Spontaneous symmetry breaking

The concept of spontaneous symmetry breaking is, in my opinion, the most important concept in modern physics. It embodies just one extremely simple idea which explains, among other things, ferromagnetism, superconductivity, Bose-Einstein condensation, the Higgs mechanism that entails mass to particles and so on. It represents a unifying concept that appears in all areas of physics.

To motivate the idea, consider the classical Ising model studied in the previous lecture notes,

$$H = -J \sum_{\langle ij \rangle} \sigma_z^i \sigma_z^j - h \sum_i \sigma_z^i \tag{1}$$

We saw that, in the mean-field approximation, the free energy of the model was given by [Eq (26) of notes 14]

$$\boxed{f = \frac{T_c m^2}{2} - T \ln\left\{ 2 \cosh\left(\frac{h + T_c m}{T}\right) \right\}} \tag{2}$$

For instance, we saw that the Curie-Weiss equation follows from

$$m = -\frac{\partial f}{\partial h} = \tanh\left(\frac{h + T_c m}{T}\right) \tag{3}$$

Now let's try to have a more detailed look on f. In particular, let us focus on T close to T_c, so that m is small. We also assume that $h=0$ for simplicity. We can then Taylor expand

$$\ln(\cosh(x)) \simeq \frac{x^2}{2} - \frac{x^4}{12} \tag{14}$$

which leads us to

$$f = \frac{T_c m^2}{2} - \frac{T}{2}\left(\frac{T_c m}{T}\right)^2 + \frac{T}{12}\left(\frac{T_c m}{T}\right)^4$$

$$= \frac{1}{2}\left(T_c - \frac{T_c^2}{T}\right) m^2 + \frac{T_c^4}{12T^3} m^4$$

We now define

$$a = T_c - \frac{T_c^2}{T} = \frac{T_c}{T}(T - T_c) \qquad b = \frac{T_c^4}{4T^3} \tag{15}$$

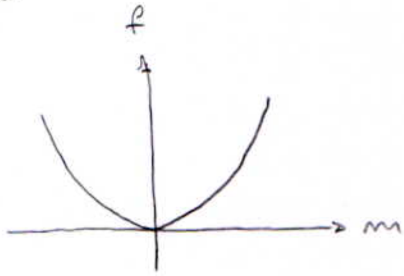
so that the free energy becomes, close to T_c

$$\boxed{f = \frac{a}{2} m^2 + \frac{b}{4} m^4} \tag{16}$$

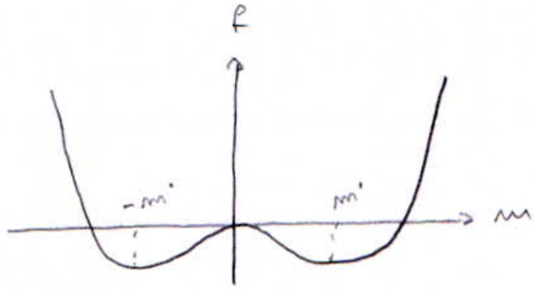
where

$$\boxed{b > 0 \qquad a \sim T - T_c} \tag{17}$$

Now, I know Eq (16) seems pretty inocent. But amount of physics contained in it is absolutely incredible. And it's all related to sign of a. A plot of $f(m)$ for $a>0$ and $a<0$ looks like this



$T > T_c$
$(a>0)$



$T < T_c$
$(a<0)$

If I had to elect what is the most important graph in all of physics, it would be this one. For sure!

Here is the idea. We already showed, many lectures ago, that equilibrium is the state which minimizes the free energy. So the equilibrium state will correspond to that value of m for which f is a minimum. If $T > T_c$, the only minimum is at $m=0$. However, if $T<T_c$, two minima appear at non-zero values $\pm m^*$, which are, of course, nothing but the solutions of the Curie-Weiss equation. Or, in terms of the parameters a and b in (16),

$$\boxed{m^* = \pm\sqrt{-\frac{a}{b}} \qquad a<0} \qquad (18)$$

For $T < T_c$ the two minima will be separated by an energy barrier which is of the order of the number of particles (recall that we are plotting here f and not $F = Nf$). Thus, the barrier that must be overcome to go from one minimum to the other is immensely large and thus, in practice, it would take an infinite amount of time to overcome it.

For this reason, if the system eventually falls down to one of the two minima, which can be induced for instance by applying a magnetic field, then it will simply stay there.

This is what we call a spontaneous symmetry breaking. The Ising model (1) with $h=0$, has a Z_2 symmetry

$$\sigma_z^i \to -\sigma_z^i \tag{19}$$

which means that the thermodynamic properties should be invariant under the symmetry

$$m \to -m \tag{20}$$

Indeed, this is true for the free energy (16), which is an even function of m. And it is also true for the minimum when $T > T_c$ ($m' = 0$). But for $T < T_c$ the system will tend to one of two solutions, $+m'$ or $-m'$. And once it's there, the symmetry is gone! It no longer has (20)

Thus, the ferromagnetic phase has a lower symmetry: the $\mathbb{Z}_2$ symmetry has been spontaneously broken.

This is it guys. This is the deal. Phase transitions are all about broken symmetries. They are about microscopic interactions which act collectively to macroscopically break microscopic symmetries.

Landau used this idea to conjecture that this should be the basic structure of any phase transition. One starts with a Hamiltonian that has some symmetry. We then define the order parameter, which is a quantity reflecting this symmetry.

