Statistical Mechanics: lecture 6

The 1D Ising model

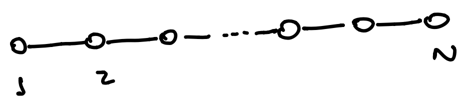
(and a 1st look at the 2D model).

Recommended reading: Salinas, Sec. 13.1.

At the beginning of the 20th century, when equilibrium Stat. Mech. was still fresh out of the oven, it was not obvious that the theory was able to describe phase transitions (or if some other ingredient was needed). Motivated by this, in 1920 W. Lenz proposed to his PhD student Ernst Ising a model that seemed like the simplest model of an interacting system that could, hopefully, present a phase transition.

Geometry

We consider here a system of N spin 1/2 particles, each described by its own set of Pauli matrices σ_i^x, σ_i^y and σ_i^z, where $i = 1 \dots N$.
We assume that these spins are displaced on a 1D lattice



The Ising model is defined by the Hamiltonian

$$\boxed{H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^z} \tag{1}$$

Here $\langle i,j \rangle$ means

$\langle i,j \rangle$ = sum over all nearest neighbor pairs

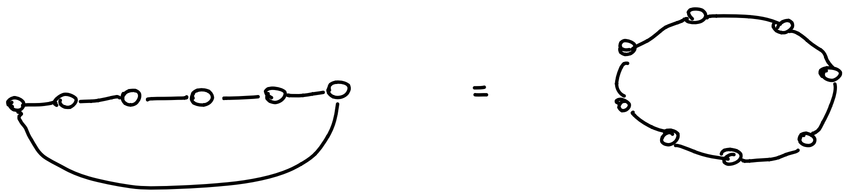
For 1D we can then write this as

$$H = -J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{N} \sigma_i^z \tag{2}$$

I just wanted to introduce the notation $\langle i,j \rangle$ now since it will be useful in 2D and 3D.

Notice also how in Eq (2) I sum up to N-1, which means I cover up to the bond $\sigma_{N-1}^z \sigma_N^z$. This is called open boundary conditions, like in the figure I drew above.

Many times it is more convenient to wrap the lattice around forming a ring



This is called periodic boundary conditions and is useful because it introduces translation invariance. It is, of course, a discrete translation, but still.

In the case of PBC the Hamiltonian (2) is modified to

$$H = -J\sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - h\sum_i \sigma_i^z \qquad (3)$$

The only difference is that now the sum goes up to N. Using PBC therefore only introduces a single extra term in the energy. When N is large, the thermodynamic properties of (2) and (3) should coincide.

Eigenstructure

The Ising Hamiltonian (1) is usually called a classical model because it is already trivially diagonal. Let $\sigma = \pm 1$ and $|\sigma\rangle$ denote the eigenstuff of a single Pauli matrix. Since the Hamiltonian (1) only depends on σ_z operators, it then follows that it will be diagonal in the basis

$$|\sigma\rangle = |\sigma_1 \dots \sigma_N\rangle = |\sigma_1\rangle \otimes \dots \otimes |\sigma_N\rangle \qquad (4)$$

i.e.,

$$H|\sigma\rangle = E(\sigma)|\sigma\rangle \qquad (5)$$

where

$$E(\sigma) = -J\sum_{\langle ij\rangle} \sigma_i \sigma_j - h\sum_i \sigma_i \qquad (6)$$

Here $\sigma_i = \pm 1$ are all c-number dichotomic variables. No operators involved.

Eq (6) is, in fact, how one usually defines the Ising model. But I prefer to use the quantum version (1) since soon we will want to generalize it to scenarios where H is not trivially diagonal (e.g. the so-called transverse field Ising model).

Thermal properties

Saying H is diagonal does not mean it is easy to find its thermal properties. We want to compute the partition function

$$Z = \mathrm{tr}\, e^{-\beta H} \tag{7}$$

As a basis for the trace, we use of course the eigenbasis $|\sigma\rangle$. Then

$$Z = \sum_{\sigma} \langle \sigma | e^{-\beta H} | \sigma \rangle = \sum_{\sigma} e^{-\beta E(\sigma)} \tag{8}$$

In this sum each σ_i should run between +1 and -1. Thus, there will be in total 2^N terms. The sum is thus not trivial at all. We will see how to do it in 1D. Later on in the course we will also see how to do it in 2D, which is much much harder. No one knows how to do it in 3D.

Solution for zero field

When $h=0$ the model becomes really easy to solve in 1D. In this case we use open BC:

$$H = -J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z \qquad (9)$$

Now remember that the trace can be decomposed into its partial traces

$$tr_{AB}(\dots) = tr_A(tr_B(\dots)) \qquad (10)$$

we thus write (7) as

$$Z = tr_1\, tr_2 \dots tr_N\, e^{-\beta H} \qquad (11)$$

Let us focus on tr_N. The σ_i^z commute, so we may write the exponential as

$$e^{-\beta H} = e^{\beta J \sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z} = e^{\beta J \sum_{i=1}^{N-2} \sigma_i^z \sigma_{i+1}^z} e^{\beta J \sigma_{N-1}^z \sigma_N^z} \qquad (12)$$

To compute tr_N in (11), we only need the last term

$$Z = tr_1 \dots tr_{N-1}\, e^{\beta J \sum_{i=1}^{N-2} \sigma_i^z \sigma_{i+1}^z}\, tr_N\, e^{\beta J \sigma_{N-1}^z \sigma_N^z} \qquad (13)$$

This tr_N is now easy to compute because it lives on a single spin space.

