Statistical Mechanics - Lecture 8

Lipkin - Meshkov - Glick (LMG) model

In these notes we introduce two "fully connected" models. that is, models where everyone interacts with everyone. The Ising version has a classical phase transition. The transverse field version, called the LMG model, has a quantum phase transition.

Recommended reading: Salinas 13.3 and appendix A.

P. Ribeiro, J. Vidal and R. Mosseri, Phys Rev. E., 78, 021106 (2008)

Laplace's asymptotic method

Consider an integral of the form

$$I = \int_a^b dx\, e^{-N f(x)} \qquad (1)$$

where $f(x)$ is an arbitrary function which we assume has a minimum at some point $x_0 \in [a, b]$.

When $N \to \infty$ the largest contribution to (1) will come from the minimum at x_0 since all other points will be really really small. Expanding f around x_0:

$$f(x) \simeq f(x_0) + \underbrace{f'(x_0)}_{=0}(x - x_0) + \frac{\overbrace{f''(x_0)}^{>0}}{2!}(x - x_0)^2 + \dots$$

Eq (1) then becomes

$$I = \int_a^b dx \exp\left\{ -N f(x_0) - \frac{N f''}{2}(x - x_0)^2 \right\}$$

$$= e^{-N f(x_0)} \int_a^b dx\, e^{-\frac{N f''}{2}(x - x_0)^2} \qquad (2)$$

Since the integral vanishes quickly as we move out of x_0, we can also extend the limits of integration to $\pm\infty$. The resulting integral is then a Gaussian

$$\int_{-\infty}^{\infty} dx \, e^{-a(x-x_0)^2} = \sqrt{\frac{\pi}{a}} \qquad (3)$$

Eq (3) then becomes

$$\int_a^b dx \, e^{-Nf(x)} \approx e^{-Nf(x_0)} \sqrt{\frac{2\pi}{N f''(x_0)}} \qquad (4)$$

This is Laplace's formula. The part of the expression you should remember is the exponential

$$\int_a^b dx \, e^{-Nf(x)} \simeq e^{-Nf(x_0)} \qquad (5)$$

The square root is just a small correction.

Example: Stirling's formula for $N!$

In terms of the Gamma function

$$N! = \Gamma(N+1) = \int_0^\infty dx\; x^N e^{-x} = \int_0^\infty dx\; e^{-x+N\ln x}$$

$$= \int_0^\infty dx\; e^{-N(x/N - \ln x)}$$

The function $f(x) = x/N - \ln x$ has a minimum at

$$f'(x) = \frac{1}{N} - \frac{1}{x} = 0 \quad \rightsquigarrow \quad x_0 = N$$

Moreover

$$f(x_0) = 1 - \ln N$$

$$f''(x_0) = \frac{1}{N^2}$$

Thus, applying (4) we get

$$N! \simeq e^{-N(1-\ln N)} \sqrt{\frac{2\pi}{N \times 1/N^2}}$$

$$= e^{-N+N\ln N} \sqrt{2\pi N}$$

or, better yet

$$\ln N! \simeq N\ln N - N + \frac{1}{2}\ln(2\pi N) \tag{6}$$

This is Stirling's formula. Again, the dominant part is $N\ln N - N$.

Fully connected classical Ising model

We start by considering a variation of the Ising model which is mean-field like, without any approximations. The model is composed of N spins where everyone interacts with everyone

$$H = -\frac{J}{2N} \sum_{i,j} \sigma_z^i \sigma_z^j - h \sum_{i=1}^{N} \sigma_z^i \qquad (7)$$

Here the lattice does not matter, since all interactions are taken to have the same strenght. Moreover, since the number of bonds is now N^2, I divided the 1st term by N to make H extensive.