Landau then conjectured, based solely on thermodynamics, that the free energy f should be an analytic function of the order parameter. Consequently, close to the critical point, where m is small, we can always expand f in a power series

$$f(m) = c_0 + c_1 m + c_2 m^2 + c_3 m^3 + \dots \tag{21}$$

However (and this is the key point), the free energy must reflect the symmetries of the Hamiltonian. So, for instance, in the case of $\mathbb{Z}_2$ symmetry ($m \to -m$), f must satisfy $f(-m) = f(m)$ and thus should be an even function of m.

Since we are close to T_c, m will be small, allowing us to retain only the leading order terms.

$$\boxed{f = \frac{a m^2}{2} + \frac{b}{4} m^4} \qquad (22)$$

Finally, we ask what are the conditions on a and b, which are necessary to capture a phase transition. The leading order term (here, b) must be positive so as to ensure that the free energy is stable. Then, in order to have something which changes behavior at T_c, as in the figures in page 3, we must then have

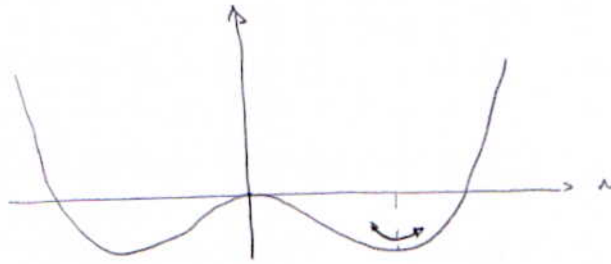
$$a \sim (T - T_c) \qquad (23)$$

This is what we call <u>Landau</u> <u>theory</u>: we identify <u>all</u> of the basic structure of a phase transition using only symmetry arguments. We didn't have to do any calculation. We only assumed $\mathbb{Z}_2$ symmetry and that was enough to fix the structure of the free energy

Oops. I forgot to say: a free energy like (22) is often nicknamed a <u>ϕ^4 theory</u>, to refer to the leading order term (the letter ϕ, instead of m, is because this name originated in quantum field theory).

Higgs and Goldstone excitations

Let us now talk about excitations above the equilibrium state. In the broken symmetry phase of the Ising model, this means pushing the system a bit away from equilibrium



We see that in this case, to create an excitation requires energy. For this reason, this is usually called a Higgs excitation, or Higgs mode, or amplitude mode

There are also excitations which do not cost energy. They are called Goldstone modes, these modes appear in situations that have a continuous symmetry. The Z_2 symmetry of the Ising model is a discrete symmetry $m \to -m$.

An example of a continuous symmetry is the $U(1)$ symmetry. Consider, for instance, a situation where the order parameter ϕ can be complex, but with a free energy given by

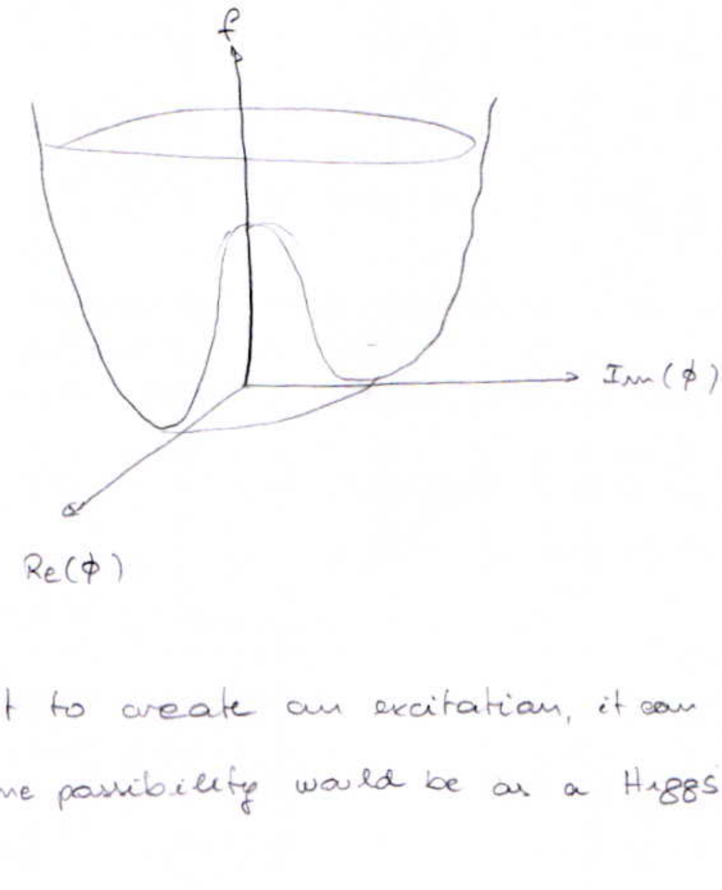
$$\boxed{f(\phi) = \frac{a}{2} |\phi|^2 + \frac{b}{4} |\phi|^4} \tag{24}$$

This free energy is invariant under the $U(1)$ symmetry

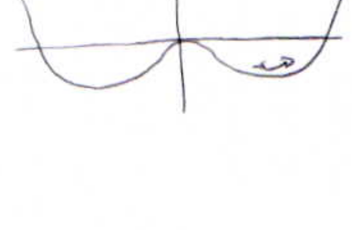
$$\boxed{\phi \to \phi e^{i\theta}} \qquad (25)$$

which is parametrized by a continuous parameter θ. An example of a system having this symmetry is the <u>Dicke</u> <u>model</u>, which will be discussed below.