Indeed

$$\mathrm{tr}_N\, e^{\beta J \sigma_{N-1}^z \sigma_N^z} = \sum_{\sigma_N} \langle \sigma_N | e^{\beta J \sigma_{N-1}^z \sigma_N^z} | \sigma_N \rangle$$

$$= \sum_{\sigma_N} e^{\beta J \sigma_{N-1}^z \sigma_N}$$

σ_{N-1}^z is still an operator.

$$= e^{\beta J \sigma_{N-1}^z} + e^{-\beta J \sigma_{N-1}^z} \qquad (14)$$

Now comes a very special pathology which is only true for this model: since $(\sigma_z)^2 = 1$, a series expansion yields

$$e^{\alpha \sigma_z} = \cosh(\alpha) + \sigma_z \sinh(\alpha) \qquad (15)$$

Thus, using this on (14) yields

$$\mathrm{tr}_N\, e^{\beta J \sigma_{N-1}^z \sigma_N^z} = 2\cosh \beta J \qquad (16)$$

which is independent of σ_{N-1}^z! (It is proportional to the identity matrix). Plugging this in (13) we then get

$$Z = (2\cosh\beta J)\, \mathrm{tr}_1 \ldots \mathrm{tr}_{N-1}\, e^{\beta J \sum_{i=1}^{N-2} \sigma_i^z \sigma_{i+1}^z} \qquad (17)$$

This remaining trace is now a replica of the original Z, but with $N-1$ terms. Thus, we can now repeat the process and trace over $N-1$. This will give us another "$2\cosh\beta J$" times the Z with $N-2$ spins.

Continuing this way, the only different term will be "tr_1"

$$Z = 2\cosh(\beta J)\, tr_1 \ldots tr_{N-1}\, e^{\beta J \sum_{i=1}^{N-2} \sigma_i^z \sigma_{i+1}^z}$$

$$= (2\cosh\beta J)^2\, tr_1 \ldots tr_{N-2}\, e^{\beta J \sum_{i=1}^{N-3} \sigma_i^z \sigma_{i+1}^z}$$

$$\vdots$$

$$= (2\cosh\beta J)^{N-1} \underbrace{tr_1 (1)}_{2} \quad \text{(trace of the identity matrix)}$$

whence

$$\boxed{Z = 2(2\cosh\beta J)^{N-1}} \tag{18}$$

This partition function is quite similar to that of N independent spins in a magnetic field

$$H = -h\sum_{i=1}^{N} \sigma_i^z \quad \rightsquigarrow \quad Z = (2\cosh\beta J)^N \tag{19}$$

One cannot, therefore, expect many exciting properties from (18). The model is easy to solve and also not very rich in terms of its properties. Things become way more interesting in 2D. I promise.

The reason why (9) looks so much like (19) is because $\sigma_i^z\sigma_{i+1}^z$ has eigenvalues ± 1, just like a single Pauli matrix.

Solution for $h \neq 0$: the transfer matrix method

Let us consider now the full Hamiltonian (1), with $h \neq 0$.
We write it as

$$H = \sum_i H_{i,i+1} \tag{20}$$

where $H_{i,i+1}$ is a Hamiltonian acting only on sites i and $i+1$.
We can shuffle the h's around to write the slightly more symmetric formula

$$H_{i,i+1} = -J\sigma_i^z \sigma_{i+1}^z - \frac{h}{2}(\sigma_i^z + \sigma_{i+1}^z) \tag{21}$$

The only exceptions may be at the boundaries. For PBC (21) is ok even for $H_{N,1}$. For OBC we change

$$H_{1,2} = -J\sigma_1^z \sigma_2^z - \frac{h}{2}(2\sigma_1^z + \sigma_2^z) \tag{22}$$

and similarly for $H_{N-1,N}$. For concreteness, we now assume PBC.

All $H_{i,i+1}$ have the same eigenvalues

$$E(\sigma_i, \sigma_{i+1}) = -J\sigma_i \sigma_{i+1} - \frac{h}{2}(\sigma_i + \sigma_{i+1}) \tag{23}$$

The partition function may then be written as

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_{\sigma} e^{-\beta E(\sigma_1,\sigma_2)} e^{-\beta E(\sigma_2,\sigma_3)} \cdots e^{-\beta E(\sigma_{N-1},\sigma_N)} e^{-\beta E(\sigma_N,\sigma_1)} \tag{24}$$

Let us introduce for convenience

$$V(\sigma_i, \sigma_{i+1}) = e^{-\beta E(\sigma_i, \sigma_{i+1})} = e^{\beta J \sigma_i \sigma_{i+1} + \frac{\beta h}{2}(\sigma_i + \sigma_{i+1})} \tag{25}$$

Eq (24) then becomes

$$Z = \sum_{\sigma_1 \ldots \sigma_N} V(\sigma_1, \sigma_2)\, V(\sigma_2, \sigma_3) \ldots V(\sigma_{N-1}, \sigma_N)\, V(\sigma_N, \sigma_1) \tag{26}$$

