Problem Set 2

1) Quantum master equation

$$\frac{d\rho}{dt} = \gamma(\bar{N}+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right] + \gamma\bar{N}\left[a^\dagger\rho a - \frac{1}{2}\{aa^\dagger, \rho\}\right]$$

(a) $$\frac{d}{dt}\mathrm{tr}(\rho) = \mathrm{tr}\left(\frac{d\rho}{dt}\right) = \gamma(\bar{N}+1)\,\mathrm{tr}\left\{a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right\} + \gamma N\,\mathrm{tr}\left\{a^\dagger\rho a - \frac{1}{2}\{aa^\dagger, \rho\}\right\}$$

Using the cyclic property of the trace, we find

$$\mathrm{tr}(a\rho a^\dagger) = \mathrm{tr}(a^\dagger a\rho) = \mathrm{tr}(\rho a^\dagger a)$$

$$\mathrm{tr}(a^\dagger\rho a) = \mathrm{tr}(aa^\dagger\rho) = \mathrm{tr}(\rho aa^\dagger)$$

Thus

$$\frac{d}{dt}\mathrm{tr}(\rho) = 0$$

This means that tr(ρ) does not change in time. If $\mathrm{tr}(\rho(0)) = 1$ then $\mathrm{tr}(\rho(t)) = 1$ for any t.

(b) We need the commutation relations

$$[a, a^\dagger] = 1 \quad \begin{array}{l} \nearrow [a^\dagger a, a] = -a \\ \searrow [a^\dagger a, a^\dagger] = a^\dagger \end{array}$$

Then, using BCH,

$$e^{\beta\hbar\omega a^\dagger a} a e^{-\beta\hbar\omega a^\dagger a} = a + \beta\hbar\omega [a^\dagger a, a] + \frac{1}{2}(\beta\hbar\omega)^2 [a^\dagger a, [a^\dagger a, a]] + \dots$$

$$= a \left[1 - \beta\hbar\omega + \frac{(\beta\hbar\omega)^2}{2} - \dots \right]$$

$$= a e^{-\beta\hbar\omega}$$

$$\therefore \quad \boxed{a e^{-\beta\hbar\omega a^\dagger a} = e^{-\beta\hbar\omega} e^{-\beta\hbar\omega a^\dagger a} a}$$

We can change the order of a and $e^{-\beta\hbar\omega a^\dagger a}$, but we pay a price of $e^{-\beta\hbar\omega}$. For $a^\dagger$ we get similarly

$$a^\dagger e^{-\beta\hbar\omega a^\dagger a} = e^{+\beta\hbar\omega} e^{-\beta\hbar\omega a^\dagger a} a^\dagger$$

Using this we now compute

$$a e^{-\beta\hbar\omega a^\dagger a} a^\dagger = e^{-\beta\hbar\omega} e^{-\beta\hbar\omega a^\dagger a} a a^\dagger$$

$$a^\dagger e^{-\beta\hbar\omega a^\dagger a} a = e^{\beta\hbar\omega} e^{-\beta\hbar\omega a^\dagger a} a^\dagger a$$

(c) Using these results we then find that

$$\mathcal{L}(e^{-\beta\hbar\omega a^\dagger a}) = \gamma(\bar{N}+1)\, e^{-\beta\hbar\omega a^\dagger a}\left(e^{-\beta\hbar\omega} a a^\dagger - a^\dagger a\right)$$

$$+ \gamma\bar{N}\, e^{-\beta\hbar\omega a^\dagger a}\left(e^{\beta\hbar\omega} a^\dagger a - a a^\dagger\right)$$

where I also used the fact that $e^{-\beta\hbar\omega a^\dagger a}$ commutes with $a^\dagger a$ and $a a^\dagger$. Combining the terms proportional to $a^\dagger a$ and $a a^\dagger$ yields

$$\mathcal{L}(e^{-\beta\hbar\omega}) = \gamma e^{-\beta\hbar\omega a^\dagger a}\Big\{\left[(\bar{N}+1)e^{-\beta\hbar\omega} - \bar{N}\right] a a^\dagger$$

$$+ \left[\bar{N} e^{\beta\hbar\omega} - (\bar{N}+1)\right] a^\dagger a\Big\}$$

But

$$\bar{N} = \frac{1}{e^{\beta\hbar\omega}-1}$$

$$\bar{N}+1 = \frac{e^{\beta\hbar\omega}}{e^{\beta\hbar\omega}-1} = \bar{N} e^{\beta\hbar\omega}$$