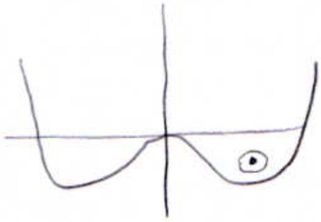
If we now plot f as a function of the real and imaginary parts of ϕ, we will get the famous mexican hat



If we now want to create an excitation, it can be created in more than one way. One possibility would be as a Higgs mode, as before



But now there is another possibility, which is to create it in the direction of the "race track" of the mexican hat



(Going into the page)

As one can see, this type of excitation has no energy cost at all. It is therefore very easy to create them. These are the Goldstone, or phase modes.

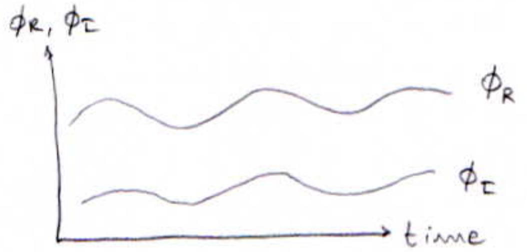
There is a fun way to spot experimentally whether a mode is Higgs or Goldstone, which consists in looking at the real and imaginary parts of the order parameter. Let us write

$$\phi = \phi_R + i\,\phi_I = r\,e^{i\lambda} \qquad (26)$$

In a Higgs mode r will change in time, as $r(t)$, whereas λ will be time-independent. Thus, we see that in this case ϕ_R and ϕ_I are always in phase

$$\phi_R = r(t)\cos\lambda$$

$$\phi_I = r(t)\sin\lambda$$

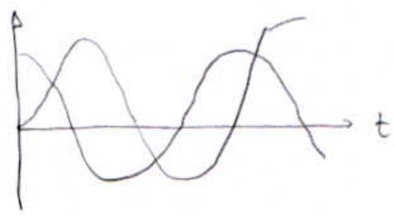


Conversely, in a Goldstone mode r is time-independent whereas $\lambda(t)$ changes in time. Then

$$\phi_R = r \cos\lambda(t)$$

$$\phi_I = r \sin\lambda(t)$$

thus ϕ_R and ϕ_I will be out-of-phase



this type of idea was used very recently in Science 358, 1415 (2017) as a tool to distinguish between Higgs and Goldstone modes.

The presence of Goldstone modes can have a dramatic effect on the physics of critical systems, as we will explore soon.

The Dicke model of superradiance

Now I want to discuss a cool model that has a complex order parameter. The model consists of N qubits interacting with a single bosonic mode:

$$\boxed{H = \omega a^\dagger a + \frac{\omega}{2}\sum_{i=1}^{N}\sigma_z^i - \frac{\lambda}{\sqrt{N}}\sum_{i=1}^{N}\left(a^\dagger\sigma_-^i + a\,\sigma_+^i\right)} \qquad (27)$$

where the factor of $1/\sqrt{N}$ is placed so as to ensure the free energy is extensive. Originally Dicke, in Physical Review, 93, 99-110 (1954), imagined this model as stemming from an optical cavity containing N atoms. The cavity was assumed to hold a single electromagnetic mode a and the atoms were assumed to be well described by 2-level systems (ie, qubits).

We can write the Hamiltonian (27) as a single sum

$$H = \sum_{i=1}^{N}\left\{ \omega\frac{a^\dagger}{\sqrt{N}}\frac{a}{\sqrt{N}} + \frac{\omega}{2}\sigma_z^i - \lambda\left(\frac{a^\dagger}{\sqrt{N}}\sigma_-^i + \frac{a}{\sqrt{N}}\sigma_+^i\right)\right\} \qquad (28)$$

which motivates us to define a new operator

$$\alpha = \frac{a}{\sqrt{N}} \qquad (29)$$

However, this operator is not bosonic. Indeed

$$[\alpha, \alpha^+] = \frac{1}{N} [a, a^+] = \frac{1}{N} \tag{30}$$

We therefore see that when N becomes large, these operators approximately commute. Consequently, they can be treated as c-numbers, so that we can write (28) simply as

$$H = \sum_{i=1}^{N} \left\{ \omega |\alpha|^2 + \frac{\omega}{2} \sigma_z^i - \lambda \left(\alpha^* \sigma_-^i + \alpha \sigma_+^i \right) \right\} \tag{31}$$

Now, I know my argument is not rigorous. But it can be made so. See, for instance, Wang and Hioe, Phys. Rev. A, 7, 831 (1973).