The reason why (7) can be treated analytically is because, if we define the collective spin operator

$$S_z = \frac{1}{2} \sum_{i=1}^{N} \sigma_z^i \qquad (8)$$

then we may write

$$\sum_{i,j} \sigma_z^i \sigma_z^j = \left(\sum_i \sigma_z^i\right)\left(\sum_j \sigma_z^j\right) = 4 S_z^2 \qquad (9)$$

Eq (7) then becomes

$$H = -\frac{2J}{N} S_z^2 - 2h S_z \qquad (10)$$

Let us now look at the eigenvalues and eigenvectors of S_z. The operators σ_z^i are all simultaneously diagonalized by

$$|\sigma\rangle = |\sigma_1, \ldots, \sigma_N\rangle \qquad \sigma_i = \pm 1 \tag{11}$$

which has 2^N states. But S_z, on the other hand, is highly degenerate. The eigenvalues of S_z are

$$M(\sigma) = \frac{1}{2}\sum_{i=1}^{N}\sigma_i \tag{12}$$

these can take on the values

$$M = S, S-1, S-2, \ldots, -S+1, -S \tag{13}$$

where

$$S = \frac{N}{2} \tag{14}$$

Each eigenvalue has degeneracy

$$\deg(M) = \binom{N}{M+N/2} = \frac{N!}{(\frac{N}{2}+M)!\,(\frac{N}{2}-M)!} = \frac{(2S)!}{(S+M)!\,(S-M)!} \tag{15}$$

which is the number of ways we can arrange $\frac{N}{2}+m$ spins up and $\frac{N}{2}-m$ down, so that the net spin is m. The eigenvalues of H may thus be written as

$$E = -\frac{2J}{N}M(\sigma)^2 - 2hM(\sigma) \tag{16}$$

The partition function may be written as

$$Z = \sum_{\{\sigma\}} \exp\left\{ \frac{2\beta J}{N} M(\sigma)^2 + 2\beta h M(\sigma) \right\} \tag{17}$$

We now use a lovely trick. Consider the integral

$$\int_{-\infty}^{\infty} dx \; e^{-x^2 + 2ax} = \sqrt{\pi}\, e^{a^2} \tag{18}$$

The RHS is quadratic in a, whereas the LHS is linear. We can use this in (17), with $a = \frac{2\beta J M^2}{N}$

$$e^{2\beta J M^2/N} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \; e^{-x^2 + 2x\sqrt{\frac{2\beta J}{N}}\, M} \tag{19}$$

We thus converted M^2 into something linear in M, at the cost of introducing an additional variable x. The partition function thus becomes

$$Z = \frac{1}{\sqrt{\pi}} \sum_{\{\sigma\}} \int dx \; e^{-x^2} \exp\left\{ \left(2\beta h + 2x\sqrt{\frac{2\beta J}{N}}\right) M(\sigma) \right\} \tag{20}$$

The advantage of this is that we can now compute first the sum over $\{\sigma\}$; and this sum we know how to do because it is linear: let $K = 2\beta h + 2x\sqrt{2\beta J/N}$. then Z can be written as

$$Z = \frac{1}{\sqrt{\pi}} \int dx\, e^{-x^2} \sum_{\{\sigma\}} e^{K M(\sigma)} = \frac{1}{\sqrt{\pi}} \int dx\, e^{-x^2} \sum_{\{\sigma\}} e^{\frac{K}{2} \sum_i \sigma_i}$$

The sum factors into N independent sums

$$Z = \frac{1}{\sqrt{\pi}} \int dx\, e^{-x^2} \left[2\cosh K/2\right]^N \tag{21}$$

or, back to N = 2S

$$Z = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, \exp\left\{-x^2 + N \ln\left[2\cosh\left(\beta h + x\sqrt{\frac{2\beta J}{N}}\right)\right]\right\} \tag{22}$$

This is an integral representation for Z. We got rid of the sums over σ and replaced it with a simple integral in x.

