Spontaneous symmetry breaking

The concept of spontaneous symmetry breaking is, in my opinion, the most important concept in modern physics. It embodies just one extremely simple idea which explains, among other things, ferromagnetism, superconductivity, Bose-Einstein condensation, the Higgs mechanism that entails mass to particles and so on. It represents a unifying concept that appears in all areas of physics.

To motivate the idea, consider the classical Ising model studied in the previous lecture notes,

$$H = -J \sum_{\langle ij \rangle} \sigma_z^i \sigma_z^j - h \sum_i \sigma_z^i \tag{1}$$

We saw that, in the mean-field approximation, the free energy of the model was given by [Eq (26) of notes 14]

$$\boxed{f = \frac{T_c m^2}{2} - T \ln\left\{ 2 \cosh\left(\frac{h + T_c m}{T}\right)\right\}} \tag{2}$$

For instance, we saw that the Curie-Weiss equation follows from

$$m = -\frac{\partial f}{\partial h} = \tanh\left(\frac{h + T_c m}{T}\right) \tag{3}$$

Now let's try to have a more detailed look on f. In particular, let us focus on T close to T_c, so that m is small. We also assume that $h=0$ for simplicity. We can then Taylor expand

$$\ln(\cosh(x)) \simeq \frac{x^2}{2} - \frac{x^4}{12} \tag{14}$$

which leads us to

$$f = \frac{T_c m^2}{2} - \frac{T}{2}\left(\frac{T_c m}{T}\right)^2 + \frac{T}{12}\left(\frac{T_c m}{T}\right)^4$$

$$= \frac{1}{2}\left(T_c - \frac{T_c^2}{T}\right) m^2 + \frac{T_c^4}{12T^3} m^4$$

We now define

$$a = T_c - \frac{T_c^2}{T} = \frac{T_c}{T}(T - T_c) \qquad b = \frac{T_c^4}{4T^3} \tag{15}$$

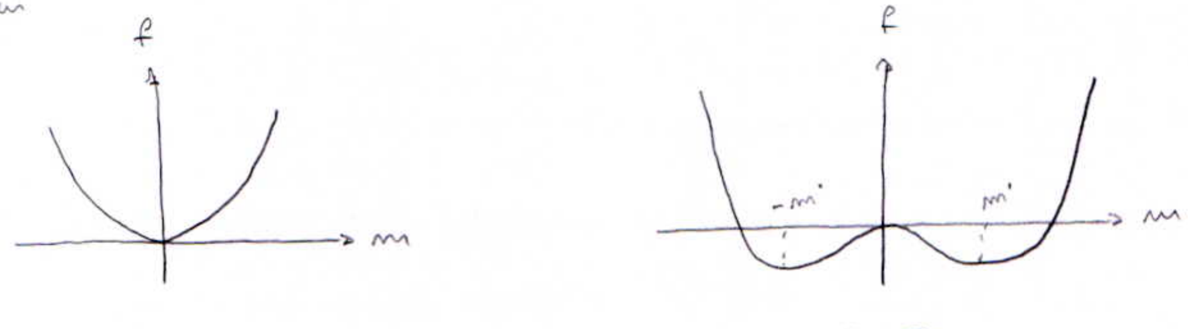
so that the free energy becomes, close to T_c

$$\boxed{f = \frac{a}{2} m^2 + \frac{b}{4} m^4} \tag{16}$$

where

$$\boxed{b > 0 \qquad a \sim T - T_c} \tag{17}$$

Now, I know Eq (16) seems pretty inocent. But amount of physics contained in it is absolutely incredible. And it's all related to sign of a. A plot of $f(m)$ for $a > 0$ and $a < 0$ looks like this



$T > T_c$
$(a > 0)$

$T < T_c$
$(a < 0)$

If I had to elect what is the most important graph in all of physics, it would be this one. For sure!

