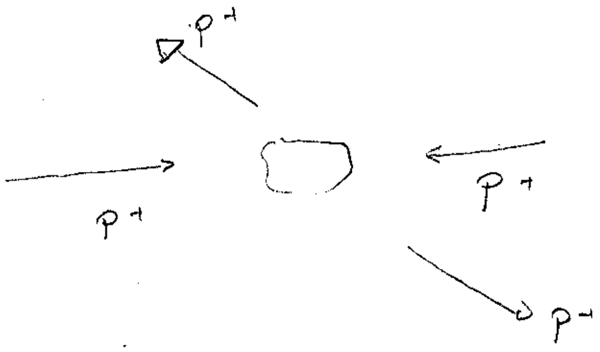
Identical Particles

In classical mechanics there is absolutely nothing interesting about identical particles, because we can always paint them in different colors or label them with numbers.

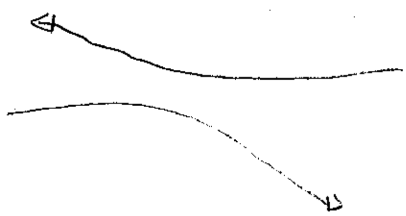
But in QM the situation is completely different. Two electrons are entirely <u>indistinguishable</u>, there is no way, nor there will ever be, to paint an electron or stamp a bar code in it. Not It is not that <u>we</u> don't know how to distinguish them. Not even nature does.

As an example, suppose we collide two protons, but with a sufficiently low energy so that they are not destroyed in any way.



Two protons go in. Two protons come out. But notice that we have two possibilities:

and



Both situations lead to exactly the same result. In fact, it doesn't even matter which of the two possibilities actually happened. The only thing that matters is that 2 protons came out.

The particle exchange operator

Let $|m\rangle$ be a set of basis states describing a single particle, for instance an electron. For instance, we could use the momentum kets, plus the spin ket : $|m\rangle = |k_x, k_y, k_z, s\rangle$, where $s = \pm 1$. Or we could use the basis vectors of the hydrogen atom: $|n, \ell, m, s\rangle$. The only point is that $|m\rangle$ completely characterizes the particle

Suppose, however, that we have 2 of these particles and that they are identical. Then a basis for the product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is $|m, m\rangle$. Hence, any ket $|\alpha\rangle$ of the product space may be written as

$$|\alpha\rangle = \sum_{m, m} \alpha_{mm} |m, m\rangle \tag{1}$$

We now define the particle exchange operator $\hat{P}$ as being the operator which exchanges the two particles

$$\boxed{\hat{P}|m, m\rangle = |m, m\rangle} \tag{2}$$

This is a symmetry operator, so it must be unitary

$$\hat{P}^{\dagger}\hat{P} = 1 \tag{3}$$

Moreover, if we apply it twice we get back where we started

$$\hat{P}^2 |m, m\rangle = |m, m\rangle$$

Hence

$$\hat{P}^2 = 1 \qquad (4)$$

which also means that $\hat{P}$ is Hermitian

$$\hat{P}^\dagger = \hat{P} \qquad (5)$$

We have learned that if $\hat{P}$ is a symmetry, then it must commute with the Hamiltonian

$$[\hat{P}, \hat{H}] = 0 \qquad (6)$$

For this particular case of particle exchange, the situation is actually more general. If $\hat{A}$ is <u>any</u> observable of the system, then we must have

$$\boxed{[\hat{P}, \hat{A}] = 0} \qquad (7)$$

So $\hat{P}$ not only commutes with $\hat{H}$, it commutes with all operators of the system of 2 particles.

The reason is the following. Suppose the system is in a state $|\alpha\rangle$ and define

$$|\alpha'\rangle = \hat{P}\,|\alpha\rangle \tag{8}$$

is the state where the two particles are exchanged.