Thus

$$(\bar{N}+1)e^{-\beta\hbar\omega} - \bar{N} = 0 = \bar{N}e^{\beta\hbar\omega} - (\bar{N}+1)$$

which implies

$$\boxed{\mathcal{L}(e^{-\beta\hbar\omega a^\dagger a}) = 0}$$

Equilibrium is thus a fixed point of the equation, but only provided the temperature β is the same as the one appearing in $\bar{N}$.

(d) From the Q Master equation

$$\frac{dp_n}{dt} = \frac{d}{dt}\langle n|\rho|n\rangle$$

$$= \gamma(\bar{N}+1)\langle n|\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right]|n\rangle$$

$$+ \gamma\bar{N}\langle n|\left[a^\dagger\rho a - \frac{1}{2}\{a a^\dagger, \rho\}\right]|n\rangle$$

$$= \gamma(\bar{N}+1)\left\{(n+1)\langle n+1|\rho|n+1\rangle - n\langle n|\rho|n\rangle\right\}$$

$$+ \gamma\bar{N}\left\{n\langle n-1|\rho|n-1\rangle - (n+1)\langle n|\rho|n\rangle\right\}$$

$\therefore$

$$\boxed{\frac{dp_n}{dt} = \gamma(\bar{N}+1)\left[(n+1)p_{n+1} - n p_n\right] + \gamma\bar{N}\left[n p_{n-1} - (n+1)p_n\right]}$$

(e) $\frac{d\langle a^\dagger a\rangle}{dt} = \mathrm{tr}\left\{a^\dagger a \frac{d\rho}{dt}\right\}$

$$= \gamma(\bar{N}+1)\,\mathrm{tr}\left\{a^\dagger a\left[a\rho a^\dagger - \tfrac{1}{2}\{a^\dagger a,\rho\}\right]\right\}$$

$$+ \gamma\bar{N}\,\mathrm{tr}\left\{a^\dagger a\left[a^\dagger\rho a - \tfrac{1}{2}\{a a^\dagger,\rho\}\right]\right\}$$

Using the cyclic property of the trace, we get

$$\frac{d\langle a^\dagger a\rangle}{dt} = \gamma(\bar{N}+1)\langle a^\dagger a^\dagger a a - a^\dagger a a^\dagger a\rangle$$

$$+ \gamma\bar{N}\langle a a^\dagger a a^\dagger - \tfrac{1}{2} a a^\dagger a^\dagger a - \tfrac{1}{2} a^\dagger a a a^\dagger\rangle$$

Using $a a^\dagger = 1 + a^\dagger a$, however, we get

$$a^\dagger a^\dagger a a - a^\dagger a a^\dagger a = a^\dagger a^\dagger a a - a^\dagger(1 + a^\dagger a)a$$

$$= -a^\dagger a$$

$$a a^\dagger a a^\dagger - \tfrac{1}{2} a a^\dagger a^\dagger a - \tfrac{1}{2} a^\dagger a a a^\dagger = a a^\dagger a a^\dagger - \tfrac{1}{2} a a^\dagger (a a^\dagger - 1) - \tfrac{1}{2}(a a^\dagger - 1) a a^\dagger$$

$$= a a^\dagger$$

Thus

$$\frac{d\langle a^\dagger a\rangle}{dt} = \gamma(\bar{N}+1)(-\langle a^\dagger a\rangle) + \gamma\bar{N}(1 + \langle a^\dagger a\rangle)$$

$$\Rightarrow \boxed{\frac{d\langle a^\dagger a\rangle}{dt} = \gamma(\bar{N} - \langle a^\dagger a\rangle)}$$

The solution of this equation is

$$\langle a^\dagger a\rangle_t = e^{-\gamma t}\langle a^\dagger a\rangle_0 + (1 - e^{-\gamma t})\bar{N}$$

The initial population $\langle a^\dagger a\rangle_0$ will thus eventually give place to the bath induced population $\bar{N}$. The relation is exponential, with rate γ. As $t \to \infty$, $\langle a^\dagger a\rangle_\infty \to \bar{N}$.