To make things a bit clearer, let us change variables to

$$\left(\frac{2\beta J}{N}\right)^{1/2} x = \beta J y$$

or

$$x = \sqrt{\frac{N\beta J}{2}}\, y \tag{23}$$

then

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \int_{-\infty}^{\infty} dy\, \exp\left\{-\frac{N\beta J y^2}{2} + N \ln\left[2\cosh\left(\beta h + \beta J y\right)\right]\right\} \tag{24}$$

Finally we define

$$g(y) = \frac{Jy^2}{2} - T\ln\left[2\cosh\left(\frac{h+Jy}{T}\right)\right] \qquad (25)$$

so that Z becomes

$$Z = \left(\frac{N\beta J}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} dy \quad e^{-N\beta g(y)} \qquad (26)$$

The significance of this expression can become clear if we use Laplace's formula (5). According to it, in the TL $N\to\infty$, the dominant contribution to (26) will come from the value of m which minimizes the function $f(y)$.

But if we look at (25) for a second we realize that $f(y)$ is exactly the free energy of the mean field Ising model we discussed in lecture 7. But what is the meaning of $g(y)$?

The free energy per particle in the thermodynamic limit is

$$f(T,h) = \lim_{N\to\infty} -\frac{T}{N}\ln Z \qquad (27)$$

and is a function of T and h, not y (since we integrate over y).

But using Laplaces method, Eq (4), we have

$$Z \simeq \left(\frac{N\beta J}{2\pi}\right)^{1/2} e^{-N\beta g(y^*)} \sqrt{\frac{2\pi}{N\beta g''(y^*)}}$$

where y^* is the minimum of $g(y)$. Then

$$-\frac{T}{N} \ln Z = g(y^*) - \frac{T}{N} \ln\left[\left(\frac{N\beta J}{2\pi}\right)^{1/2}\left(\frac{2\pi}{N\beta g''(y^*)}\right)\right]$$

In the limit $N \to \infty$ the last term vanishes and Eq (27) reduces to

$$f(T,h) = g(y^*) \tag{28}$$

where $y^* = y^*(T,h)$ is a function of T and h. The actual thermodynamic free energy is thus the Landau free energy g evaluated at y^*.

What is the meaning of y in Eqs (25) and (26)? To answer that, let us compute the magnetization

$$m = 2\frac{\langle S_z \rangle}{N} = -2\frac{\partial f}{\partial(2h)} \tag{29}$$

where the factor of 2 is put simply because of the 2 in Eq (10).

We have

$$m = \frac{T}{Z}\frac{\partial}{\partial h} Z = \frac{T}{Z}\left(\frac{N\beta J}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} dy \, \frac{\partial}{\partial h} \, e^{-N\beta g(y)}$$

$$= -\frac{N}{Z}\left(\frac{N\beta J}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty} dy \, \frac{\partial g}{\partial h} \, e^{-N\beta g}$$

Using (25) we get

$$\frac{\partial g}{\partial h} = -\tanh(\beta h + \beta J y)$$

Thus the average magnetization can be written as

$$m = N \frac{\int dy \, \tanh(\beta h + \beta J y) \, e^{-N\beta g(y)}}{\int dy \, e^{-N\beta g(y)}} \tag{30}$$

this introduces a neat interpretation. Clearly this looks like an average over a distribution of y

$$P(y) = \frac{e^{-N\beta g(y)}}{\tilde{Z}} \qquad (31)$$

where $\tilde{Z} = \int dy \, e^{-N\beta g(y)}$ is a normalization constant.

Thus, instead of thinking about a distribution for all possible spin configurations, we can think about a distribution for the order parameter y itself. Of course, y is not exactly the magnetization, but its pretty close to it. After all, from (30) we can think of the magnetization as being $\tanh(\beta h + \beta J y)$.

The Lipkin-Meshkov-Glick model

The LMG model is the transverse-field analog of the fully connected Ising model (7)

$$H = -\frac{J}{2N} \sum_{i,j} \sigma_z^i \sigma_z^j - g \sum_{i=1}^{N} \sigma_x^i \tag{32}$$

what changes is the last term, which is now proportional to σ_x^i. I also call the transverse field as g instead of h.