Now, if the particles are indistinguishable, then <u>all</u> <u>experimental</u> <u>observations</u> of the systems <u>must</u> be invariant of the exchange of the two particles. For, otherwise, we would be able to use these observations to distinguish the particles. Hence, if $\hat{A}$ is any observable, then it must be true that

$$\boxed{\langle\alpha'|\hat{A}|\alpha'\rangle = \langle\alpha|\hat{A}|\alpha\rangle} \tag{9}$$

but

$$\langle\alpha'|\hat{A}|\alpha'\rangle = \langle\alpha|\hat{P}^{+}\hat{A}\hat{P}|\alpha\rangle$$

so that

$$\hat{P}^{+}\hat{A}\hat{P} = \hat{A}$$

Since $\hat{P}$ is unitary, this implies Eq (7).

Eigenvalues and eigenvectors of $\hat{P}$

Let us write the eigenvalue / eigenvector Eq for $\hat{P}$ as

$$\hat{P}\,|\chi\rangle = \chi\,|\chi\rangle \tag{10}$$

Since $\hat{P}^2 = 1$ we have

$$\hat{P}^2\,|\chi\rangle = |\chi\rangle = \chi^2\,|\chi\rangle$$

Thus

$$\boxed{\chi = \pm 1} \tag{11}$$

We know that the eigenvalues must be +1 or -1. But we don't know how many are +1 and how many are -1.

Now, to look for the eigenvectors, we use Eq (2). Starting with $|m,m\rangle$, we see that

$$\hat{P}\,|m,m\rangle = |m,m\rangle \tag{12a}$$

so $|m,m\rangle$ is an eigenvector with $\chi = +1$. Next consider the linear combination

$$\hat{P}\left[\frac{|m,m\rangle + |m,m\rangle}{\sqrt{2}}\right] = \frac{|m,m\rangle + |m,m\rangle}{\sqrt{2}} \tag{12b}$$

This is also an eigenvector with $\chi = +1$.

Alternatively, we may look at

$$\hat{P}\left[\frac{|n,m\rangle - |m,n\rangle}{\sqrt{2}}\right] = \frac{|m,n\rangle - |n,m\rangle}{\sqrt{2}} \qquad (12c)$$

So these are eigenvectors with $\lambda = -1$.

We can see that these are all the eigenvectors as follows. Suppose $\mathcal{H}_1$ has dimension d, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ will have dimension d^2. There are d vectors of the form $|n,n\rangle$. The number of eigenvectors of the form in (12b) or (12c) is, avoiding double counting, $\frac{d(d-1)}{2}$. Thus in total we have

$$d + \frac{d(d-1)}{2} + \frac{d(d-1)}{2} = d + d^2 - d = d^2$$

Hence, the eigenvectors in (12) are all the eigenvectors of $\hat{P}$.

We write

$$\boxed{|n,m\rangle_S = \begin{cases} |n,n\rangle & m = n \\ \dfrac{|n,m\rangle + |m,n\rangle}{\sqrt{2}} & m \neq n \end{cases}} \qquad (13)$$

and

$$\boxed{|n,m\rangle_A = \frac{|n,m\rangle - |m,n\rangle}{\sqrt{2}}} \qquad (14)$$

The states $|m,m'\rangle_S$ are eigenvectors of $\hat{P}$ with $\epsilon = +1$. They are called <u>symmetric states</u> and there are $d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$ of them.

The states $|m,m'\rangle_A$ have $\epsilon = -1$ and are called <u>anti-symmetric</u> states. There are $\frac{d(d-1)}{2}$ of them.

<u>Example</u>: spin 1/2 particles. $(d=2)$

The symmetric states are the triplets $\left(\frac{2(2+1)}{2} = 3\right)$

$$|++\rangle_S = |++\rangle$$

$$|+-\rangle_S = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}$$

$$|--\rangle_S = |--\rangle$$

and the anti-symmetric is the singlet $\left(\frac{2(2-1)}{2} = 1\right)$

$$|+-\rangle_A = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$$

<u>Exercise</u>: write the symmetric and anti-symmetric states of two spin 1 particles.

The eigenvalues of $\hat{P}$ are enormously degenerate. So any linear combination of the $|n,m\rangle_S$ will also be an eigenvector of $\hat{P}$ with $\chi = +1$:

$$|\alpha\rangle_S = \sum_{n,m} \alpha_{nm}^{S} \; |n,m\rangle_S \qquad (15)$$

Similarly, all vectors of the form

$$|\alpha\rangle_A = \sum_{n,m} \alpha_{nm}^{A} \; |n,m\rangle_A \qquad (16)$$

will be an eigenvector of $\hat{P}$ with $\chi = -1$.