2) Spontaneous emission

$$\frac{d\rho}{dt} = \gamma(\bar{N}+1)\left[\sigma\rho\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \rho\}\right] + \gamma\bar{N}\left[\sigma^\dagger\rho\sigma - \frac{1}{2}\{\sigma\sigma^\dagger, \rho\}\right]$$

The operator $\sigma = |0\rangle\langle 1|$ satisfies $\sigma^2 = \sigma^{\dagger 2} = 0$ and $(\sigma^\dagger\sigma)^2 = \sigma^\dagger\sigma$.

Moreover, $\sigma^\dagger\sigma = |1\rangle\langle 1|$ and $\sigma\sigma^\dagger = |0\rangle\langle 0|$ so $\sigma\sigma^\dagger = 1 - \sigma^\dagger\sigma$.

(a)

Proceeding as in ex. 2(e), we get

$$\frac{d}{dt}\langle\sigma^\dagger\sigma\rangle = \gamma(\bar{N}+1)\langle \overbrace{\sigma^\dagger\sigma^\dagger}^{0}\sigma\sigma - \overbrace{\sigma^\dagger\sigma\sigma^\dagger\sigma}^{\sigma^\dagger\sigma}\rangle + \gamma\bar{N}\langle \underbrace{\sigma\sigma^\dagger\sigma\sigma^\dagger}_{\sigma\sigma^\dagger = 1-\sigma^\dagger\sigma} - \frac{1}{2}\sigma\underbrace{\sigma^\dagger\sigma^\dagger}_{0}\sigma - \frac{1}{2}\sigma^\dagger\underbrace{\sigma\sigma}_{0}\sigma^\dagger\rangle$$

$$= -\gamma(\bar{N}+1)\langle\sigma^\dagger\sigma\rangle + \gamma\bar{N}(1 - \langle\sigma^\dagger\sigma\rangle)$$

$$= \gamma\bar{N} - \gamma(2\bar{N}+1)\langle\sigma^\dagger\sigma\rangle$$

or

$$\boxed{\frac{d\langle\sigma^\dagger\sigma\rangle}{dt} = \gamma(2\bar{N}+1)\left(\frac{\bar{N}}{2\bar{N}+1} - \langle\sigma^\dagger\sigma\rangle\right)}$$

The steady state is

$$\langle \sigma^\dagger \sigma \rangle_{ss} = \frac{\bar{N}}{2\bar{N}+1}$$

But

$$2\bar{N}+1 = \frac{2}{e^{\beta\epsilon}-1} + 1 = \frac{e^{\beta\epsilon}+1}{e^{\beta\epsilon}-1}$$

Thus

$$\boxed{\langle \sigma^\dagger \sigma \rangle = \frac{\bar{N}}{2\bar{N}+1} = \frac{1}{e^{\beta\epsilon}+1}}$$

which is nothing but the Fermi-Dirac occupation of a qubit. This makes sense: in the steady-state the system tends to thermal equilibrium.

(b) The solution of $\langle \sigma^\dagger \sigma \rangle_t$ is, as in ex. 2(c),

$$\langle \sigma^\dagger \sigma \rangle_t = e^{-\Gamma t} \langle \sigma^\dagger \sigma \rangle_0 + (1 - e^{-\Gamma t}) \frac{\bar{N}}{2\bar{N}+1}$$

where

$$\Gamma = \gamma(2\bar{N}+1)$$

Thus, the effective relaxation rate Γ will depend on T. When $\bar{N}=0$, $\Gamma \to \gamma$ and the solution reduces to

$$\langle \sigma^\dagger \sigma \rangle_t = e^{-\gamma t} \langle \sigma^\dagger \sigma \rangle_0$$

which is the spontaneous emission.