Here is the idea. We already showed, many lectures ago, that equilibrium is the state which minimizes the free energy. So the equilibrium state will correspond to that value of m for which f is a minimum. If $T > T_c$, the only minimum is at $m = 0$. However, if $T < T_c$, two minima appear at non-zero values $\pm m^*$, which are, of course, nothing but the solutions of the Curie-Weiss equation. Or, in terms of the parameters a and b in (16),

$$\boxed{m^* = \pm\sqrt{-\frac{a}{b}} \qquad a < 0} \tag{18}$$

For $T < T_c$ the two minima will be separated by an energetic barrier which is of the order of the number of particles (recall that we are plotting here f and not $F = Nf$). Thus, the barrier that must be overcome to go from one minimum to the other is immensely large and thus, in practice, it would take an infinite amount of time to overcome it.

For this reason, if the system eventually falls down to one of the two minima, which can be induced for instance by applying a magnetic field, then it will simply stay there.

This is what we call a spontaneous symmetry breaking. The Ising model (1) with $h=0$, has a $\mathbb{Z}_2$ symmetry

$$\sigma_z^i \to -\sigma_z^i \tag{19}$$

which means that the thermodynamic properties should be invariant under the symmetry

$$m \to -m \tag{20}$$

Indeed, this is true for the free energy (16), which is an even function of m. And it is also true for the minimum when $T > T_c$ ($m'=0$). But for $T < T_c$ the system will tend to one of two solutions, $+m'$ or $-m'$. And once it's there, the symmetry is gone! It no longer has (20)

Thus, the ferromagnetic phase has a lower symmetry: the $\mathbb{Z}_2$ symmetry has been spontaneously broken.

This is it guys. This is the deal. Phase transitions are all about broken symmetries. They are about microscopic interactions which act collectively to macroscopically break microscopic symmetries.

Landau used this idea to conjecture that this should be the basic structure of any phase transition. One starts with a Hamiltonian that has some symmetry. We then define the order parameter, which is a quantity reflecting this symmetry.

Landau then conjectured, based solely on thermodynamics, that the free energy f should be an analytic function of the order parameter. Consequently, close to the critical point, where m is small, we can always expand f in a power series

$$f(m) = c_0 + c_1 m + c_2 m^2 + c_3 m^3 + \dots \tag{21}$$

However (and this is the key point), the free energy must reflect the symmetries of the Hamiltonian. So, for instance, in the case of $\mathbb{Z}_2$ symmetry ($m \to -m$), f must satisfy $f(-m) = f(m)$ and thus should be an even function of m.

Since we are close to T_c, m will be small, allowing us to retain only the leading order terms.

$$\boxed{f = \frac{a m^2}{2} + \frac{b}{4} m^4} \tag{22}$$

Finally, we ask what are the conditions on a and b, which are necessary to capture a phase transition. The leading order term (here, b) must be positive so as to ensure that the free energy is stable. Then, in order to have something which changes behavior at T_c, as in the figures in page 3, we must then have

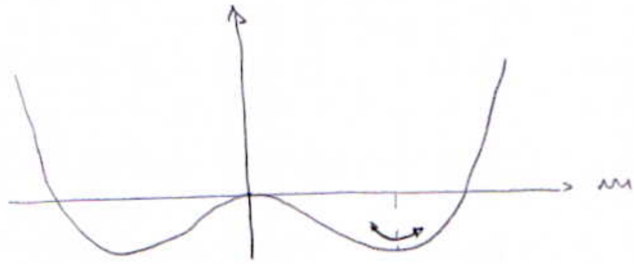
$$a \sim (T - T_c) \tag{23}$$

This is what we call Landau theory: we identify all of the basic structure of a phase transition using only symmetry arguments. We didn't have to do any calculation. We only assumed $\mathbb{Z}_2$ symmetry and that was enough to fix the structure of the free energy

Oops. I forgot to say: a free energy like (22) is often nicknamed a ϕ^4 theory, to refer to the leading order term (the letter ϕ, instead of m, is because this name originated in quantum field theory).

Higgs and Goldstone excitations

Let us now talk about excitations above the equilibrium state. In the broken symmetry phase of the Ising model, this means pushing the system a bit away from equilibrium



We see that in this case, to create an excitation requires energy. For this reason, this is usually called a Higgs excitation, or Higgs mode, or amplitude mode

There are also excitations which do not cost energy. They are called Goldstone modes, these modes appear in situations that have a continuous symmetry. The Z_2 symmetry of the Ising model is a discrete symmetry $m \to -m$.