Now that the α's are treated as numbers, we can compute the partition function. In this case, since the α's vary continuously, we treat it as a classical variable, so

$$Z = \int d^2\alpha \; tr_{1\ldots N} \, e^{-\beta H} \tag{32}$$

where $d^2\alpha = d\alpha_R \, d\alpha_I$ is an integral over all values of $\alpha = \alpha_R + i\alpha_I$. Moreover, $tr_{1\ldots N}$ now stands for a trace over the spin components

The spin trace can now be computed easily because the spins are independent. Thinking in terms of 2×2 matrices

$$H_1 = \frac{\omega}{2}\sigma_z - \lambda(\alpha^* \sigma_-^i + \alpha \sigma_+^i) = \begin{pmatrix} \omega/2 & \lambda\alpha \\ -\lambda\alpha^* & -\omega/2 \end{pmatrix}$$

The eigenvalues of this matrix are

$$\text{eigs}(H) = \pm \frac{1}{2}\sqrt{\omega^2 + 4\lambda^2|\alpha|^2}$$

Thus

$$\operatorname{tr}_{1\ldots N} e^{-\beta H} = e^{-\beta\omega|\alpha|^2 N}\left[2\cosh\left(\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2|\alpha|^2}\right)\right]^N \qquad (33)$$

The partition function then becomes

$$Z = \int d^2\alpha \; e^{-\beta N\omega|\alpha|^2}\left[2\cosh\left(\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2|\alpha|^2}\right)\right]^N$$

$$= \int d^2\alpha \; \exp\left\{-\beta N\omega|\alpha|^2 + N\ln\left[2\cosh\left(\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2|\alpha|^2}\right)\right]\right\}$$

We now define a free energy (I'll explain why its a free energy in a second)

$$\boxed{f(\alpha) = \omega|\alpha|^2 - T\ln\left[2\cosh\left(\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2|\alpha|^2}\right)\right]} \qquad (34)$$

so that the partition function becomes

$$\boxed{Z = \int d^2\alpha \; e^{-\beta N f(\alpha)}} \tag{35}$$

we now use Laplace's asymptotic method. The idea is that since we have an N in (35), when $N \to \infty$ the only contribution to this integral will come from the global minimum of $f(\alpha)$. The reason is that the exponential and the factor of N immensely amplify this global minimum, thus

$$\boxed{Z \sim e^{-\beta N f(\alpha')}} \tag{36}$$

where α' is the global minimum of $f(\alpha)$. [Note how $f(\alpha) = -\frac{T}{N}\ln Z$].

You see, we essentially recover the same spirit of the mean-field approximation. In fact, this free energy is almost identical to the Ising model.

Defining

$$\alpha = r e^{i\theta} \tag{37}$$

we get

$$f = \omega r^2 - T \ln\left[2\cosh\left(\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2 r^2}\right)\right] \tag{38}$$

To find the minimum, we compute

$$\frac{\partial f}{\partial r} = 2\omega r - 2\lambda^2 r \frac{1}{\sqrt{\omega^2+\lambda^2 r^2}} \tanh\left[\frac{\beta}{2}\sqrt{\omega^2+\lambda^2 r^2}\right] = 0$$

which yields

$$\boxed{\sqrt{\omega^2+4\lambda^2 r^2} = \frac{\lambda^2}{\omega}\tanh\left(\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2 r^2}\right)} \tag{39}$$

For concreteness, let us investigate the case $T=0$. Then $\tanh(\,) \approx 1$ and we get

$$\omega^2 + 4\lambda^2 r^2 = \frac{\lambda^4}{\omega^2}$$

or

$$\boxed{r^* = \pm \frac{\sqrt{\lambda^4-\omega^4}}{2\lambda\omega}} \tag{40}$$

Thus, we see that there will be a non-zero solution provided that $\lambda > \omega$

This defines the critical parameter of the phase transition

$$\boxed{\lambda_c = \omega} \tag{41}$$

For $\lambda < \lambda_c$ the only minimum is $r=0$, or $\alpha = 0$. But as λ crosses λ_c, we enter a phase with $\alpha \neq 0$. But recall that

$$\boxed{r^2 = |\alpha|^2 = \langle a^\dagger a \rangle N} \tag{42}$$

Thus, the order parameter here is the number of photons in the cavity. When $\lambda > \lambda_c$ the system enters into a superradiant phase in which the ground-state has a macroscopic number of photons

Returning now to the free energy (34), when $T \to 0$, $\beta \to \infty$, we can approximate

$$\cosh\left(\frac{\beta}{2}\sqrt{\omega^2 + 4\lambda^2|\alpha|^2}\right) \approx \frac{e^{\frac{\beta}{2}\sqrt{\omega^2+4\lambda^2|\alpha|^2}}}{2}$$

which yields

$$f = \omega|\alpha|^2 - \frac{1}{2}\sqrt{\omega^2 + 4\lambda^2|\alpha|^2}$$

$$= \omega\left\{ |\alpha|^2 - \frac{1}{2}\sqrt{1 + \frac{4\lambda^2}{\omega^2}|\alpha|^2} \right\}$$

close to the critical point we can expand the square root using

$$\sqrt{1+x^2} \simeq 1 + \frac{x^2}{2} - \frac{x^4}{8}$$

We then get

$$f = \omega\left\{ |\alpha|^2 - \frac{1}{2} - \frac{1}{4}\frac{4\lambda^2}{\omega^2}|\alpha|^2 + \frac{1}{16}\frac{4^2\lambda^4}{\omega^4}|\alpha|^4 \right\}$$

Neglecting constants we then get

$$\boxed{f = \frac{a}{2}|\alpha|^2 + \frac{b}{4}|\alpha|^4} \tag{43}$$

where

$$b = \frac{4\lambda^4}{\omega^3} \qquad a = \omega - \frac{\lambda^2}{\omega} = \frac{\omega^2 - \lambda^2}{\omega} \tag{44}$$

Thus, we see that we can recover a Landau theory with a complex order parameter. Indeed, if we go back to the original Hamiltonian (27), we see that it has a $U(1)$ symmetry

$$\boxed{\begin{aligned} a &\to a\, e^{i\theta} \\ \sigma_+^i &\to \sigma_+^i\, e^{-i\theta} \end{aligned}} \qquad (45)$$

(Recall that $\sigma_z^i \sim \sigma_+^i \sigma_-^i$, and so is invariant under this symmetry). And, as Landau anticipated, the free energy (43) reflects this symmetry.

