Quantum Langevin Equation

Throughout this course we have assumed that thermal equilibrium could always be reached by placing the system in contact with an environment. However, we have never really discussed how this relaxation may actually take place.

The reason is that relaxation is problem specific, whereas the equilibrium state is universal.

But in order for you to have some idea of how this relaxation takes place, in these notes I want to discuss one concrete model of system environment interaction.

In our model the system is described by a single bosonic mode a, whereas the environment are described by an infinite number of bosonic modes b_k. The total S+E Hamiltonian is assumed to have the form

$$H = \omega a^\dagger a + \sum_k \Omega_k b_k^\dagger b_k + \sum_k g_k (a^\dagger b_k + b_k^\dagger a) \qquad (1)$$

The bath modes b_k do not interact directly, but they all interact with the system, with an interaction strength g_k. Moreover, Ω_k is the frequency of mode b_k. Although we represent them as a discrete set of frequencies, in the end we will assume the Ω_k vary quasi-continuously.

In these notes we shall work in the Heisenberg picture. So here is a quick review. the state ρ in the Schrödinger picture obeys the von Neumann Eq

$$\frac{d\rho}{dt} = -i[H,\rho] \tag{2}$$

whose solution is

$$\rho(t) = e^{-iHt}\rho(0)\,e^{iHt} \tag{3}$$

Expectation values of observables then become

$$\langle A\rangle_t = \mathrm{tr}\{A\rho(t)\} = \mathrm{tr}\{A\,e^{-iHt}\rho(0)\,e^{iHt}\} \tag{4}$$

But using the cyclic property of the trace we can also write this as

$$\langle A\rangle_t = \mathrm{tr}\{A(t)\rho(0)\} \tag{5}$$

where I defined the <u>Heisenberg picture operator</u>

$$\boxed{A(t) = e^{iHt}A\,e^{-iHt}} \tag{6}$$

Some people write this as $A_H(t)$, to remember in which picture we are in. I always put the (t), so that A is in the Schrödinger picture and $A(t)$ in Heisenberg's

Differentiating (6) with respect to t, we find that $A(t)$ obeys

$$\boxed{\frac{dA(t)}{dt} = i\,[H, A(t)]} \qquad (7)$$

with the initial condition

$$A(0) = A = \text{Schrödinger picture operator} \qquad (8)$$

In the Heisenberg picture the operators evolve in time and the state remains fixed. This can be seen in Eq (5), where the trace is over the initial state $\rho(0)$.

Another comment: because of (6), the commutator in (7) may be written as

$$[H, A(t)] = e^{iHt}\,[H, A]\,e^{-iHt} \qquad (9)$$

so to compute $[H, A(t)]$, we can first compute $[H, A]$ and then put (t) everywhere.

Heisenberg equations for the operators a and b_k

Let us now apply Eq (7) to the operators a and b_k, with H given by Eq (1). We find

$$\frac{da(t)}{dt} = -i\omega a(t) - i\sum_k g_k b_k(t) \tag{10a}$$

$$\frac{db_k(t)}{dt} = -i\Omega_k b_k(t) - i g_k a(t) \tag{10b}$$

These are a system of coupled ordinary differential equations for $a(t)$ and $b_k(t)$. Once we solve them, we can compute expectation values using Eq (5).

For concreteness, we shall assume that S and E were initially uncorrelated and the bath modes were in equilibrium at a temperature T. Thus

$$\rho(0) = \rho_S(0) \otimes \rho_E(0) \tag{11}$$

$$\rho_E(0) = \prod_k (1 - e^{-\beta\Omega_k})\, e^{-\beta\Omega_k b_k^\dagger b_k} \tag{12}$$

[the state $\rho_S(0)$ of the system is arbitrary.].

Before we continue, one thing that is important to realize is that the Heisenberg picture may mess up the commutation relations.

Recall that, from (6)

$$a(t) := e^{iHt} a e^{-iHt}$$
$$b_u(t) := e^{iHt} b_u e^{-iHt} \qquad (13)$$

commutation relations at equal times remain fine

$$[a(t), a^\dagger(t)] = [e^{iHt} a e^{-iHt}, e^{iHt} a^\dagger e^{-iHt}]$$
$$= e^{iHt} [a, a^\dagger] e^{-iHt}$$
$$= 1 \qquad (14)$$

or

$$[a(t), b_u(t)] = e^{iHt} [a, b_u] e^{-iHt} = 0 \qquad (15)$$

and so on. However, commutation relations at different times become non-trivial. For instance,

$$[a(t), a^\dagger(t')] = [e^{iHt} a e^{-iHt}, e^{iHt'} a^\dagger e^{-iHt'}]$$
$$= e^{iHt} a e^{-iH(t-t')} a^\dagger e^{-iHt'}$$
$$- e^{iHt'} a^\dagger e^{-iH(t'-t)} a e^{-iHt}$$
$$= ? \qquad (16)$$

the result of this commutation relation will depend on H and is in general non-trivial.