We thus reach a cool conclusion: the Hilbert space $\mathcal{H}$ factors into two subspaces. The first, $\mathcal{H}_S$, has all vectors of the form (15) and the second, $\mathcal{H}_A$, has all vectors of the form (16). Mathematically we write

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_A \qquad (17)$$

Here $\oplus$ is called a <u>direct sum</u>. (this is different from $\otimes$). All this means is that we can write a basis for $\mathcal{H}$ in which the vectors are either symmetric or anti-symmetric. It doesn't mean any vector $|\alpha\rangle$ will be divided in this way.

Since $[\hat{A}, \hat{P}] = 0$, this means that we may always write the eigenvectors of $\hat{A}$ such that they will also be eigenvectors of $\hat{P}$; ie, such that they will either be symmetric or anti-symmetric.

Next consider matrix elements of an observable $\hat{A}$ in the basis $|m, m\rangle_S$ and $|m, m\rangle_A$. In particular, we look at the cross terms

$$ {}_S\langle m', m' | \hat{A} \hat{P} | m, m \rangle_A = - {}_S\langle m', m' | \hat{A} | m, m \rangle_A $$

On the other hand

$$ {}_S\langle m', m' | \hat{P} \hat{A} | m, m \rangle_A = {}_S\langle m', m' | \hat{A} | m, m \rangle_A $$

But $\hat{A} \hat{P} = \hat{P} \hat{A}$. Thus we must conclude that

$$ \boxed{ {}_S\langle m' m' | \hat{A} | m, m \rangle_A = 0 } \qquad (18) $$

In words: no observable will ever have matrix elements connecting the $\pi = 1$ and $\pi = -1$ subspaces. The matrix $\hat{A}$ will look like

$$ \hat{A} = \begin{array}{cc} \begin{array}{cc} |m, m\rangle_S & |m, m\rangle_A \end{array} & \\ \left[\begin{array}{c|c} & 0 \\ \hline 0 & \end{array} \right] & \begin{array}{c} {}_S\langle m, m| \\ \\ {}_A\langle m, m| \end{array} \end{array} \qquad (19) $$

It factors into blocks which do not communicate.

The spin-statistics theorem

What our calculation shows is that the whole thing separates into two subspaces and there is no communication between them. Thus, if the particles start out in a symmetric state, they will remain in a symmetric state for all eternity. And similarly for an anti-symmetric state. Thus, from our results, the system has the possibility of living in $\mathcal{H}_S$ and in $\mathcal{H}_A$.

However, nature does not work that way. It is an experimental fact that a pair of particles will either live only in a symmetric space or only in an anti-symmetric space. And the choice of which space they will live depends only on the spin of the particles

Particles which live in the symmetric space ($\varkappa = +1$) are called Bosons. They include the photon, Mesons and the Higgs. Also, they have integer spin $0, 1, 2, \dots$

Particles who live in the anti-symmetric space ($\varkappa = -1$) are called Fermions. They have half-integer spin, $1/2$, $3/2$, $5/2$, ... and include the electron, proton and neutron.

Thus, a system of two electrons will always be in an anti-symmetric state. Always.

This relation between what space each particle lives and their spin is called the <u>spin-statistics theorem</u>. It can be demonstrated in relativistic quantum field theory. So for us it should be taken as an experimental fact.

For Fermions, in particular, Eq. (14) implies that

$$\boxed{|m, m\rangle_A = 0} \tag{20}$$

This is the famous <u>Pauli exclusion principle</u>: no two Fermions will ever occupy the same state.