Landau - Ginzburg theory

Landau's theory was based on the idea of universality. It essentially asks what has to be the general structure of the free energy close to a critical. And, as we have, this will be essentially determined by the symmetries of the model. For instance Z_2 symmetry ($m \to -m$) means the free energy will look like

$$f(m) = \frac{a}{2} m^2 + \frac{b}{4} m^4 \qquad (46)$$

where $b>0$ and a changes sign at the transition, leading to minima at

$$m^* = \begin{cases} 0 & a>0 \\ \pm\sqrt{-\frac{a}{b}} & a<0 \end{cases} \qquad (47)$$

The idea of Landau - Ginzburg is to keep this sort of simplistic and phenomenological approach, but ask what happens when the order parameter can also be spatially inhomogeneous. That is, when

$$m \to m(\mathbf{r})$$

Instead of talking about the magnetization as a number, we now think about it as a function of position (you can imagine the sample is very large that we don't have to worry about the boundaries)

Thus, the idea is that we will now consider a free energy of the form

$$F = \int d^d\mathbf{r} \left\{ \frac{a}{2} m(\mathbf{r})^2 + \frac{b}{4} m(\mathbf{r})^4 + \dots \right\} \qquad (48)$$

where ... means new terms that we could add to explore the new spatial dependence of $m(\mathbf{r})$. That is, we want things such as ∇m, which mean changes in energy that appear when the magnetization is not homogeneous.

Since we always assume that the underlying model is translationally invariant, the true equilibrium should be spatially homogeneous, $m^e(\mathbf{r}) = m^e$. Thus, the new terms we are looking for in (48) should always increase F.

A reasonable assumption is that the system is also <u>isotropic</u> that is, invariant under rotations. It turns out that this is true when we are close to the critical point, since the fluctuations become large.

Thus, in this case, the ideal object is ∇m, since this guy is rotationally invariant.

But we cannot have ∇m alone, as this would not be Z_2 symmetric, $\nabla m \to -\nabla m$. What we could have instead, is $(\nabla m)^2$, or $(\nabla m)^4$, and so on. But to leading order, assuming that both $m(\mathbf{r})$ is small and that the fluctuations in $\mathbf{r}$ are also small, then $(\nabla m)^2$ will be the dominant term.

Thus, a minimalistic ansatz for the Landau-Ginzburg energy (48) would be

$$\boxed{F = \int d^d\mathbf{r} \left\{ \kappa (\nabla m)^2 + \frac{a}{2} m^2 + \frac{b}{4} m^4 \right\}} \tag{49}$$

Another possibility would be to try something with a Laplacian $\nabla^2 m$. But to comply with Z_2, we would have to combine it as something like $m\nabla^2 m$. However, integrating by parts

$$\int d^d r \; m \nabla^2 m = - \int d^d\mathbf{r} \, (\nabla m)^2 + \text{surface term}$$

Thus, this term is really no different than what we are already using in (49).

Oh, just to clarify, sorry:

$$\nabla m = \left(\frac{\partial m}{\partial x_1}, \dots, \frac{\partial m}{\partial x_d} \right) \tag{50}$$

$$(\nabla m)^2 = \left(\frac{\partial m}{\partial x_1} \right)^2 + \dots + \left(\frac{\partial m}{\partial x_d} \right)^2 \tag{51}$$

* Oh, I forgot to say: κ in (60) is just a constant. For simplicity I will set it to $\kappa = 1$.

The term $(\nabla m)^2$ is always non-negative. Thus, any spatial inhomogeneity in $m(\mathbf{r})$ will only increase the free energy. Hence, the equilibrium configuration continues to be a homogeneous magnetization profile

$$m(\mathbf{r}) = \pm m^* \tag{52}$$

where

$$m^* = \begin{cases} 0 & a > 0 \\ \sqrt{-a/b} & a < 0 \end{cases} \tag{53}$$

But now the fun part about Ginzburg-Landau is to look at the excitations around this equilibrium value. Let us then define

$$m(\mathbf{r}) = m^* + \phi(\mathbf{r}) \tag{54}$$

where $\phi(\mathbf{r})$ represents a perturbation that we assume to be small. Expanding F in a power series in ϕ, we get up to 2nd order

$$\begin{aligned} \nabla m &= \nabla \phi \\ m^2 &= m^{*2} + 2m^*\phi + \phi^2 \\ m^4 &= m^{*4} + 4m^{*3}\phi + 6m^{*2}\phi^2 + O(\phi)^3 \end{aligned} \tag{55}$$