As before, we can write this in terms of collective spin operators as

$$H = -\frac{J S_z^2}{S} - 2g S_x \tag{33}$$

This Hamiltonian is now no longer diagonal because of the S_x term. Diagonalizing it for arbitrary spin size is actually a hard problem. But we are only interested in the thermodynamic limit $N \to \infty$, or $S \to \infty$. In this case we can divide the diagonalization in two steps. First we look at the classical ground state. Then we analyze the quantum fluctuations around this ground state.

Classical Ground state

Let us first assume that the S_α are actually classical spins. That is, they can be written as

$$S_x = S\sin\theta\cos\phi \qquad S_z = S\cos\theta \tag{34}$$

This will give us some intuition on how to treat the quantum case. The Hamiltonian (33) then becomes an energy

$$E(\theta,\phi) = -S\left(J\cos^2\theta + 2g\sin\theta\cos\phi\right) \tag{35}$$

We then look for the angles which minimize E:

$$\frac{\partial E}{\partial\phi} = 2Sg\sin\theta\sin\phi = 0$$

$$\frac{\partial E}{\partial\theta} = -S\left[-2J\sin\theta\cos\theta + 2g\cos\theta\cos\phi\right] = 0$$

which implies

$$\sin\theta\sin\phi = 0 \tag{36a}$$

$$\sin\theta\cos\theta = \frac{g}{J}\cos\theta\cos\phi \tag{36b}$$

To get the minima, we need to look at the 2^{nd} derivatives as well. For simplicity, I will just give you the answer. If $g > J$ the minimum occurs at

$$\phi = 0 \quad \text{and} \quad \theta = \pi/2 \tag{37}$$

If $g < J$, then a new minimum appears at

$$\phi = 0 \quad \text{and} \quad \sin\theta = g/J \tag{38}$$

This defines the critical field

$$g_c = J$$

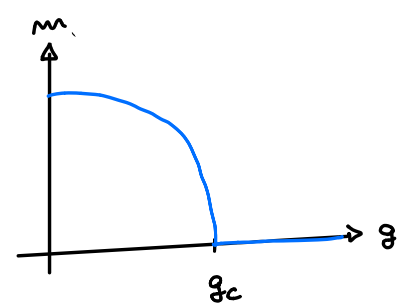
at which the nature of the solution changes. The magnetization in the z direction then reads

$$m = \cos\theta = \sqrt{1 - g^2/J^2}$$

for $g < J$ and $m = 0$ otherwise. We can also write this in a more suggestive way as

$$m = \cos\theta = \frac{1}{J}\sqrt{g_c^2 - g^2} \qquad (39)$$

which is of the form of the mean-field solution we saw before.



This result points to the existence of a broken symmetry phase when $g < J$. But this is all at the classical level. Now we have to see how to carry this over to the quantum regime.

Landau free energy

Going back to the energy (35), since the minima are always on $\phi=0$, let focus only on the θ part:

$$E(\theta) = -S\left(J\cos^2\theta + 2g\sin\theta\right) \qquad (40)$$

This actually represents the Landau free energy of the system at zero temperature. To see this let us express it in terms of the order parameter $m=\cos\theta$. Assuming $g>0, m>0$ we get

$$E(m) = -S\left(J m^2 + 2g\sqrt{1-m^2}\right)$$

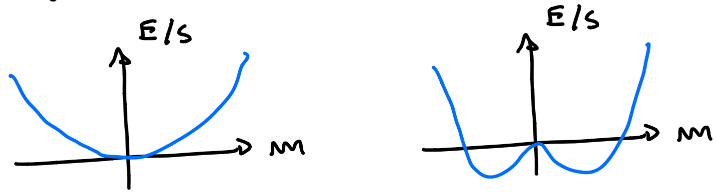
Close to $g=g_c=J$, we will have $m \ll 1$, so we can expand $\sqrt{1-m^2} \simeq 1 - \frac{m^2}{2} - \frac{m^4}{8}$, leading to

$$E(m) \simeq -S\left[J^2 m^2 + 2g - g m^2 + \frac{g m^4}{4}\right]$$

Omitting the constant term, we find

$$\frac{E(m)}{S} = (g-J)m^2 + \frac{g m^4}{4} \qquad (41)$$

which is again the typical Landau ϕ^4 theory, with the first term changing sign at $g=g_c$



$g>g_c$ $\qquad\qquad$ $g<g_c$

Holstein-Primakoff transformation

The key to understanding quantum fluctuations is a neat transformation between spin operators and harmonic oscillator creation and annihilation operators.