Spinless particles (spin 0)

The only particle we know that has spin 0 is the Higgs Boson. In this case the particle will only inhabit the symmetric subspace. Thus the most general ket of the space of two spin 0 particles has the form

$$|\alpha\rangle_S = \sum_{n,m} \alpha^{S}_{nm} \, |n,m\rangle_S .$$

The entire anti-symmetric space is never used. It is just abandoned. In terms of wave-functions

$$\psi_m(x = \langle x|m\rangle \qquad (21)$$

the typical symmetric states are of the form

$$\boxed{\Psi_S(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_n(x_1)\,\psi_m(x_2) + \psi_m(x_1)\,\psi_n(x_2)\right]} \qquad (22)$$

typical anti-symmetric wavefunction would be of the form

$$\boxed{\Psi_A(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_n(x_1)\,\psi_m(x_2) - \psi_m(x_1)\,\psi_n(x_2)\right]} \qquad (23)$$

Spin 1/2 particles

Spin 1/2 particles are by far the most important case. Here we must be careful to distinguish between the spatial part of the state and the spin part.

Since the spin part is sepparated from the spatial part, the single-particle base states can always be written as $|m,s\rangle$ instead of $|m\rangle$, where m now refers only to the spatial part.

The basis for the product state now reads

$$|m\,s,\ m,s'\rangle$$

the effect of $\hat{P}$ is to exchange both spatial and spin parts

$$\hat{P}\,|ms,\ ms'\rangle = |ms',\ ms\rangle \tag{23}$$

We may, alternatively, divide both the spatial and the spin parts in symmetric and anti-symmetric groups. The division of the spatial part is the same as Eq. (13). And the division of the spin part is given by the triplet and singlet states. Thus we have four possibilities

$$\begin{aligned}
&|m,m\rangle_S\ |s,s'\rangle_S \\
&|m,m\rangle_S\ |s,s'\rangle_A \\
&|m,m\rangle_A\ |s,s'\rangle_S \\
&|m,m\rangle_A\ |s,s'\rangle_A
\end{aligned} \tag{24}$$

we may now sepparately examine the effect of $\hat{P}$ in each part. what is important is that the overall state should be anti-symmetric. Thus

$$\hat{P}\, |\rangle_S \, |\rangle_S = |\rangle_S \, |\rangle_S$$

This is not anti-symmetric so it doesn't work. We will never find two electrons in a state where both spatial and spin parts are symmetric. The same is true for $|\rangle_A |\rangle_A$. Hewer, we may either have $|\rangle_A |\rangle_S$ or $|\rangle_S |\rangle_A$, which means hat if the spin part corresponds to a singlet state (which is A), then the spatial wavefunction will be symmetric. conversely, for a triplet state it will be anti-symmetric

$$\boxed{\begin{array}{lcl} \text{Singlet} & \Leftrightarrow & \Psi_S(x_1, x_2) \\ \text{Triplet} & \Leftrightarrow & \Psi_A(x_1, x_2) \end{array}} \qquad (25)$$

So the total Hilbert space has 4 parts, but the S-S and 1-A parts are completely abandoned due to the spin tatestics theorem.

The exchange interaction

Now I want to work out an example to show you that the mere fact that the particles are indistinguishable implies to them very peculiar properties.

What I want to do is compute

$$\langle (\hat{x}_2-\hat{x}_1)^2\rangle = \langle \hat{x}_1^2\rangle + \langle \hat{x}_2^2\rangle - 2\langle \hat{x}_1 \hat{x}_2\rangle \qquad (26).$$

I will do this for 3 cases: (i) if they are distinguishable, in a state $|n,m\rangle$. (ii) Bosons in a state $|n,m\rangle_S$ and (iii) Fermions in a state $|n,m\rangle_A$.

Distinguishable case

$$\langle \hat{x}_1^2\rangle = \langle n,m|\hat{x}_1^2|n,m\rangle = \left[\langle n|\otimes\langle m|\right]\left[\hat{x}^2\otimes 1\right]\left[|n\rangle\otimes|m\rangle\right]$$

$$= \langle n|\hat{x}^2|n\rangle \otimes \langle m|m\rangle$$

Thus

$$\langle \hat{x}_1^2\rangle = \langle \hat{x}^2\rangle_n \qquad (27a)$$

Similarly

$$\langle \hat{x}_2^2\rangle = \langle \hat{x}^2\rangle_m \qquad (27b)$$

Moreover

$$\langle \hat{x}_1\hat{x}_2\rangle = \left[\langle n|\otimes\langle m|\right]\left[\hat{x}\otimes\hat{x}\right]\left[|n\rangle\otimes|m\rangle\right]$$