For $a>0$, $m'=0$. For $a<0$ we get instead

$$\frac{a}{2}m^2 + \frac{b}{4}m^4 = \frac{a}{2}m'^2 + \frac{b}{4}m'^4 +$$

$$+ \frac{a}{2}2m'\phi + \frac{b}{4}4m'^3\phi$$

$$+ \frac{a}{2}\phi^2 + \frac{b}{4}6m'^2\phi^2$$

$$= \frac{a}{2}\left(-\frac{a}{b}\right) + \frac{b}{4}\left(-\frac{a}{b}\right)^2$$

$$+ \left[a\left(-\frac{a}{b}\right)^{1/2} + b\left(-\frac{a}{b}\right)^{3/2}\right]\phi$$

$$+ \left[\frac{a}{2} + \frac{3b}{2}\left(-\frac{a}{b}\right)\right]\phi^2$$

the first line is simply the equilibrium free energy per unit volume

$$\frac{a}{2}\left(-\frac{a}{b}\right) + \frac{b}{4}\left(-\frac{a}{b}\right)^2 = -\frac{a^2}{4b} \qquad (56)$$

the 2^{nd} line, on the other hand, cancels out

$$\left(-\frac{a}{b}\right)^{1/2}\left[a + b\left(-\frac{a}{b}\right)\right] = 0 \qquad (57)$$

Finally, the third line simplifies to

$$\frac{a}{2} + \frac{3b}{2}\left(-\frac{a}{b}\right) = -a \qquad (58)$$

thus, we get

$$\frac{a}{2} m^2 + \frac{b}{4} m^4 = -\frac{a^2}{4b} - a\,\psi^2 \tag{59}$$

the free energy (49) then becomes

$$\boxed{F = F_{eq} + \int d^d r \left\{ (\nabla\psi)^2 + |a|\,\psi^2 \right\}} \qquad a<0 \tag{60}$$

where

$$F_{eq} = \int d^d r \left(-\frac{a^2}{4b} \right) = -\frac{a^2}{4b} V \tag{61}$$

where V is the volume of the system.

Conversely, if $a>0$ then $m^* = 0$ and we get, up to 2^{nd} order, only $m^2 = \psi^2$, leading to

$$F = \int d^d r \left\{ (\nabla\psi)^2 + \frac{a}{2}\psi^2 \right\} \tag{62}$$

We see that the parameter in front of ψ^2 is different from (60)

Next we introduce a Fourier transform

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \psi_{\mathbf{k}} \tag{63}$$

We then get

$$\int d^d\mathbf{r}\ \psi(\mathbf{r})^2 = \frac{1}{V} \int d^d\mathbf{r} \sum_{\mathbf{k}\mathbf{k}'} e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \psi_{\mathbf{k}} \psi_{\mathbf{k}'}$$

Integrating over space and using

$$\frac{1}{V} \int d^d\mathbf{r}\ e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} = \delta_{\mathbf{k},-\mathbf{k}'}$$

we get, as expected

$$\int d^d\mathbf{r}\ \psi(\mathbf{r})^2 = \sum_{\mathbf{k}} \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \tag{64}$$

Next

$$\nabla \psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} (i\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}} \psi_{\mathbf{k}}$$

so

$$\int d^d\mathbf{r}\ (\nabla\psi)^2 = \frac{1}{V} \int d^d\mathbf{r} \sum_{\mathbf{k},\mathbf{k}'} (i\mathbf{k})\cdot(i\mathbf{k}')\ e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} \psi_{\mathbf{k}} \psi_{\mathbf{k}'}$$

$$= \sum_{\mathbf{k}} k^2 \psi_{\mathbf{k}} \psi_{-\mathbf{k}}$$

where I used the fact that, since we get a $\delta_{\mathbf{k},-\mathbf{k}'}$, then the constant in front becomes $(i\mathbf{k})\cdot(-i\mathbf{k}) = k^2$

Eq (60) is then transformed as

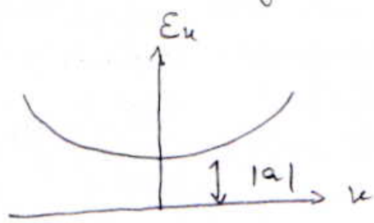
$$\boxed{F = F_{eq} + \sum_{k} \left(k^2 + |a| \right) \psi_k \psi_{-k}} \tag{63}$$

We therefore see that in momentum space the excitations will have a dispersion relation of the form

$$\boxed{\mathcal{E}_k = k^2 + |a|} \tag{64}$$

The result for $a > 0$ will be similar because (62) is just like (60). The only difference is that the dispersion relation will look like $\mathcal{E}_k = k^2 + a/2$.

The parameter $|a|$ in (64) therefore represents the <u>energy gap</u>. It is the minimum energy required to create an excitation



A <u>gap</u> is a signature of a Higgs mode. In quantum field theory the gap becomes the mass of the particles. Thus, people use both terms interchangeably

$$\boxed{\text{gap} = \text{mass}}$$

Finally, we see that both phases are gapped: when $a < 0$ the gap is $|a|$ and when $a > 0$ it is $a/2$. Thus, we also see that at the transition the gap closes

$$\boxed{\text{gap} = 0 \text{ @ the transition}}$$

Landau - Ginzburg for systems with continuous symmetries

Let us now consider a system with a complex order parameter ϕ, which is invariant under a $U(1)$ symmetry $\phi \to \phi e^{i\theta}$. The Landau free energy was given by Eq (24). In the same spirit that led us to (49), let us study a Landau-Ginzburg functional of the form

$$F = \int d^d r \left\{ |\nabla \phi|^2 + \frac{a}{2} |\phi|^2 + \frac{b}{4} |\phi|^4 \right\} \tag{65}$$

The equilibrium configuration is a spatially homogeneous order parameter $|\phi(r)| = \phi_0$, where

$$\phi_0 = \begin{cases} 0 & a > 0 \\ \sqrt{-a/b} & a < 0 \end{cases} \tag{66}$$

with the phase being arbitrary

As before, let us then consider excitations above equilibrium, by defining

$$\phi(r) = (\phi_0 + \rho(r))\, e^{i\theta(r)} \tag{67}$$

where $\rho(r)$ and $\theta(r)$ represent the amplitude and phase of the excitations, respectively.