QHO operators satisfy

$$[a, a^\dagger] = 1$$

whereas spin operators satisfy

$$[S_x, S_y] = i S_z \qquad (42)$$

(plus cyclic permutations). We can also introduce the ladder operators

$$S_\pm = S_x \pm i S_y \qquad (43)$$

They satisfy

$$[S_z, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = 2 S_z \qquad (44)$$

Consider now the following mapping

$$S_z = S - a^\dagger a \qquad (45a)$$

$$S_+ = \sqrt{2S - a^\dagger a}\; a \qquad (45b)$$

$$S_- = S_+^\dagger = a^\dagger \sqrt{2S - a^\dagger a} \qquad (45c)$$

This is the Holstein-Primakoff transformation. The idea is that we can convert QHO operators into spin operators in this weird but beautiful way.

The reason why this makes sense is because it preserves the algebra. For instance

$$[S_+, S_-] = S_+S_- - S_-S_+$$

$$= \sqrt{2S-a^\dagger a}\; \underbrace{aa^\dagger}_{a^\dagger a+1} \sqrt{2S-a^\dagger a} - a^\dagger \sqrt{2S-a^\dagger a}\sqrt{2S-a^\dagger a}\, a$$

↳ commutes with $\sqrt{\;}$

$$= aa^\dagger(2S-a^\dagger a) - a^\dagger(2S-a^\dagger a)a$$

$$= \underbrace{(aa^\dagger - a^\dagger a)}_{1} 2S - \underbrace{aa^\dagger}_{a^\dagger a+1} a^\dagger a + a^\dagger a^\dagger aa$$

$$= 2S - a^\dagger a - a^\dagger \underbrace{aa^\dagger}_{a^\dagger a+1} a + a^\dagger a^\dagger aa$$

$$= 2S - 2a^\dagger a \cancel{- a^\dagger a^\dagger aa} \cancel{+ a^\dagger a^\dagger aa}$$

$$= 2(S - a^\dagger a)$$

$$= 2S_z.$$

which is the 2^{nd} Eq in (44). You may check that the 1^{st} Eq is also satisfied.

Another way to think about the HP transformation is in terms of the ladder of states. Spins and QHOs both have equally spaced levels. The only difference is that the spin ladder is finite on both sides, whereas the QHO ladder is infinite in one direction.

Introduce the QHO states $|m\rangle$, defined from

$$a^\dagger a\,|m\rangle = m\,|m\rangle \tag{46}$$

Then from (45a) we get

$$S_z\,|m\rangle = (S-m)\,|m\rangle \tag{47}$$

Thus, $|m\rangle$ is an eipenstate of S_z with magnetization

$$M(m) = S - m \tag{48}$$

The state $|0\rangle$ corresponds to the spin all the way up, $M=S$. Indeed, if we were to try to go up even more by applying S_+, we get zero:

$$S_+\,|0\rangle = \sqrt{2S - a^\dagger a}\;a\,|0\rangle = 0 \tag{49}$$

The other states $|1\rangle, |2\rangle, |3\rangle, \ldots$, corresponds to eower values of M. The lowest value should, of course, be $M=-S$, which corresponds to $m = 2S$. In this case, if we try to eower even more we get

$$S_-\,|2S\rangle = a^\dagger\sqrt{2S - a^\dagger a}\;|2S\rangle = 0 \tag{50}$$

This is the reason for the weird square root in the HP map.