If we think about it for a second now, we see that we may interpret V as a 2×2 matrix, with entries

$$V = \begin{pmatrix} V(1,1) & V(1,-1) \\ V(-1,1) & V(-1,-1) \end{pmatrix} = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h} \end{pmatrix} \tag{27}$$

This is called the transfer matrix. Eq (26) now looks like matrix multiplication:

$$(A^2)_{ij} = \sum_k A_{ik} A_{kj} \tag{28}$$

$$(A^3)_{ij} = \sum_{k,l} A_{ik} A_{kl} A_{lj}$$

and so on. Even more, we see what the PBC is doing: $V(\sigma_N, \sigma_1)$ is matching the last index with the first; it is making a trace!

$$\mathrm{tr}\, A^3 = \sum_i (A^3)_{ii} = \sum_{ikl} A_{ik} A_{kl} A_{li} \tag{29}$$

just like (26).

Thus, we conclude from this that Eq (26) may be written as

$$\boxed{Z = \mathrm{tr}\, V^N} \tag{30}$$

where "tr" here is simply a trace over a single 2×2 system

The eigenvalues of the transfer matrix (27) are

$$\boxed{\lambda_\pm = e^{\beta J}\cosh \beta h \pm \sqrt{e^{-2\beta J} + e^{2\beta J}\sinh^2 \beta h}} \tag{31}$$

Since the eigenvalues of V^N are λ_+^N and λ_-^N, then

$$Z = \lambda_+^N + \lambda_-^N \tag{32}$$

But we are interested in the thermodynamic limit, $N \to \infty$. The eigenvalues $\lambda_\pm \geq 0$ and $\lambda_+ > \lambda_-$. Thus we may write

$$Z = \lambda_+^N \left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N \right] \tag{33}$$

Since $\lambda_- < \lambda_+$, in the limit $N \to \infty$ the last term becomes negligible and we are left with

$$\boxed{Z = \lambda_+^N} \quad (\text{for } N \to \infty) \tag{34}$$

There are several models that can be solved by this transfer matrix technique. And the logic remains identical

$$\boxed{Z = (\text{largest eigenvalue of the transfer matrix})^N} \tag{35}$$

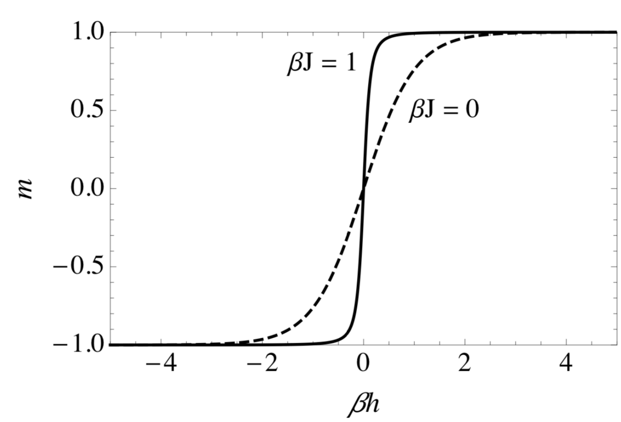
The free energy is then

$$\boxed{F = -T\ln Z = -NT\ln\lambda_+} \tag{36}$$

from which the magnetization readily follows

$$\boxed{M = -\frac{\partial F}{\partial h} = N\frac{e^{\beta J}\sinh\beta h}{\sqrt{e^{-2\beta J} + e^{2\beta J}\sinh^2(\beta h)}}} \tag{37}$$

The magnetization look like this



Sanity check

If $\beta J = 0$ we get

$M = N\tanh\beta h$

It is similar to the one with $\beta J = 0$. Thus, there is no unusual behavior, like a phase transition of something of the sort.

Correlation functions

From the partition function (34) all basic properties, like internal energy, heat capacity, entropy and etc. can be readily computed. I will leave that to you as a (fun) exercise. The formulas are a bit ugly and the results are not particularly interesting.

A much more interesting quantity is the correlation function defined as

$$G_r = \langle \sigma_i^z \sigma_{i+r}^z \rangle - \langle \sigma_i^z \rangle \langle \sigma_{i+r}^z \rangle \tag{38}$$

where $r = 1, 2, 3, \ldots$ this function measures how spin at a position i is correlated with the spin at a distance $i+r$. the reason why I only wrote G_r on the LHS (and not $G_{m,r}$) is because for PBC the system is manifestly translation invariant, so the result cannot depend on m, but only on the distance r between the spins.

The last term in (38) is included to subtract from $\langle \sigma_m^z \sigma_{m+r}^z \rangle$ the part related the natural magnetization of the system. In fact, again due to translation invariance, we already know that

$$\langle \sigma_m^z \rangle = \langle \sigma_{m+r}^z \rangle = \frac{M}{N} \tag{39}$$

where M is given in (37).

