sigma_-^i \sigma_-^{i+1}) \qquad (5c)$$

$$+ (J_x + J_y)(\sigma_+^i \sigma_-^{i+1} + \sigma_-^i \sigma_+^{i+1})$$

It is convenient to parametrize

$$J_x = J\,\frac{(1+\gamma)}{2} \qquad \gamma = 0 \to \text{XX model}$$

$$(6)$$

$$J_y = J\,\frac{(1-\gamma)}{2} \qquad \gamma = 1 \to \text{TFIM}$$

We then get $J_x - J_y = J\gamma$ and $J_x + J_y = J$, so that (5c) becomes

$$J_x \sigma_x^i \sigma_x^{i+1} + J_y \sigma_y^i \sigma_y^{i+1} = J \left\{ \sigma_+^i \sigma_-^{i+1} + \sigma_-^i \sigma_+^{i+1} + \right. \tag{7}$$
$$\left. + \gamma \left(\sigma_+^i \sigma_+^{i+1} + \sigma_-^i \sigma_-^{i+1} \right) \right\}$$

We also get rid of σ_z:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$\hookrightarrow \sigma_+ \sigma_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8a}$$

Thus

$$\sigma_z = 2\sigma_+\sigma_- - 1 \tag{8b}$$

Hence, we may finally write the Hamiltonian (1) as

$$\boxed{H = -h \sum_{i=1}^{L} \sigma_+^i \sigma_-^i - \frac{J}{2} \sum_{i=1}^{L-1} \left\{ \sigma_+^i \sigma_-^{i+1} + \sigma_-^i \sigma_+^{i+1} + \gamma \left(\sigma_+^i \sigma_+^{i+1} + \sigma_-^i \sigma_-^{i+1} \right) \right\}} \tag{9}$$

where I neglected the constant that comes from the "1" in Eq (8)

Notice how, at least the term without γ, look a bit like a tight-binding chain. It essentially flips a spin at site i and then does the opposite flip on $i+1$. (the term with γ makes things a bit weirder, I admit).

However, there is one fundamental difference which distinguishes this from a tight-binding model: the spin algebra is messed up! Bosons and Fermions are clean

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{10}$$

$$\{c_i, c_j^\dagger\} = \delta_{ij} \tag{11}$$

But for Pauli matrices we get something mixed. Recall that

$$\sigma_+^i = 1 \otimes \ldots \otimes \sigma_+ \otimes \ldots \otimes 1 \tag{12}$$

thus if $i \neq j$ then the $\sigma_\pm$ commute

$$\boxed{(i \neq j) \qquad [\sigma_+^i, \sigma_-^j] = [\sigma_+^i, \sigma_+^j] = 0} \tag{13}$$

But if $i = j$ then they satisfy the same algebra of the 2×2 matrices in Eq (8a), which is an anti-commutation relation

$$\boxed{\{\sigma_+^i, \sigma_-^i\} = 1} \tag{14}$$

Isn't this awkward? they behave like bosons when $i \neq j$ but like fermions when $i = j$. This messes up everything. At first we could be tempted to think about $\sigma_\pm^i$ as fermionic operators $c_i^\dagger$ and c_i, since both only have 2 levels. But they are not fermionic, as they commute for $i \neq j$ (whereas fermionic operators should anti-commute).

A day may come when there is nothing we can do to move forward. But it is not this day! this day we use an idea developed by Jordan and Wigner to map Pauli operators into fermionic operators:

$\boxed{\text{Jordan-Wigner: exact mapping between spin } 1/2 \text{ Pauli operators and fermionic operators}}$

the map looks like this

$$\boxed{c_i = \left[\prod_{m=1}^{i-1} (-\sigma_z^m) \right] \sigma_-^i} \qquad \text{Jordan-Wigner} \qquad (15)$$

Yeah, it's weird I know. Let me explain. Recall the tensor structure of the Pauli operators [Eq (12)]. What (15) means is

$$c_i = (-\sigma_z) \otimes (-\sigma_z) \otimes \dots \otimes (-\sigma_z) \otimes \sigma_- \otimes 1 \otimes 1 \dots \otimes 1 \qquad (16)$$

we get the operator σ_- and we multiply it by a <u>string</u> of $(-\sigma_z)$ operators on all sites to the left.

The operator c_i^+ is just the adjoint of (15). Since Pauli operators for different sites commute we can write

$$c_i^+ = \left[\prod_{m=1}^{i-1} (-\sigma_z^m) \right] \sigma_+^i \qquad (17)$$

Let's now check that these guys satisfy the fermionic algebra (11) and in doing so we will understand why this transformation works.