An example of a continuous symmetry is the $U(1)$ symmetry. Consider, for instance, a situation where the order parameter ϕ can be complex, but with a free energy given by

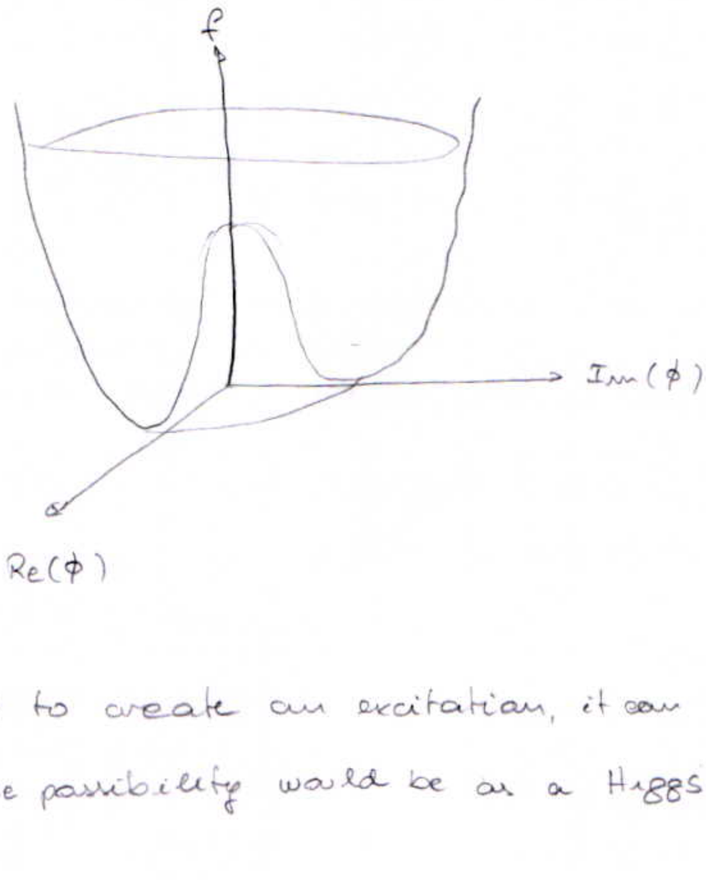
$$\boxed{f(\phi) = \frac{a}{2}|\phi|^2 + \frac{b}{4}|\phi|^4} \tag{24}$$

This free energy is invariant under the $U(1)$ symmetry

$$\boxed{\phi \to \phi e^{i\theta}} \tag{25}$$

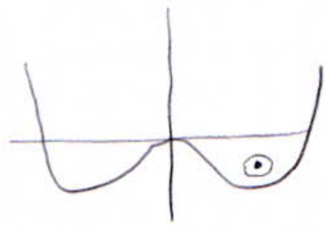
which is parametrized by a continuous parameter θ. An example of a system having this symmetry is the Dicke model, which will be discussed below.

If we now plot f as a function of the real and imaginary parts of ϕ, we will get the famous mexican hat



If we now want to create an excitation, it can be created in more than one way. One possibility would be as a Higgs mode, as before

But now there is another possibility, which is to create it in the direction of the "race track" of the mexican hat



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As one can see, this type of excitation has no energy cost at all. It is therefore very easy to create them. These are the Goldstone, or phase modes.

There is a fun way to spot experimentally whether a mode is Higgs or Goldstone, which consists in looking at the real and imaginary parts of the order parameter. Let us write

$$\phi = \phi_R + i\,\phi_I = r\,e^{i\lambda} \tag{26}$$

In a Higgs mode r will change in time, as $r(t)$, whereas λ will be time-independent. Thus, we see that in this case ϕ_R and ϕ_I are always in phase

$$\phi_R = r(t)\cos\lambda$$

$$\phi_I = r(t)\sin\lambda$$