As a sanity check, however, let us also compute this directly. We have

$$\langle \sigma_m^z \rangle = \frac{1}{Z} \operatorname{tr} \left(\sigma_m^z e^{-\beta H} \right)$$

$$= \frac{1}{Z} \operatorname{tr} \left\{ e^{-\beta H_{12}} e^{-\beta H_{23}} \cdots e^{-\beta H_{m-1,m}} \sigma_m^z e^{-\beta H_{m,m+1}} \cdots e^{-\beta H_{N1}} \right\} \qquad (40)$$

which I am allowed to do, because everything commutes. In terms of the eigenvalues σ_i and the corresponding transfer matrices $V(\sigma_i, \sigma_{i+1})$ (see Eq (26)) this becomes

$$\langle \sigma_m^z \rangle = \frac{1}{Z} \sum_{\sigma_1 \cdots \sigma_N} V(\sigma_1, \sigma_2) \cdots V(\sigma_{m-1}, \sigma_m)\, \sigma_m\, V(\sigma_m, \sigma_{m+1}) \cdots V(\sigma_N, \sigma_1) \qquad (41)$$

This is now again starting to look like a trace. In fact, notice that for any 2×2 matrices A, B,

$$\sum_{\sigma_1 \sigma_2 \sigma_3} A(\sigma_1, \sigma_2)\, \sigma_2\, B(\sigma_2, \sigma_3) = \sum_{\sigma_1 \sigma_2 \sigma_3} A(\sigma_1, \sigma_2) (\sigma_z)_{\sigma_2 \sigma_2'} B(\sigma_2', \sigma_3)$$

$$= \operatorname{tr} \left(A \sigma_z B \right) \qquad (42)$$

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is just the 2×2 Pauli matrix. Thus we conclude that Eq (41) may be written as

$$\langle \sigma_m^z \rangle = \frac{1}{Z} \operatorname{tr} \left\{ V^{m-1} \sigma_z V^{N-m+1} \right\} \qquad (43)$$

But using the cyclic property of the trace this simplifies to

$$\boxed{\langle \sigma_m^z \rangle = \frac{1}{Z} \operatorname{tr}\{ \sigma_z V^N \}} \tag{44}$$

which is independent of the site N. The cyclic property reflects translation invariance!

Before we carry out this trace to get an explicit formula, let us find an analogous expression for $\langle \sigma_m^z \sigma_{m+r}^z \rangle$. Just like in (41), we get

$$\langle \sigma_m^z \sigma_{m+r}^z \rangle = \frac{1}{Z} \sum_{\sigma_1 \ldots \sigma_N} V(\sigma_1, \sigma_2) \ldots V(\sigma_{m-1}, \sigma_m)\, \sigma_m\, V(\sigma_m, \sigma_{m+1}) \ldots V(\sigma_{m+r-1}, \sigma_{m+r}) \cdot \sigma_{m+r}\, V(\sigma_{m+r}, \sigma_{m+r+1}) \ldots V(\sigma_N, \sigma_1) \tag{45}$$

or

$$\langle \sigma_m^z \sigma_{m+r}^z \rangle = \frac{1}{Z} \operatorname{tr}\{ V^{m-1} \sigma_z V^r \sigma_z V^{N-m-r+1} \} \tag{46}$$

Again, using the cyclic property, this simplifies to

$$\boxed{\langle \sigma_m^z \sigma_{m+r}^z \rangle = \frac{1}{Z} \operatorname{tr}\{ \sigma_z V^r \sigma_z V^{N-r} \}} \tag{47}$$

Isn't this formula beautiful? It really shows where the transfer matrix gets its name from: it transfers the solution from site m to site r.

The trace in (43) is now easy to compute because all matrices are 2×2. It is convenient to also introduce the eigenvectors of V. They read

$$|\lambda_+\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \qquad |\lambda_-\rangle = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \tag{48}$$

where

$$\tan\theta = \frac{e^{-2\beta J}}{\sinh\beta h} \tag{49}$$

I write $|\lambda_\pm\rangle$ with Dirac's notation for a nice reason, which I will explain soon. But please keep in mind that these are just 2D vectors.

Using $|\lambda_\pm\rangle$ as a basis for the trace, the magnetization (44) becomes

$$\langle\sigma_m^z\rangle = \frac{1}{Z}\left\{ \lambda_+^N \langle\lambda_+|\sigma_z|\lambda_+\rangle + \lambda_-^N \langle\lambda_-|\sigma_z|\lambda_-\rangle \right\} \tag{50}$$

In the thermodynamic limit $Z \simeq \lambda_+^N$ so we get

$$\langle\sigma_m^z\rangle = \langle\lambda_+|\sigma_z|\lambda_+\rangle + \left(\frac{\lambda_-}{\lambda_+}\right)^N \langle\lambda_-|\sigma_z|\lambda_-\rangle$$

As before, the last term vanishes and we are left with

$$\langle\sigma_m^z\rangle = \langle\lambda_+|\sigma_z|\lambda_+\rangle \tag{51}$$

Using (48) this further reduces to

$$\langle \sigma_M^z \rangle = \cos^2\theta/2 - \sin^2\theta/2 = \cos\theta \tag{52}$$

From (49)

$$\cos\theta = \frac{\sinh \beta h}{\sqrt{e^{-4\beta J} + \sinh^2\beta h}} \tag{53}$$

We can now compare this with (37) and thus conclude that (39) is indeed true.