The expansion of $\frac{a}{2}|\phi|^2 + \frac{b}{4}|\phi|^4$ is exactly as before [Eq (59)] and does not involve $\theta(r)$:

$$\frac{a}{2}|\phi|^2 + \frac{b}{4}|\phi|^4 = \begin{cases} \frac{a}{2}\rho^2 & a>0 \\ -\frac{a^2}{4b} + |a|\rho^2 & a<0 \end{cases} \tag{68}$$

As for the term $|\nabla\phi|^2$, we now get

$$\nabla\phi = (\nabla\rho)\, e^{i\theta} + (\phi_0 + \rho)(i\nabla\theta)\, e^{i\theta}$$

But the term $\rho(\nabla\theta)$ will be small compared to $\phi_0(\nabla\theta)$, since ρ is taken to be small. Thus, we can approximate

$$\nabla\phi = [\nabla\rho + i\phi_0 \nabla\theta]\, e^{i\theta}$$

and, thus

$$|\nabla\phi|^2 \simeq (\nabla\rho)^2 + \phi_0^2(\nabla\theta)^2 \tag{69}$$

Using (66) we get

$$|\nabla\phi|^2 \simeq \begin{cases} (\nabla\rho)^2 & a>0 \\ (\nabla\rho)^2 + \frac{|a|}{b}(\nabla\theta)^2 & a<0 \end{cases} \tag{70}$$

Thus, the LG free energy (65) becomes (neglecting constants)

$$\boxed{F = \int d^d r \left\{ (\nabla\rho)^2 + \frac{a}{2}\rho^2 \right\}} \qquad a>0 \qquad (71)$$

or

$$\boxed{F = \int d^d r \left\{ (\nabla\rho)^2 + |a|\rho^2 + \frac{|a|}{b}(\nabla\theta)^2 \right\}} \qquad a<0 \qquad (72)$$

We see that if $a>0$ there is no dependence on θ. that is, there is no Goldstone mode. But if $a<0$ it is there.

We now go to Fourier space, exactly as we did in (63):

$$\rho(r) = \frac{1}{\sqrt{V}}\sum_{k} e^{ik\cdot r}\rho_k$$
$$\theta(r) = \frac{1}{\sqrt{V}}\sum_{k} e^{ik\cdot r}\theta_k \qquad (73)$$

All calculations remain identical as those in page 25, so that the F's in (71) and (72) become

$$\boxed{F = \sum_{k} (k^2 + a/2)\rho_k\rho_{-k}} \qquad a>0 \qquad (74)$$

or

$$\boxed{F = \sum_{k} \left\{ (k^2+|a|)\rho_k\rho_{-k} + k^2\theta_k\theta_{-k} \right\}} \qquad a<0 \qquad (75)$$

The Higgs mode behaves exactly as before, with a gap/mass proportional to a. However, we now see that the Goldstone mode θ has is massless: there is no minimum energy cost (gap) for creating a Goldstone excitation. The name "Goldstone mode" is a synonym for "massless excitation".

We just showed that a free energy with a continuous symmetry implies the existence of a Goldstone mode in the excitation spectrum. This result is actually quite general and is called Goldstone's theorem.

Correlation functions

One of the most useful applications of the Landau-Ginzburg theory is in the calculation of correlation functions. For instance, going back to the Z_2 free energy (49),

$$F[m(\mathbf{r})] = \int d^d\mathbf{r} \left\{ (\nabla m)^2 + \frac{a}{2} m^2 + \frac{b}{4} m^4 \right\} \tag{76}$$

we can ask "what is the prob. of finding the system in a given configuration $m(\mathbf{r})$?" In the spirit of the Gibbs state, it is reasonable to suppose this will be given by

$$P[m(\mathbf{r})] = \frac{e^{-F[m(\mathbf{r})]}}{Z} \tag{77}$$

where
$$Z = \int \mathcal{D}m(\mathbf{r})\, e^{-F[m(\mathbf{r})]} \tag{78}$$

This is a bit ugly because $\mathcal{D}m(\mathbf{r})$ is a functional integral. That is, it is an integral over all possible functions $m(\mathbf{r})$. But if you are not familiar with functional integrals, don't worry. We won't have to use them at all. All I want you to remember is that the prob. of a given configuration $m(\mathbf{r})$ will be proportional to $e^{-F[m(\mathbf{r})]}$

Also, by the way, there is no need to write $e^{-\beta F[m(\mathbf{r})]}$ since the actual temperature is already contained in F (for instance, in the parameters a and b)

A typical 2-point correlation function is then given by

$$G(\mathbf{r}-\mathbf{r}') = \langle m(\mathbf{r})\, m(\mathbf{r}')\rangle - \langle m(\mathbf{r})\rangle \langle m(\mathbf{r}')\rangle \tag{79}$$