The only thing we need to remember is that

$$(\sigma_z)^2 = 1 \tag{18}$$

We then get, for the same site $(i=j)$

$$c_i c_i^\dagger = \left[\prod_{m=1}^{i-1} (-\sigma_z^m)\, \sigma_-^i \right] \left[\prod_{m=1}^{i-1} (-\sigma_z^m)\, \sigma_+^i \right]$$

Note how we can move the right-most block of σ_z's accross σ_-^i, as the block acts only on $1, \dots, i-1$. This block will then match up with the other block and give 1 due to (18)

$$\left[\prod_{m=1}^{i-1} (-\sigma_z^m) \right] \left[\prod_{m=1}^{i-1} (-\sigma_z^m) \right] = 1 \tag{19}$$

Thus we get simply

$$c_i c_i^\dagger = \sigma_-^i \sigma_+^i \tag{20a}$$

And, by the same arguments

$$c_i^\dagger c_i = \sigma_+^i \sigma_-^i \tag{20b}$$

Thus, the "same-site" algebra remains perfect

$$\boxed{\{ \sigma_-^i, \sigma_+^i \} = \{ c_i, c_i^\dagger \} = 1} \tag{21}$$

Next, let us talk about different sites, which is where the problem is. Suppose, for concreteness, that $j > i$. Then we get

$$c_i c_j^{\dagger} = \left[\prod_{m=1}^{i-1} (-\sigma_z^m)\, \sigma_-^i \right] \left[\prod_{m=1}^{j-1} (-\sigma_z^m)\, \sigma_+^j \right] \tag{22}$$

Now the σ_z block cannot go through the σ_-^i because there will be a $(-\sigma_z^i)$ in the right block. But now comes the magic. Are you ready? Using only the 2×2 matrices in Eq (8a), one may check that

$$\boxed{\begin{aligned} \sigma_+ (-\sigma_z) &= \sigma_+ & \qquad \sigma_- (-\sigma_z) &= -\sigma_- \\ (-\sigma_z)\, \sigma_+ &= -\sigma_+ & \qquad (-\sigma_z)\, \sigma_- &= \sigma_- \end{aligned}} \tag{23}$$

Thus, going back to (22) we can, in the end, take the right-most σ_z block through σ_-^i. But all we need to know is that, when $(-\sigma_z^i)$ crosses σ_-^i, we will get a -1:

$$\boxed{\sigma_-^i (-\sigma_z^i) = - (-\sigma_z^i)\, \sigma_-^i} \tag{24}$$

As a result, we will cancel most σ_z's, except those between i and j

$$(j>i) \qquad c_i c_j^{\dagger} = -\left[\prod_{m=i}^{j-1} (-\sigma_z^m) \right] \sigma_-^i \sigma_+^j \tag{25}$$

Next we do it the other way around

$$c_j^\dagger c_i = \left[\prod_{m=1}^{j-1} (-\sigma_z^m)\, \sigma_+^j \right] \left[\prod_{m=1}^{i-1} (-\sigma_z^m)\, \sigma_-^i \right]$$

But now we have no problems moving the σ_z's to the left because $j > i$. Thus we get

$$c_j^\dagger c_i = \left[\prod_{m=i}^{j-1} (-\sigma_z^m) \right] \sigma_+^j \sigma_-^i \qquad (26)$$

Note how the key is the minus sign in (25). It converts a anti-commutator into a commutator

$$\{c_i, c_j^\dagger\} = \left[\prod_{m=1}^{j-1} (-\sigma_z^m) \right] \underbrace{\left(-\sigma_-^i \sigma_+^j + \sigma_+^j \sigma_-^i \right)}_{0 \text{ because } \sigma\text{'s commute.}} = 0 \qquad (27)$$

Thus, as our big conclusion, we find that the Jordan-Wigner transformation (15) indeed produces fermionic operators from Pauli matrices

In terms of states, we can also map the computational basis $|\sigma_1 \dots \sigma_L\rangle$ into a fermionic Fock basis. From (15) it is quite clear that the vacuum corresponds to

$$|0\rangle = |\downarrow \dots \downarrow\rangle \tag{28}$$

as the σ_-^i annihilate any $\downarrow$ state. This motivates us to try a correspondence of the form

$$|0\rangle = |\downarrow\rangle \qquad |1\rangle = |\uparrow\rangle \tag{29}$$

Let's see if this works. Recall that Fermionic operators should be anti-symmetric under sign exchange. thus

$$c_i\, |m_1, \dots, m_i, \dots, m_L\rangle = (-1)^{\Sigma_\ell}\, m_i\, |m_1 \dots, m_i - 1, \dots, m_L\rangle \tag{30}$$

where

$$\Sigma_\ell = \sum_{i=1}^{\ell-1} m_i \tag{31}$$

In words, every time we jump a c_i over each state, we get a (-1).

Now it becomes even more evident why the factors of $(-\sigma_z^m)$ in (15) are so important: recall that

$$\sigma_z\, |\sigma\rangle = \sigma\, |\sigma\rangle \qquad \sigma = \pm 1 \tag{32}$$

In terms of $m = 0, 1$ we then have

$$\sigma = 2m - 1 \qquad m = 0 \leftrightarrow \sigma = -1$$
$$m = \frac{\sigma + 1}{2} \qquad m = 1 \leftrightarrow \sigma = 1 \tag{33}$$