Next, we turn to (47):

$$\langle \sigma_M^z \sigma_{M+r}^z \rangle = \frac{1}{Z} \left\{ \lambda_+^{N-r} \langle \lambda_+ | \sigma_z V^r \sigma_z | \lambda_+ \rangle + \lambda_-^{N-r} \langle \lambda_- | \sigma_z V^r \sigma_z | \lambda_- \rangle \right\}$$

$$\simeq \frac{1}{\lambda_+^r} \left\{ \langle \lambda_+ | \sigma_z V^r \sigma_z | \lambda_+ \rangle + \left(\frac{\lambda_-}{\lambda_+}\right)^{N-r} \langle \lambda_- | \sigma_z V^r \sigma_z | \lambda_- \rangle \right\}$$

whence, when $N \to \infty$ we get

$$\boxed{\langle \sigma_M^z \sigma_{M+r}^z \rangle = \frac{\langle \lambda_+ | \sigma_z V^r \sigma_z | \lambda_+ \rangle}{\lambda_+^r}} \tag{54}$$

Let us write

$$V^r = \lambda_+^r |\lambda_+\rangle\langle\lambda_+| + \lambda_-^r |\lambda_-\rangle\langle\lambda_-| \tag{55}$$

Then we get

$$\langle \sigma_m^z \sigma_{m+r}^z \rangle = \underbrace{\langle \lambda_+ | \sigma_z | \lambda_+ \rangle \langle \lambda_+ | \sigma_z | \lambda_+ \rangle}_{\cos^2\theta/2 - \sin^2\theta/2 = \cos\theta} + \left(\frac{\lambda_-}{\lambda_+}\right)^r \underbrace{\langle \lambda_+ | \sigma_z | \lambda_- \rangle \langle \lambda_- | \sigma_z | \lambda_+ \rangle}_{2 \sin\frac{\theta}{2} \cos\frac{\theta}{2} = \sin\theta}$$

$\therefore$

$$\boxed{\langle \sigma_m^z \sigma_{m+r}^z \rangle = \cos^2\theta + \left(\frac{\lambda_-}{\lambda_+}\right)^r \sin^2\theta} \qquad (56)$$

Note that we cannot throw away the last term in this case because r is not necessarily large, like N.

Notice how the 1^{st} term is just the magnetization squared. Thus, the "connected" correlation function (38) becomes

$$\boxed{G_r = \langle \sigma_m^z \sigma_{m+r}^z \rangle - \langle \sigma_m^z \rangle \langle \sigma_{m+r}^z \rangle = \left(\frac{\lambda_-}{\lambda_+}\right)^r \sin^2\theta} \qquad (57)$$

Let us analyze this for the more interesting case of $h \to 0$. Due to (49), $\theta \to \pi/2$. Moreover

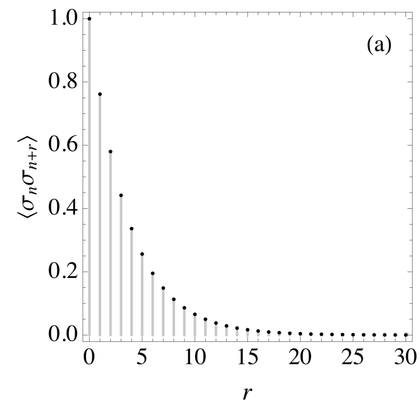
$$\lambda_\pm = e^{\beta J} \pm e^{-\beta J} = \begin{cases} 2\cosh\beta J \\ 2\sinh\beta J \end{cases}$$

We then get

$$\boxed{G_r = \tanh^r(\beta J)} \qquad (58)$$

which is pretty cute.

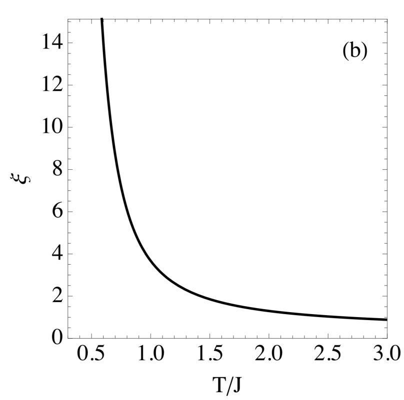
Eq (58) looks like a typical exponential decay



It is therefore suggestive to write (49) as

$$\boxed{G_r = e^{-r/\xi}, \qquad \xi = \frac{-1}{\ln \tanh \beta J}} \qquad (59)$$

The quantity ξ is called the correlation length of the system.
It looks like this:



When $T \to 0$ ($\beta J \to \infty$) ξ diverges. The GS of the model is to have all spins up or all down. The correlation thus has infinite range: two spins arbitrarily far apart will be perfectly correlated.

classical - quantum mapping

We started with a classical 1D Ising model. But when we introduce the transfer matrix, everything reduces to 2×2 systems. It is as if we are now dealing with a single spin 1/2 particle.

We can actually make this connection more concrete as follows. Start with the transfer matrix (27):

$$V = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h} \end{pmatrix} \tag{60}$$

We now decompose it in terms of two matrices

$$V_1 = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \quad \text{or} \quad V_1(\sigma_1\sigma_2) = e^{\beta J \sigma_1 \sigma_2} \tag{61}$$

and

$$V_2 = \begin{pmatrix} e^{\beta h} & 0 \\ 0 & e^{-\beta h} \end{pmatrix} \quad \text{or} \quad V_2(\sigma_1,\sigma_2) = e^{\beta h(\sigma_1 + \sigma_2)} \tag{62}$$

It is not true that $V = V_1 V_2$. Instead, I will leave it for you to check that

$$\boxed{V = V_2^{1/2}\, V_1\, V_2^{1/2}} \tag{63}$$

where

$$V_2^{1/2} = \begin{pmatrix} e^{\beta h/2} & 0 \\ 0 & e^{-\beta h/2} \end{pmatrix} \tag{64}$$

is the matrix square root of V_2.