(c)

$$\frac{d\langle\sigma\rangle}{dt} = \gamma(\bar{N}+1)\,\mathrm{tr}\left\{\sigma\left[\sigma\rho\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma,\rho\}\right]\right\}$$

$$+\gamma\bar{N}\,\mathrm{tr}\left\{\sigma\left[\sigma^\dagger\rho\sigma - \frac{1}{2}\{\sigma\sigma^\dagger,\rho\}\right]\right\}$$

$$= \gamma(\bar{N}+1)\left\langle \sigma^\dagger\overbrace{\sigma\sigma}^{0} - \frac{1}{2}\sigma^\dagger\overbrace{\sigma\sigma}^{0} - \frac{1}{2}\sigma\sigma^\dagger\sigma\right\rangle$$

$$+\gamma\bar{N}\left\langle \underbrace{\sigma\sigma}_{0}\sigma^\dagger - \frac{1}{2}\sigma\sigma^\dagger\sigma - \frac{1}{2}\sigma^\dagger\underbrace{\sigma\sigma}_{0}\right\rangle$$

using

$$\sigma\sigma^\dagger\sigma = \sigma(1-\sigma\sigma^\dagger) = \sigma$$

we get

$$\frac{d\langle\sigma\rangle}{dt} = -\frac{\gamma}{2}(\bar{N}+1)\langle\sigma\rangle - \frac{1}{2}\gamma\bar{N}\langle\sigma\rangle$$

or

$$\boxed{\frac{d\langle\sigma\rangle}{dt} = -\frac{\Gamma}{2}\langle\sigma\rangle} \qquad \left[\Gamma = \gamma(2\bar{N}+1)\right]$$

The coherences thus evolve according to

$$\boxed{\langle\sigma\rangle_t = e^{-\Gamma t/2}\langle\sigma\rangle_0}$$

which is an exponential relaxation.

(d) we have from (b) and (c)

$$p_t = e^{-\Gamma t} p_0 + (1-e^{-\Gamma t}) p^* \qquad\qquad p^* = \frac{\bar{N}}{2\bar{N}+1}$$

$$q_t = e^{-\Gamma t/2} q_0 .$$

Matching the parametrizations

$$\rho = \begin{pmatrix} 1-p & q \\ q^* & p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+s_z & s_x - i s_y \\ s_x + i s_y & 1-s_z \end{pmatrix}$$

we get

$$p = \frac{1-s_z}{2} \rightsquigarrow s_z = 1-2p$$

$$q = \frac{s_x - i s_y}{2} \rightsquigarrow \begin{array}{l} s_x = 2\,\mathrm{Re}(q) \\ s_y = -2\,\mathrm{Im}(q) \end{array}$$

Thus, s_x, s_y, s_z will evolve according to

$$s_z(t) = 1 - 2p_t = 1 - 2e^{-\Gamma t}\left(\frac{1-s_z(0)}{2}\right) - 2(1-e^{-\Gamma t})p^*$$

$$= e^{-\Gamma t} s_z(0) + (1-e^{-\Gamma t}) s_z^*$$

where

$$s_z^* = 1-2p^* = 1 - \frac{2\bar{N}}{2\bar{N}+1} = \frac{1}{2\bar{N}+1}$$

← from (a)