$$= \langle n|\hat{x}|n\rangle \otimes \langle m|\hat{x}|m\rangle$$

or

$$\langle \hat{x}_1 \hat{x}_2 \rangle = \langle \hat{x} \rangle_n \langle \hat{x} \rangle_m \tag{27c}$$

thus we conclude that

$$\langle (\hat{x}_1 - \hat{x}_2)^2 \rangle = \langle \hat{x}^2 \rangle_n + \langle \hat{x}^2 \rangle_m - 2 \langle \hat{x} \rangle_n \langle \hat{x} \rangle_m \tag{28}$$

this is quite what we'd expect from two non-interacting particles.

Indistinguishable case : $m \neq n$.

Now consider

$$|n, m\rangle_{S,A} = \frac{|n, m\rangle \pm |m, n\rangle}{\sqrt{2}}$$

Note that this is an entangled state. We now have

$$\langle \hat{x}_1^2 \rangle = \frac{1}{2}\Big[\langle n, m| \hat{x}_1^2 | n, m\rangle + \langle m, n| \hat{x}_1^2 | m, n\rangle +$$

$$\pm \langle n, m| \hat{x}_1^2 | m, n\rangle \pm \langle m, n| \hat{x}_1^2 | n, m\rangle \Big]$$

$$= \frac{1}{2}\Big[\langle \hat{x}^2 \rangle_n + \langle \hat{x}^2 \rangle_m \Big] \tag{29a}$$

the last two terms are zero because

$$\langle n, m| \hat{x}_1^2 | m, n\rangle = \langle n| \hat{x}^2 |m\rangle \otimes \underbrace{\langle m | n \rangle}_{0}$$

since I am assuming that $n \neq m$.

Similarly

$$\langle \hat{x}_2^2 \rangle = \frac{1}{2} \left[\langle \hat{x}^2 \rangle_n + \langle \hat{x}^2 \rangle_m \right] \qquad (29b)$$

Finally:

$$\langle \hat{x}_1 \hat{x}_2 \rangle = \frac{1}{2} \Big[\langle n,m | \hat{x}_1 \hat{x}_2 | n,m \rangle + \langle m,n | \hat{x}_1 \hat{x}_2 | m,n \rangle$$

$$\pm \langle n,m | \hat{x}_1 \hat{x}_2 | m,n \rangle \pm \langle m,n | \hat{x}_1 \hat{x}_2 | n,m \rangle \Big]$$

The first two terms are $\langle \hat{x} \rangle_n \langle \hat{x} \rangle_m$, the last term is

$$\langle n,m | \hat{x}_1 \hat{x}_2 | m,n \rangle = \left[\langle n | \otimes \langle m | \right] \left[\hat{x} \otimes \hat{x} \right] \left[| m \rangle \otimes | n \rangle \right]$$

$$= \langle n | \hat{x} | m \rangle \langle m | \hat{x} | n \rangle.$$

$$= \langle \hat{x} \rangle_{nm} \left[\langle \hat{x} \rangle_{nm} \right]^*$$

$$= | \langle \hat{x} \rangle_{nm} |^2$$

$\langle \hat{x} \rangle_{nm}$ is the matrix element of $\hat{x}$ in the single particle basis $|m\rangle$. The other term has exactly the same result. Thus, instead of (27c), we now get

$$\langle \hat{x}_1 \hat{x}_2 \rangle = \langle \hat{x} \rangle_n \langle \hat{x} \rangle_m \pm | \langle \hat{x} \rangle_{nm} |^2 \qquad (29c)$$

Hence we finally arrive at

$$\boxed{\langle(\hat{x}_1-\hat{x}_2)^2\rangle = \langle\hat{x}^2\rangle_n + \langle\hat{x}^2\rangle_m - 2\langle\hat{x}\rangle_n\langle\hat{x}\rangle_m \mp 2\,|\langle\hat{x}\rangle_{nm}|^2} \quad (30)$$