Now let's do the following. The matrix V_2 can clearly be written as

$$V_2 = e^{\beta h \sigma_z} \tag{65}$$

for a 2×2 Pauli matrix $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It would be great to write a similar formula for V_1. To do that, we first write V_1 as

$$V_1 = e^{\beta J} + e^{-\beta J}\sigma_x = e^{\beta J}(1 + e^{-2\beta J}\sigma_x) \tag{66}$$

we also know that because $\sigma_x^2 = 1$,

$$e^{\alpha \sigma_x} = \cosh\alpha + \sigma_x \sinh\alpha = \cosh\alpha\,(1 + \sigma_x \tanh\alpha) \tag{67}$$

Let us then introduce a new constant J^* such that

$$\boxed{\tanh(\beta J^*) = e^{-2\beta J}} \tag{68}$$

Then by comparing (66) and (67), we can write

$$V_1 = e^{\beta J}(1 + e^{-2\beta J}\sigma_x) = e^{\beta J}(1 + \sigma_x \tanh\beta J')$$
$$= e^{\beta J}\frac{e^{\beta J'\sigma_x}}{\cosh\beta J'} \tag{69}$$

To clean up these constants we play with (68). This relation between J and J' turns out to be really really cute. First, you may check that

$$\tanh \beta J = e^{-2\beta J'} \tag{70}$$

(it's symmetric!) And second

$$\sinh 2\beta J \sinh 2\beta J' = 1 \tag{71}$$

It is also useful to know that

$$f(x) = \operatorname{arctanh} e^{-2x} \tag{72}$$

is monotonically decreasing in x (for $x>0$). So if βJ goes up, $\beta J'$ goes down.

Playing with these trigonometric identities, one finds

$$\frac{e^{\beta J}}{\cosh \beta J'} = (2 \sinh 2\beta J)^{1/2}$$

whence,

$$\boxed{V_J(\sigma,\sigma') = e^{\beta J \sigma\sigma'} \longrightarrow V_1 = (2\sinh 2\beta J)^{1/2} e^{\beta J' \sigma_x}} \tag{73}$$

For simplicity, let us assume that $h=0$. Then $V_2 = 1$ and $V = V_1$ [Eq. (63)]. Recall also Eq (47) for the correlation function:

$$\langle \sigma_n^z \sigma_{n+r}^z \rangle = \frac{1}{Z} \operatorname{tr} \{ \sigma_z V^r \sigma_z V^{N-r} \}$$

we can write this as

$$\langle \sigma_n^z \sigma_{n+r}^z \rangle = \frac{1}{Z} \operatorname{tr} \{ V^{-r} \sigma_z V^r \sigma_z V^{N-r} \} \tag{75}$$

Finally, looking at $V = V_1$ in (73), let us define an effective "Hamiltonian"

$$H_e = -J^* \sigma_x \tag{76}$$

Then

$$V^r = e^{-r\beta H_e} \tag{77}$$

The reason why I am doing this is to draw attention to the relation between Eq (75) and a two-time correlation function of the form

$$\langle A(t) B(0) \rangle = \operatorname{tr} \{ e^{iHt/\hbar} A e^{-iHt/\hbar} B \rho \} \tag{80}$$

where A and B are arbitrary operators. The logic is that the system is prepared at ρ and then (80) measures how B at time 0 is correlated with A at time t.

Eq (75) is very similar, but operates in imaginary time

$$-\frac{i\,t}{\hbar} = -\beta r \qquad (90)$$

Moreover, "time" here is actually discrete because $r = 0, 1, 2, 3, \ldots$.
Thus, "time" runs discretely in steps of β. This gives another way of thinking about why the transfer matrix transfers: it propagates the solution along the lattice, just like e^{-iHt} propagates the solution in time.

What is most important to realize about this discussion, however, is that we have reduced the classical Ising model in 1D into a quantum model in $(1-1=0)$D. The reason why I say "quantum" is because an effective Hamiltonian now involves σ_x, which is not trivially diagonal like the original model. This kind of mapping actually turns out to be more general: a d dimensional classical model maps into a $d-1$ dimensional quantum model. This is exactly the logic behind path integrals in general, actually. We will talk more about it later on.

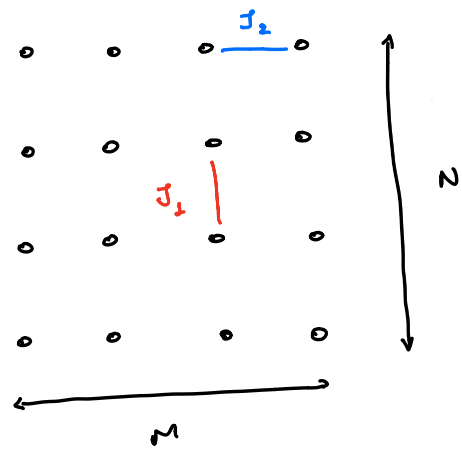
Actually, let me be even more precise. Looking at (79), we see that the largest eigenvalue λ_+ of V is actually the smallest eigenvalue of H_e. That is, λ_+ is the ground-state of H_e.