$$= \frac{e^{\beta\epsilon}-1}{e^{\beta\epsilon}+1}$$

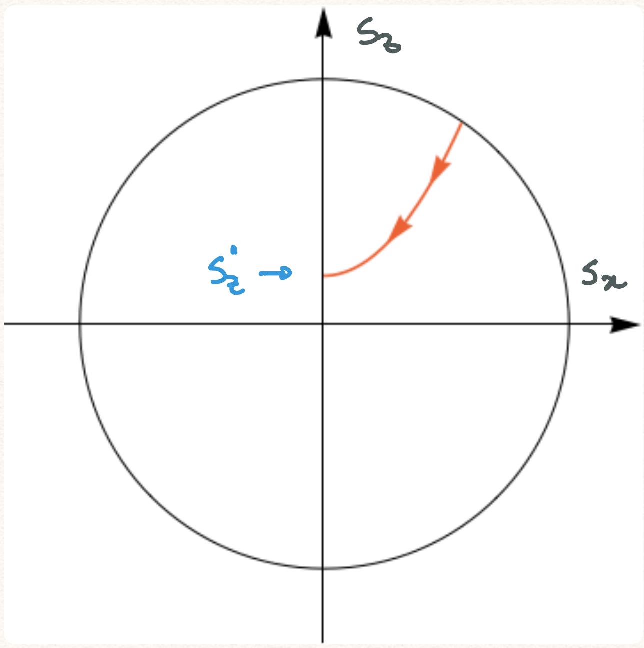
$$= \tanh\left(\frac{\beta\epsilon}{2}\right)$$

Thus

$$S_z(t) = e^{-\Gamma t} S_z(0) + (1 - e^{-\Gamma t}) S_z' \qquad S_z' = \tanh\left(\frac{\beta\delta}{2}\right)$$

As for S_x and S_y, they will simply evolve as

$$S_x(t) = e^{-\Gamma t/2} S_x(0), \qquad S_y(t) = e^{-\Gamma t/2} S_y(0)$$



Trajectory in Bloch's sphere

3) Entropy production and decoherence

I will study this problem in 2 ways: one clumsy but direct. The other elegant but abstract.

In exercises 2(a) and 2(c) we found that

$$p_t = e^{-\Gamma t} p_0 + (1 - e^{-\Gamma t}) p^* \qquad p^* = \frac{\bar{N}}{2\bar{N}+1}$$

$$q_t = e^{-\Gamma t/2} q_0 . \qquad \Gamma = \gamma(2\bar{N}+1)$$

The thermal state in this case is

$$\rho_{th} = \begin{pmatrix} 1-p^* & 0 \\ 0 & p^* \end{pmatrix}$$

Let us write

$$S(\rho \| \rho_{th}) = \mathrm{tr}(\rho \ln \rho - \rho \ln \rho_{th})$$

$$= - S(\rho) - \mathrm{tr}(\rho \ln \rho_{th})$$

Then

$$\Pi = -\frac{d}{dt} S(\rho \| \rho_{th}) = \frac{dS(\rho)}{dt} + \mathrm{tr}\left\{ \frac{d\rho}{dt} \ln \rho_{th} \right\}$$

The last term is easy if we compute the trace in the basis $|0\rangle$, $|1\rangle$ where ρ_{th} is already diagonal. Then

$$\mathrm{tr}\left\{ \frac{d\rho}{dt} \ln \rho_{th} \right\} = \langle 0 | \frac{d\rho}{dt} | 0 \rangle \ln(1-p^*) + \langle 1 | \frac{d\rho}{dt} | 1 \rangle \ln p^*$$

$$= \frac{d}{dt}(1 - p_t) \ln(1-p^*) + \frac{d}{dt} p_t \ln p^*$$

$$= -\frac{dp_t}{dt} \ln(1-p^*) + \frac{dp_t}{dt} \ln p^*$$

Thus

$$\mathrm{tr}\left\{\frac{d\rho}{dt}\ln\rho_{th}\right\} = \frac{d p_t}{dt}\ln\frac{p'}{1-p'}$$

$$= \Gamma\,(p' - p_t)\ln\frac{p'}{1-p'}$$

where I used Eq (6) of the problem set which, in terms of Γ and p', reads

$$\frac{dp}{dt} = \Gamma(p' - p)$$

Notice that this term $\mathrm{tr}\left\{\frac{d\rho}{dt}\ln\rho_{th}\right\}$ is completely independent of the coherences. We can also write

$$\frac{p'}{1-p'} = \frac{\bar{N}}{2\bar{N}+1}\,\frac{2\bar{N}+1}{\bar{N}+1} = e^{-\beta\epsilon}$$