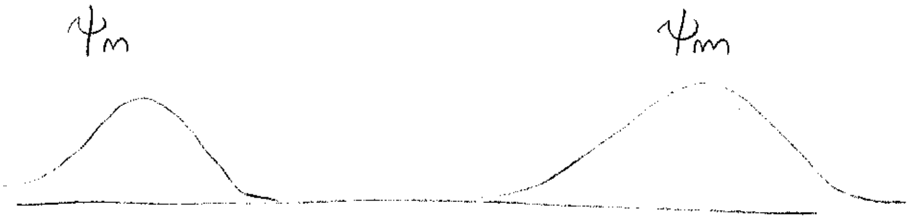
If we compare this with (28) we see the appearance of a new term. For Bosons (symmetric states) the term appears with a minus sign, and hence tends to <u>attract</u> the two particles. Conversely, for Fermions this term has a plus sign so it has the tendency to <u>repel</u> the particles.

This phenomena is called the <u>exchange</u> <u>interaction</u>. It is not really a force, but it acts just like one. It looks like a force which repels Fermions and attracts Bosons. This is yet another crazy thing of QM. Note also that it is a manifestation of entanglement since the indistinguishability imposes that the particles will live in an entangled state.

The new term in Eq (30) depends on $\langle\hat{x}\rangle_{nm}$. In terms of wave functions this reads

$$\langle\hat{x}\rangle_{nm} = \langle n|\hat{x}|m\rangle = \int \psi_n^*(x)\, x\, \psi_m(x)\, dx \quad (31)$$

Thus, note how it will only be significant if the wavefunctions overlap substantially



$\langle x \rangle_{nm} \approx 0$

$\langle x \rangle_{nm} \neq 0.$

Hence, the exchange effect will only be meaningful if the particles are close together; ie, if there is a meaningful overlap in their wave functions.

If one electron is in London and the other in Tokyo, the exchange interaction will be negligible.

As an example consider the H_2 molecule

There will be a net attraction between the two electrons only if they are in a symmetric state. But since they are Fermions, this must mean they are in a singlet. Indeed, this is what happens, as can be verified experimentally.

Systems with N particle

When we have N particles we need to consider all possible permutations. The number of allowed permutations is N! so it grows *very* quickly with N (N! grows faster than e^N).

We now define a set of operator $P_{n,m}$ which commute particles n and m. For instance [I'm writting $|e_i e_j e_k e_l\rangle = |1234\rangle$]

$$\hat{P}_{12}\,|1234\rangle = |2134\rangle$$

$$\hat{P}_{24}\,|1234\rangle = |1432\rangle$$

etc.

However, these operators do not commute. To see this consider, for instance

$$\hat{P}_{12}\hat{P}_{23}\,|123\rangle = \hat{P}_{12}\,|132\rangle = |312\rangle$$

$$\hat{P}_{23}\hat{P}_{12}\,|123\rangle = \hat{P}_{23}\,|213\rangle = |231\rangle$$

the two orders give different results, so that

$$[\hat{P}_{12}, \hat{P}_{23}] \neq 0$$

This means that the $\hat{P}_{n,m}$ operators cannot be simultaneously diagonalized.

However, there exists certain states, which are eigenvect[ors] of all the $\hat{P}_{n,m}$ simultaneously. For instance consider the state, for $N=3$

$$|123\rangle_S = \frac{1}{\sqrt{6}}\left[|123\rangle + |312\rangle + |231\rangle + |132\rangle + |213\rangle + |32\right.$$

This state contains all $3! = 6$ permutations of $|123\rangle$. It can be verified that this state is an eigenstate of all 3 operators $\hat{P}_{12}$, $\hat{P}_{13}$, $\hat{P}_{23}$, with eigenvalue $+1$. We call this a completely symmetric state.

Similarly, consider

$$|123\rangle_A = \frac{1}{\sqrt{6}}\left[|123\rangle + |312\rangle + |231\rangle - |132\rangle - |213\rangle - |321\rangle\right]$$

This state is an eigenvector of $\hat{P}_{12}, \hat{P}_{13}$ and $\hat{P}_{23}$ with eigenvalu[e] -1. We call it a completely anti-symmetric state. Note that we put a $+1$ when we have a cyclic permutation of 123 and a -1 for a non-cyclic.