What we actually learned, therefore, is that the (finite temperature) partition function of a d-dimensional classical model was actually mapped into the ground-state of a $(d-1)$ dimensional quantum model!

The 2D Ising model

Ising studied the 1D Ising model and concluded it did not have a phase transition. He then wrongly presumed that this was also true in higher dimensions. But that is not true! In 2D and higher dimensions, the Ising model does have a transition.

We consider here the model in a square lattice, with each site (n,m) being associated with a set of Pauli operators σ^x_{nm}, σ^y_{nm}, σ^z_{nm}.



It is convenient for bookkeeping, to assume that the lattice is asymmetric, with M spins in each row and N spins in each column. Moreover, the spin couplings are also assumed to be different: J_1 for horizontal couplings and J_2 for vertical ones. The Hamiltonian is thus taken to be

$$H = -\sum_{n=1}^{N} \sum_{m=1}^{M} \left(J_1 \sigma^z_{nm} \sigma^z_{n+1,m} + J_2 \sigma^z_{nm} \sigma^z_{n,m+1} \right) \qquad (91)$$

We assume no magnetic fields. Notice how, by counting one to the right and one up, we never count the bonds twice.

We also assume PBCs in (91). This means

$$\sigma^z_{N+1,m} = \sigma_{1,m} \qquad \forall\, m$$
$$\sigma^z_{n,M+1} = \sigma^z_{n,1} \qquad \forall\, n. \tag{92}$$

Effectively, therefore, we are wrapping the square lattice to form a torus.

We want the partition function, like before:

$$Z = \operatorname{tr} e^{-\beta H} = \sum_{\{\sigma\}} e^{-\beta E(\sigma)}$$
$$= \sum_{\{\sigma\}} \exp\left\{ \beta \sum_{n,m} \left(J_1 \sigma_{nm} \sigma_{n+1,m} + J_2 \sigma_{nm} \sigma_{nm+1} \right) \right\} \tag{93}$$

where $\{\sigma\}$ is a shorthand notation for the sum over all spin configurations.

Define now a function

$$V(\sigma_n, \sigma_{n+1}) = \exp\left\{ \beta \sum_{m=1}^{M} \left(J_1 \sigma_{nm} \sigma_{n+1\,m} + \frac{J_2}{2} \left(\sigma_{nm} \sigma_{nm+1} + \sigma_{n+1,m} \sigma_{n+1,m+1} \right) \right) \right\} \tag{94}$$

where $\sigma_n = (\sigma_{n1}, \sigma_{n2}, \ldots, \sigma_{nM})$ is a shorthand for the spin configuration of row n. The factor of $1/2$ in (94) is put to make V more symmetric, exactly like we did in the 1D model [Eq (25)].

Eq (93) is then written as

$$\boxed{Z = \sum_{\{\sigma\}} V(\sigma_1, \sigma_2)\, V(\sigma_2, \sigma_3) \dots V(\sigma_{N-1}, \sigma_N)\, V(\sigma_N, \sigma_1)} \qquad (95)$$

We see that this is looking again like a transfer matrix. But now the transfer matrix transfers entire rows! Each $V(\sigma_m, \sigma_{m+1})$ can be viewed as one entry of the matrix. But since σ_m can take on 2^M values, the transfer matrix will now have dimensions $2^M \times 2^M$.

The problem is therefore mathematically much more difficult. But the logic still applies: we can write (95) as

$$\boxed{Z = \operatorname{tr} V^N} \qquad (96)$$

and, in the thermodynamic limit $N \to \infty$, the dominant contribution will be the largest eigenvalue of V:

$$Z = \lambda_{max}^N, \qquad \lambda_{max} = \text{largest eig of } V \qquad (97)$$

Finding λ_{max} is tough. But we will do it later on in the course! :)

It is convenient to split V as we did in (63)

$$V = V_2^{1/2} \, V_1 \, V_2^{1/2} \tag{98}$$

where V_2 is a diagonal matrix

$$V_2(\sigma_m, \sigma_{m+1}) = \delta_{\sigma_m \sigma_{m+1}} \exp\left\{ \beta J_2 \sum_{n} \sigma_{nm} \sigma_{n\,m+1} \right\} \tag{99}$$

It is diagonal because it does not mix σ_m with σ_{m+1}. then V_1 in (98) is the remainder

$$V_1(\sigma_m, \sigma_{m+1}) = \exp\left\{ \beta J_1 \sum_{n} \sigma_{nm} \sigma_{m+1, n} \right\} \tag{100}$$

We can check that the decomposition (98) is indeed true. To do this, the Kronecker δ in (99) is essential:

$$
\begin{aligned}
(V_2^{1/2} V_1 V_2^{1/2})_{\sigma_m, \sigma_{m+1}} &= \sum_{\sigma'} (V_2^{1/2})_{\sigma_m \sigma'} \; (V_1 V_2^{1/2})_{\sigma' \sigma_{m+1}} \\
&= (V_2^{1/2})_{\sigma_m \sigma_m} \; (V_1 V_2^{1/2})_{\sigma_m \sigma_{m+1}} \\
&= \sum_{\sigma'} (V_2^{1/2})_{\sigma_m \sigma_m} \; (V_1)_{\sigma_m \sigma'} \; (V_2^{1/2})_{\sigma' \sigma_{m+1}} \\
&= (V_2^{1/2})_{\sigma_m \sigma_m} \; (V_1)_{\sigma_m \sigma_{m+1}} \; (V_2^{1/2})_{\sigma_{m+1} \sigma_{m+1}}
\end{aligned}
$$

the combination of the 3 terms then gives us back (93).