Fluctuations over the ground state

We can use the HP transformation to describe the fluctuations over the classical ground state. But to do that, we must first be sure that the vacuum $|0\rangle$ is the actual GS. As we saw in (37) and (38), the classical GS corresponds to non-zero values of θ. Thus, to set $|0\rangle$ to be the GS, we must first rotate the Hamiltonian (33) by θ. Let

$$H' = e^{i\theta S_y} H e^{-i\theta S_y} \tag{51}$$

We use the formulas

$$\begin{aligned} e^{i\theta S_y} S_z e^{-i\theta S_y} &= S_z \cos\theta - S_x \sin\theta \\ e^{i\theta S_y} S_x e^{-i\theta S_x} &= S_z \sin\theta + S_x \cos\theta \end{aligned} \tag{52}$$

We then get

$$H' = -\frac{J}{S} \left(S_z \cos\theta - S_x \sin\theta \right)^2 - 2g \left(S_z \sin\theta + S_x \cos\theta \right) \tag{53}$$

Let us first focus on $g > J$. Then we have to rotate by $\theta = \pi/2$, leading to

$$H' = -\frac{J}{S} S_x^2 - 2g S_z \tag{54}$$

This is exactly like the original Hamiltonian (33), but with x and z exchanged.

we now apply the HP transformation on (54). we have

$$S_x = \frac{S_+ + S_-}{2} \tag{55}$$

However, we can write (45b) as

$$S_+ = \sqrt{2S - a^\dagger a}\; a = \sqrt{2S}\sqrt{1 - \frac{a^\dagger a}{2S}}\; a \tag{56}$$

If we are only interested in large S, we can approximate the square root to unity, leading to

$$S_+ \simeq \sqrt{2S}\, a \tag{57}$$

then

$$S_x \simeq \frac{\sqrt{2S}}{2}(a + a^\dagger) \tag{58}$$

and (54) becomes

$$H' = -\frac{J}{S}\frac{2S}{4}(a+a^\dagger)^2 - 2g(S - a^\dagger a)$$

or

$$H' = -2gS + 2g a^\dagger a - \frac{J}{2}(a + a^\dagger)^2 \tag{59}$$

Let us introduce position and momentum operators

$$q = \frac{a + a^\dagger}{\sqrt{2}} \qquad p = \frac{i}{\sqrt{2}}(a^\dagger - a) \tag{60}$$

Then

$$a^\dagger a = \frac{p^2 + q^2}{2} \tag{61}$$

and so (59) becomes

$$H' = -2gS + g(q^2+p^2) - J q^2$$

Neglecting the constant $-2gS$, we can finally write this as

$$H' = gp^2 + (g-J)q^2 \tag{62}$$

this is just a harmonic oscillator

$$H' = \frac{p^2}{2\mu} + \frac{1}{2}\mu \omega^2 q^2 \tag{63}$$

with mass $\mu = 1/2g$ and frequency

$$\frac{1}{2}\mu\omega^2 = g - J$$

or

$$\omega^2 = 4g(g-J) \tag{64}$$

the excitation energies are therefore of the QHO type

$$E_m = \hbar\omega(m+1/2) = \hbar\sqrt{4g(g-J)}\,(m+1/2) \tag{65}$$

the energy gap between the ground-state and the first excited state is thus

$$\Delta = \hbar\omega = 2\hbar\sqrt{g(g-J)} \tag{66}$$

We see that as we approach the critical point $g = J$, the gap closes. This is a signature of quantum phase transitions. At the critical point the Hamiltonian (62) becomes that of a free particle, $p^2/2m$. There is no cost to create an excitation: you kick it and it starts moving.

For $g < J$ the frequency in (62) would become imaginary. This is because the rotation angle is no longer $\theta = \pi/2$. Instead, we have to use $\sin\theta = g/J$.