As another interesting example, consider

$$[a, b_u(t)] = [a, e^{iHt} b_u e^{-iHt}] = ? \tag{17}$$

At $t=0$, $b_u(0) = b_u$ commutes with a. But for $t>0$ this is no longer true. So even though a and b_u live in separate Hilbert spaces ($a = a \otimes I_E$ and $b_u = I_S \otimes b_u$), the same is not true for a and $b_u(t)$. We say that $b_u(t)$ will have a finite support on the Hilbert space of a

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Returning now to Eq (10), we shall solve them formally using the following naughty trick. Instead of trying to find an actual solution, we will assume that $a(t)$ is some known function, so that we can formally write down a solution to (10b). Recall that the ODE

$$\frac{dx}{dt} = \lambda x + g(t) \tag{18}$$

has the solution

$$x(t) = e^{\lambda t} x(0) + \int_0^t dt' \, e^{\lambda(t-t')} g(t') \tag{19}$$

Thus, the solution of (10b) will be

$$b_n(t) = e^{-i\Omega_n t} b_n(0) - i\, g_n \int_0^t dt'\, e^{-i\Omega_n (t-t')} a(t') \tag{20}$$

This solution is exact, although at first it may not seem very useful since we don't know what $a(t)$ is.

Next we plug this into (10a), which gives

$$\begin{aligned} \frac{da(t)}{dt} = -i\,\omega\, a(t) - i \sum_n g_n\, e^{-i\Omega_n t}\, b_n \; + \\ - \sum_n g_n^2 \int_0^t dt'\, e^{-i\Omega_n (t-t')} a(t') \end{aligned} \tag{21}$$

where I already used the fact that $b_n(0) = b_n$, is the Schrödinger picture operator.

Next it is convenient to define

$$\tilde{a}(t) = e^{i\omega t} a(t) \tag{22}$$

then

$$\frac{d\tilde{a}(t)}{dt} = i\omega\,\tilde{a}(t) + e^{i\omega t}\frac{da(t)}{dt}$$

$$= i\omega\,\tilde{a}(t) - i\omega\,\tilde{a}(t) - i\sum_n g_n\, e^{i(\omega-\Omega_n)t}\, b_n$$

$$- \sum_n g_n^2 \int_0^t dt'\, e^{i(\omega-\Omega_n)(t-t')}\, \tilde{a}(t')$$

Thus $\tilde{a}(t)$ obeys

$$\frac{d\tilde{a}}{dt} = -i\sum_n g_n e^{i(\omega-\Omega_n)t} b_n + - \sum_n g_n^2 \int_0^t dt' \, e^{i(\omega-\Omega_n)(t-t')} \tilde{a}(t') \tag{23}$$

we can make this equation clearer by defining the noise operator

$$\boxed{f(t) = -i\sum_n g_n e^{i(\omega-\Omega_n)t} b_n} \tag{24}$$

and the memory kernel

$$\boxed{M(t-t') = \sum_n g_n^2 e^{i(\omega-\Omega_n)(t-t')}} \tag{25}$$

Eq (23) then finally becomes

$$\boxed{\frac{d\tilde{a}(t)}{dt} = f(t) - \int_0^t dt' \, M(t-t')\tilde{a}(t')} \tag{26}$$

This is the quantum Langevin equation. The operator $f(t)$ is a noise because it depends only on the Schrödinger operators b_n of the bath. Hence, it is affected by the initial fluctuations in $\rho_B(0)$.

The last term in (26), on the other hand, is like a damping term $-\gamma a(t)$. The difference is that a term like $-\gamma a(t)$ damps based on the current value of the operator at time t. The term in (26), on the other hand, damps based on the entire history $a(t')$, with $t'<t$, weighted by a kernel $M(t-t')$, which reflects the influence of $a(t')$ in damping at time t. For this reason, Eq (26) is said to describe a non-Markovian evolution (Markovian evolutions are memoryless, like fish!)