We can now finally introduce a matrix representation for V_1 and V_2. For V_2 its easy because it is already diagonal. We simply interpret σ_m as the eigenvalues of a set of M Pauli matrices σ_m^z, $m = 1, \dots, M$ (one for each element of the row).

Then

$$\boxed{V_2 = \exp\left\{ \beta J_2 \sum_{m=1}^{M} \sigma_m^z \sigma_{m+1}^z \right\}} \tag{101}$$

As for V_1 in Eq (100), we need to remember (73):

$$V_1(\sigma, \sigma') = e^{\beta J \sigma, \sigma'} \rightsquigarrow V_1 = (2 \sinh 2\beta J)^{1/2} e^{\beta J^* \sigma_x}$$

The matrix V_1 in (100) is simply a product of M such terms

$$\begin{aligned} V_1(\sigma, \sigma') &= \exp\left\{ \beta J_1 \sum_{m=1}^{M} \sigma_m \sigma_m' \right\} \\ &= e^{\beta J_1 \sigma_1 \sigma_1'} e^{\beta J_1 \sigma_2 \sigma_2'} \dots e^{\beta J_1 \sigma_M \sigma_M'} \end{aligned} \tag{102}$$

Thus

$$V_1 = (2 \sinh \beta J_1)^{M/2} e^{\beta J_1^* \sigma_1^x} e^{\beta J_1^* \sigma_2^x} \dots e^{\beta J_1^* \sigma_M^x} \tag{103}$$

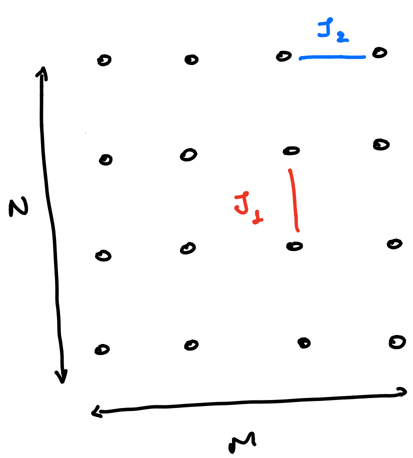
where σ_m^x are a set of M Pauli x matrices. Since they commute, we can also write

$$\boxed{V_1 = (2 \sinh \beta J_1)^{M/2} e^{\beta J_1^* \sum_{m=1}^{M} \sigma_m^x}} \tag{104}$$

We have therefore, like before, mapped a 2D classical model into a 1D quantum model.

We are not gonna solve this model now. To do so will require some additional techniques that we will develop later on. So let me summarize our results, so we can come back to this problem later:

Partition function of the 2D Ising model using the transfer matrix method



$$H = -\sum_{n=1}^{N}\sum_{m=1}^{M}\left(J_1\,\sigma^z_{nm}\,\sigma^z_{n+1,m} + J_2\,\sigma^z_{nm}\,\sigma^z_{n,m+1}\right)$$

$$Z = \mathrm{tr}\,V^N \to \lambda_{max}^N$$

$$V = V_2^{1/2}\,V_1\,V_2^{1/2}$$

$$V_2 = \exp\left\{\beta J_2 \sum_{m=1}^{M} \sigma^z_m\,\sigma^z_{m+1}\right\}$$

$$V_1 = (2\sinh 2\beta J_1)^{M/2}\, e^{\beta J_1^* \sum_{m=1}^{M} \sigma^x_m} \qquad \left(\tanh \beta J_1^* = e^{-2\beta J_1}\right)$$

Remaining task: find λ_{max}, the largest eigenvalue of V.

Onsager showed (and so will we!) that this model does present a phase transition. Assuming $J_1 = J_2 = J$, the equilibrium magnetization is found to be given by the insanely simple formula

$$\boxed{m = \begin{cases} \left(1 - (\sinh 2\beta J)^{-4}\right)^{1/8} & T < T_c \\ 0 & T > T_c \end{cases}} \tag{105}$$

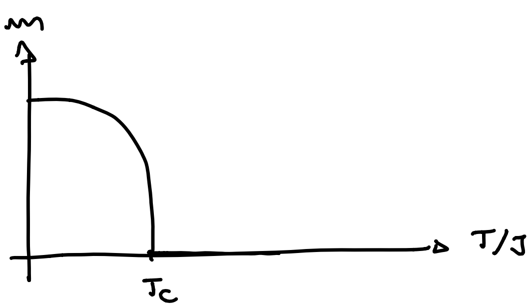
where T_c is the critical temperature, defined as the point where

$$\sinh(J/T_c) = 1 \tag{106}$$

which gives

$$\frac{T_c}{J} = \frac{2}{\ln(1+\sqrt{2})} = 2.26919 \tag{107}$$

this formula looks like this



This is really a phase transition: $m \neq 0$ in one phase and $m \equiv 0$ in the other, it is not small, it's zero, zero zero, zero.