Thus

$$\mathrm{tr}\left\{\frac{d\rho}{dt}\ln\rho_{th}\right\} = -\beta\epsilon\Gamma\,(p' - p_t)$$

and therefore

$$\Pi = \frac{dS}{dt} + \mathrm{tr}\left\{\frac{d\rho}{dt}\ln\rho_{th}\right\}$$

$$= \frac{dS}{dt} - \beta\epsilon\Gamma\,(p' - p_t)$$

The contribution from decoherence will come from dS/dt.

In lecture 2 we saw that for a qubit

$$S(\rho) = -\left(\frac{1+s}{2}\right)\ln\left(\frac{1+s}{2}\right) - \left(\frac{1-s}{2}\right)\ln\left(\frac{1-s}{2}\right)$$

where

$$s^2 = s_x^2 + s_y^2 + s_z^2 = 4|q_t|^2 + (1-2p_t)^2$$

Since $S(\rho)$ only depends on $|q|$, I will henceforth take $q \in \mathbb{R}$. this will not affect the final result. Then

$$\frac{dS}{dt} = \frac{\partial S}{\partial q}\frac{dq}{dt} + \frac{\partial S}{\partial p}\frac{dp}{dt}$$

$$= \frac{\partial S}{\partial s}\frac{\partial s}{\partial q}\frac{dq}{dt} + \frac{\partial S}{\partial s}\frac{\partial s}{\partial p}\frac{dp}{dt}$$

we have:

$$\frac{\partial s}{\partial q} = \frac{4q}{s} \qquad \frac{\partial s}{\partial p} = \frac{-4(1-2p)}{s} \qquad \frac{\partial S}{\partial s} = \frac{1}{2}\ln\left(\frac{1-s}{1+s}\right)$$

Thus

$$\frac{dS}{dt} = \frac{1}{2}\ln\left(\frac{1-s}{1+s}\right)\left\{\frac{4q}{s}\frac{dq}{dt} - \frac{4(1-2p)}{s}\frac{dp}{dt}\right\}$$

$$= \frac{2}{s}\ln\left(\frac{1-s}{1+s}\right)\left\{q\frac{dq}{dt} - (1-2p)\frac{dp}{dt}\right\}$$

For what is asked in the exercise, all we need to focus on is the coherence part. Recall that

$$\frac{dq}{dt} = -\Gamma q$$

$$\frac{dp}{dt} = \Gamma(p^*-p)$$

Moreover, note that for $s \in [0,1]$, $\ln\left(\frac{1-s}{1+s}\right) < 0$. Thus let us write

$$\Pi = \frac{2}{s}\ln\left(\frac{1+s}{1-s}\right)\Gamma|q|^2 + \frac{2}{s}\ln\left(\frac{1+s}{1-s}\right)(1-2p)\Gamma(p^*-p) - \beta\epsilon\Gamma(p^*-p_t)$$

Coherence only appears in the 1^{st} and 2^{nd} terms above. The 1^{st} term shows that q gives an always non-negative contribution to Π. In the 2^{nd} term, q only increases s. Moreover, for $p^* \in [0, 1/2]$ (positive temperatures), $(1-2p)(p^*-p) > 0$. Thus, q also increases the 2^{nd} term.

3) (Elegant solution).

Given a density matrix ρ, define the relative entropy of coherence

$$C(\rho) = S(\rho \| \rho_d)$$

where

$$\rho_d = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|$$

is a DM composed only of the diagonal entries of ρ. Note that

$$C(\rho) = \mathrm{tr}(\rho \ln \rho - \rho \ln \rho_d)$$

$$= -S(\rho) - \mathrm{tr}(\rho \ln \rho_d)$$

$$= -S(\rho) - \sum_i \langle i|\rho|i\rangle \ln \langle i|\rho|i\rangle$$

thus

$$C(\rho) = \underline{S(\rho_d) - S(\rho)} \;//$$

The relative entropy in this case is just a difference of entropies.