What emerges out of this is that the total Hilbert space $\mathcal{H}$ actually contains 3 parts. One part $\mathcal{H}_S$ has completely symmetric states only. The other part, $\mathcal{H}_A$, has only completely anti-symmetric. Then there is the remainder $\mathcal{H}_R$ which has everything else, from states which are partially symmetric or anti-symmetric, to states with no particular symmetry. In symbols

$$\boxed{\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_A \oplus \mathcal{H}_R}$$

The spin-statistics case now says that <u>Bosons</u> <u>only</u> <u>occupy</u> $\mathcal{H}_S$ and <u>Fermions</u> <u>only</u> <u>occupy</u> $\underline{\mathcal{H}_A}$. For a system of N identical particles the space $\mathcal{H}_R$ is completely abandoned. You will never find the system there.

Symmetrizer and anti-symmetrizer

Now I want to discuss an optional way of looking at $\hat{P}$, for the case of 2 particles. We define two operators

$$\hat{Q}_+ = \frac{1 + \hat{P}}{2}$$

$$\hat{Q}_- = \frac{1 - \hat{P}}{2}$$

These are called the symmetrizer and anti-symmetrizer. Let us look at some properties:

$$\hat{Q}_+ + \hat{Q}_- = 1$$

$$\hat{Q}_\pm = \hat{Q}_\pm^\dagger$$

$$Q_\pm^2 = \frac{(1 \pm \hat{P})}{2}\frac{(1 \pm \hat{P})}{2} = \frac{1 + \hat{P}^2 \pm \hat{P} \pm \hat{P}}{4} = \frac{2(1 \pm \hat{P})}{4} = Q_\pm$$

$$\therefore \quad \hat{Q}_\pm^2 = \hat{Q}_\pm$$

[If you symmetrize twice, nothing happens].

$$\hat{Q}_+ \hat{Q}_- = \frac{(1 + \hat{P})}{2}\frac{(1 - \hat{P})}{2} = \frac{1 - \hat{P} + \hat{P} - \hat{P}^2}{4} = 0$$

$$\therefore \quad \hat{Q}_+ \hat{Q}_- = \hat{Q}_- \hat{Q}_+$$

Let $|\alpha\rangle$ be any ket. Since $\hat{Q}_+ + \hat{Q}_- = 1$ we can always write

$$|\alpha\rangle = \hat{Q}_+|\alpha\rangle + \hat{Q}_-|\alpha\rangle$$

Now let us apply $\hat{P}$ on $|\alpha\rangle$. We have

$$\hat{P}\hat{Q}_+ = \frac{\hat{P}(1+\hat{P})}{2} = \frac{\hat{P}+\hat{P}^2}{2} = \frac{1+\hat{P}}{2} = \hat{Q}_+$$

$$\hat{P}\hat{Q}_- = \frac{\hat{P}(1-\hat{P})}{2} = \frac{\hat{P}-\hat{P}^2}{2} = -\frac{(1-\hat{P})}{2} = -\hat{Q}_-$$

Thus

$$\hat{P}\hat{Q}_\pm = \pm\hat{Q}_\pm$$

We then have

$$\hat{P}\left(\hat{Q}_+|\alpha\rangle\right) = \hat{Q}_+|\alpha\rangle$$

$$\hat{P}\left(\hat{Q}_-|\alpha\rangle\right) = -\hat{Q}_-|\alpha\rangle$$

Thus, for any $|\alpha\rangle$, $\hat{Q}_+|\alpha\rangle$ and $\hat{Q}_-|\alpha\rangle$ are eigenvectors of $|\alpha\rangle$ with eigenvalue ± 1.

Thus, we see that any vector $|\alpha\rangle$ may be decomposed into a symmetric and a anti-symmetric part. And we do this by applying $\hat{Q}_\pm$. Moreover, since $\hat{Q}_+\hat{Q}_- = 0$ the two parts do not communicate

$$\left(\langle\alpha|\hat{Q}_+\right)\left(\hat{Q}_-|\alpha\rangle\right) = 0$$