To treat $g < J$ we have to go back to H' in (53) and rotate by the right angle. We then get

$$H' = -\frac{J}{S}\left(S_z\cos\theta - S_x\sin\theta\right)^2 - 2g\left(S_z\sin\theta + S_x\cos\theta\right)$$

$$= -\frac{J}{S}\left(S_z\sqrt{1-g^2/J^2} - S_x\frac{g}{J}\right)^2 - 2g\left(S_z\frac{g}{J} + S_x\sqrt{1-g^2/J^2}\right)$$

$$= -\frac{J}{S}\left[(S-a^\dagger a)\sqrt{1-g^2/J^2} - \frac{\sqrt{2S}}{2}(a+a^\dagger)\frac{g}{J}\right]^2 + \tag{67}$$

$$- \frac{2g^2}{J}(S-a^\dagger a) - 2g\frac{\sqrt{2S}}{2}(a+a^\dagger)\sqrt{1-g^2/J^2}$$

The quadratic term, when expanded, becomes

$$[\dots]^2 = (s - a^\dagger a)^2 (1 - g^2/J^2) + \frac{2s}{4} (a + a^\dagger)^2 \frac{g^2}{J^2}$$

$$+ \sqrt{1 - g^2/J^2}\, \frac{g}{J} \frac{\sqrt{2s}}{2} \left[(s - a^\dagger a)(a + a^\dagger) + (a + a^\dagger)(s - a^\dagger a) \right]$$

$$= (1 - g^2/J^2) \left[s^2 - 2 s a^\dagger a + (a^\dagger a)^2 \right] + \frac{s}{2} \frac{g^2}{J^2} (a + a^\dagger)^2$$

$$+ \frac{\sqrt{2s}}{2} \frac{g}{J} \sqrt{1 - g^2/J^2} \left[2s(a + a^\dagger) - \{a^\dagger a, a + a^\dagger\} \right]$$

Multiplying this by $-J/s$, Eq (67) becomes

$$H' = -\frac{J}{s}(1 - g^2/J^2) s^2 - \frac{2g^2}{J} s^2$$

$$+ \left[-\frac{J}{s} \frac{\sqrt{2s}}{2} 2s \frac{g}{J} \sqrt{1 - g^2/J^2} - 2g \frac{\sqrt{2s}}{2} \sqrt{1 - g^2/J^2} \right] (a + a^\dagger)$$

$$+ \left[-\frac{J}{s} (1 - g^2/J^2)(-2s) + \frac{2g^2}{J} \right] a^\dagger a \tag{68}$$

$$+ \left[-\frac{J}{s} \frac{s}{2} \frac{g^2}{J^2} \right] (a + a^\dagger)^2$$

$$-\frac{J}{s} (1 - g^2/J^2)(a^\dagger a)^2 + \frac{J}{s} \frac{\sqrt{2s}}{2} \frac{g}{J} \sqrt{1 - g^2/J^2} \{a^\dagger a, a + a^\dagger\}$$

I know it is ugly, but this result is very important. Have a look. The 1st line is just a constant. The 2nd line is linear in $(a + a^\dagger)$. However, if you look at the coefficients, this term vanishes! Finally, the last term in grey may seem quite complicated. But it's proportional to $1/s$ and thus becomes negligible when $s \to \infty$!

As a result, omitting the constant term, we are only left with the term in bracket

$$H' = \left[-\frac{J}{S}\left(1 - g^2/J^2\right)(-2S) + \frac{2g^2}{J} \right] a^\dagger a + \left[-\frac{J}{S}\,\frac{S}{2}\,\frac{g^2}{J^2} \right](a + a^\dagger)^2$$