Since ρ_{th} is diagonal we now write

$$S(\rho \| \rho_{th}) = -S(\rho) - \mathrm{tr}(\rho \ln \rho_{th})$$

Because ρ_{th} is diagonal.

$$= C(\rho) - S(\rho_d) - \mathrm{tr}(\rho_d \ln \rho_{th})$$

thus

$$\boxed{S(\rho \| \rho_{th}) = S(\rho_d \| \rho_{th}) + C(\rho)}$$

The 1st term is exactly the classical relative entropy between two distributions $p_i = \langle i|\rho|i\rangle$ and $p_i^{th} = \langle i|\rho_{th}|i\rangle$. The entropy production then becomes

$$\boxed{\Pi = -\frac{d}{dt} S(\rho_d \| \rho_{th}) - \frac{dC}{dt}}$$

the 1st term is the classical entropy production that we saw in lecture 2. The last term, on the other hand, is a new contribution related to the loss of quantum coherence.

Since the relaxation destroys the coherence, it follows that

$$\frac{dC}{dt} < 0 \qquad \forall t$$

Thus, burning quantum coherence adds an extra contribution to irreversibility.

4) Superradiance

$$\frac{d\rho}{dt} = \kappa \left[S_- \rho S_+ - \frac{1}{2} \{ S_+ S_-, \rho \} \right]$$

(a) $$\frac{d\rho_m}{dt} = \frac{d}{dt} \langle m | \rho | m \rangle$$

$$= \kappa \langle m | \left[S_- \rho S_+ - \frac{1}{2} S_+ S_- \rho - \frac{1}{2} \rho S_+ S_- \right] | m \rangle$$

Using

$$S_+ |m\rangle = \sqrt{(N-m)(m+1)} \, |m+1\rangle$$

$$S_- |m\rangle = \sqrt{m(N-m+1)} \, |m-1\rangle$$

we get

$$\langle m | S_- \rho S_+ | m \rangle = (N-m)(m+1) \langle m+1 | \rho | m+1 \rangle$$

$$= (N-m)(m+1) \rho_{m+1}$$

$$S_+ S_- |m\rangle = \sqrt{m(N-m+1)} \, S_+ |m-1\rangle$$

$$= m(N-m+1) |m\rangle$$

Thus

$$\boxed{\frac{d\rho_m}{dt} = \kappa \left[(N-m)(m+1) \rho_{m+1} - m(N-m+1) \rho_m \right]}$$

Note that the prob. flows are only downwards: there is nothing coming in from ρ_{m-1}.

(b)

$$\frac{d\langle m\rangle}{dt} = \sum_{m=0}^{N} m \frac{dp_m}{dt}$$

$$= \kappa \sum_{m=0}^{N} m \left[(N-m)(m+1) p_{m+1} - m(N-m+1) p_m \right]$$

Change to $m = m+1$ only in the first term:

← since $m=-1$ and $m=N$ give zero anyway

$$\sum_{m=0}^{N} m(N-m)(m+1) p_{m+1} = \sum_{m=-1}^{N-1} m(N-m)(m+1) p_{m+1}$$

$$= \sum_{m=0}^{N} (m-1)(N-m+1) m p_m$$

Now go back to m:

$$\frac{d\langle m\rangle}{dt} = \kappa \sum_{m=0}^{N} \left\{ (m-1)(N-m+1)m - m^2 (N-m+1) \right\} p_m$$

$$\therefore \quad \frac{d\langle m\rangle}{dt} = -\kappa \sum_{m=0}^{N} (N-m+1) m p_m$$

We can also write this more neatly as

$$\boxed{\frac{d\langle m\rangle}{dt} = -\kappa (N+1) \langle m\rangle + \kappa \langle m^2\rangle}$$

5) Anomalous heat flow

This problem is solved in the accompanying Mathematica notebook.