Instead of working with $m(\mathbf{r})$, let us work with the fluctuations $\psi(\mathbf{r}) = m(\mathbf{r}) - m^*$. Since $\langle m(\mathbf{r})\rangle = m^*$, it follows that $\langle \psi(\mathbf{r})\rangle = 0$, so the correlation function becomes

$$G(\mathbf{r}-\mathbf{r}') = \langle \psi(\mathbf{r})\, \psi(\mathbf{r}')\rangle \tag{80}$$

Going to Fourier space using (63) we get

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{V} \sum_{\mathbf{k},\mathbf{q}} e^{i(\mathbf{k}\cdot\mathbf{r} + \mathbf{q}\cdot\mathbf{r}')} \langle \psi_{\mathbf{k}} \psi_{\mathbf{q}} \rangle \tag{81}$$

But in Fourier space the free energy will be given by Eq (63)

$$F[\psi_{\mathbf{k}}] = \sum_{\mathbf{k}} (k^2 + |a|)\, \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \tag{82}$$

In terms of these discrete Fourier modes, the functional integrals become regular integrals

$$P[\psi_{\mathbf{k}}] = \frac{e^{-F[\psi_{\mathbf{k}}]}}{Z} \tag{83}$$

$$Z = \int \left[\prod_{\mathbf{k}} d\psi_{\mathbf{k}} \right] e^{-F[\psi_{\mathbf{k}}]} \tag{84}$$

Since $\phi(\mathbf{r})$ is real, it follows from (63) that

$$\phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^{*} \tag{85}$$

Thus we can also write (82) as

$$F[\phi_{\mathbf{k}}] = \sum_{\mathbf{k}} (k^2 + |a|) \, |\phi_{\mathbf{k}}|^2 \tag{86}$$

We therefore see that $P[\phi_{\mathbf{k}}]$ is nothing but a set of <u>independent</u> <u>Gaussian</u> <u>distributions</u>

$$Z = \prod_{k} \int d|\phi_k| \; e^{-(k^2 + |a|)|\phi_k|^2} \tag{87}$$

Now, going back to (81), since the Gaussians are independent the only non-zero contribution to $\langle \phi_{\mathbf{k}} \phi_{\mathbf{q}} \rangle$ will be when $\mathbf{q} = -\mathbf{k}$. Thus

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{V} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{r} - \mathbf{r}')} \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle \tag{88}$$

Moreover,

$$\frac{\int_{-\infty}^{\infty} dx \; e^{-\lambda x^2} x^2}{\int_{-\infty}^{\infty} dx \; e^{-\lambda x^2}} = \frac{1}{2\lambda} \tag{89}$$

Thus, forgetting about the factor of 2, we finally conclude that

$$\langle \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \rangle = \frac{1}{k^2 + |a|}$$

this is something which is very useful to remember:

$$\boxed{F = \sum_{\mathbf{k}} (k^2 + |a|)\, \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \quad \rightsquigarrow \quad \langle \psi_{\mathbf{k}} \psi_{-\mathbf{k}} \rangle = \frac{1}{k^2 + |a|}} \tag{90}$$

Plugging this in Eq (88) then gives

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{V} \sum_{\mathbf{k}} \frac{e^{i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}}{k^2 + |a|} \tag{91}$$

we can convert this to an integral as

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^d} \int d^d\mathbf{k} \; \frac{e^{i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}}{k^2 + |a|} \tag{92}$$

This integral is a bit tough. But all I want from it is its asymptotic behavior, which can be shown to read

$$\boxed{G(r) \sim \frac{\xi^{(3-d)/2}}{r^{(d-1)/2}} e^{-r/\xi}} \tag{93}$$

where ξ is called the correlation length and reads

$$\boxed{\xi = \frac{1}{\sqrt{|a|}} \sim \frac{1}{|T-T_c|^{1/2}}} \tag{94}$$

The important part of (93) is the exponential behavior $e^{-r/\xi}$ and this measures the statistical correlation between $m(r)$ and $m(r')$ which are a distance r apart. We see that this correlation decays exponentially, but with a typical length scale ξ that grows as $T \to T_c$. As we approach T_c, $\xi \to \infty$ meaning spins infinitely far apart become statistically correlated.

It is also interesting to see how the correlation length is related to the mass/gap

$$\boxed{\text{correlation length} \sim \frac{1}{\text{mass}}} \tag{95}$$

Let us evaluate Eq (93) at $r = \xi$:

$$G(\xi) = \frac{\xi^{(3-d)/2}}{\xi^{(d-1)/2}} e^{0} \sim \xi^{2-d} = \frac{1}{|a|^{(2-d)/2}} \qquad (96)$$

this gives a typical measure of the fluctuations in the system. On the other hand, the average magnetization scales as

$$(m^*)^2 \sim |a| \qquad (97)$$

thus, the ratio between the average and the fluctuations will be

$$\frac{G(\xi)}{(m^*)^2} \sim |a|^{(d-4)/2} \qquad (98)$$

close to the critical point, $|a| \to 0$ and we therefore get

$$\boxed{\frac{G(\xi)}{(m^*)^2} \sim \begin{cases} 0 & d > 4 \\ \infty & d < 4 \end{cases}} \qquad (99)$$

We therefore see that for $d < 4$ the fluctuations are dominant, whereas for $d > 4$ they become irrelevant. It is for this reason that mean-field becomes exact when $d \geq 4$ (the upper critical dimension). This is called the Ginzburg criterion