or

$$H' = 2J a^\dagger a - \frac{g^2}{2J}(a + a^\dagger)^2 \tag{69}$$

In terms of quadrature operators we get

$$H' = J(p^2 + q^2) - \frac{g^2}{J} q^2$$

or

$$H' = J p^2 + (J - g^2/J) q^2 \tag{70}$$

Thus, the mass is

$$\mu = \frac{1}{2J}$$

and the frequency is

$$\omega^2 = 4J\,(J - g^2/J)$$

or

$$\omega = 2\sqrt{J^2 - g^2} \tag{71}$$

The energy gap in the ordered phase is thus

$$\Delta = \hbar\omega = 2\hbar\sqrt{J^2 - g^2} \tag{72}$$

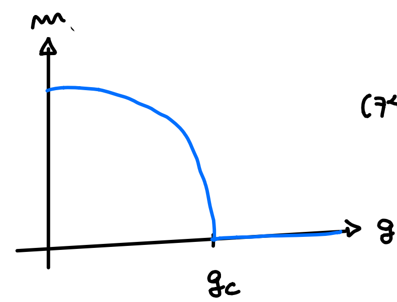
What did we learn?

Let us summarize what we learned. In the limit $S \to \infty$ we obtain a classical spin described by the energy (40):

$$E(\theta) = -S\left(J\cos^2\theta + 2g\sin\theta\right) \qquad (73)$$

This energy is extensive and represents the Landau free energy of the system. Minimizing it leads to the order parameter

$$m = \cos\theta = \begin{cases} \frac{1}{J}\sqrt{g_c - g^2} & g < g_c = J \\ 0 & g > g_c \end{cases} \qquad (74)$$



This represents the behavior of the system in the thermodynamic limit. On top of that, however, we have the quantum fluctuations. To include this, we write the total Hamiltonian as

$$H' = E(\theta) + \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 q^2 \qquad (75)$$

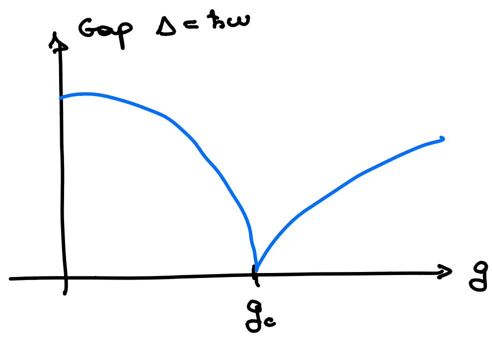
where the mass and frequency are

$$\mu = \begin{cases} 1/2J & g < g_c \\ 1/2g & g > g_c \end{cases} \qquad \omega = \begin{cases} 2\sqrt{J^2 - g^2} & g < g_c \\ 2\sqrt{g(g-J)} & g > g_c \end{cases} \qquad (76)$$

The first term in (75) is extensive ($E \propto S$) whereas the 2^{nd} is intensive (independent of S). Generally, therefore, the fluctuations are a kind of correction to the Landau energy.

The Hamiltonian (75) shows that the fluctuations are like a harmonic oscillator, in the sense that the excitations have equally spaced energies.

The energy gap between the ground state and the first excited state is essentially $\hbar\omega$, so it's given by (76). It looks as follows



The gap closes at the critical point. This is a signature of a quantum phase transition. At the vicinity of the critical point it becomes very easy to create an excitation. As a consequence, even though the fluctuations are not extensive, they become quite significant at the vicinity of g_c.

This procedure is quite useful: we first look at the Landau energy for $N \to \infty$ and then we analyze the fluctuations on top of that, by looking at the energy gap.

Alternative derivation

Alternatively, we can also compute the partition function without the Gaussian integral trick. Instead of writing Z as a sum over all σ, we write it as a sum over M, taking into account the degeneracy (15):

$$Z = \sum_{M=-S}^{S} \frac{(2S)!}{(S+M)!\,(S-M)!} \exp\left\{ \frac{\beta J M^2}{S} + 2\beta h M \right\} \tag{32}$$

Using Stirling's formula (6) we can write

$$\ln\left[\frac{(2S)!}{(S+M)!\,(S-M)!} \right] \simeq 2S\ln 2S - (S+M)\ln(S+M) - (S-M)\ln(S-M) \tag{33}$$

Thus, (32) can be written as

$$Z = \sum_{M=-S}^{S} \exp\left\{ \frac{\beta J M^2}{S} + 2\beta h M + 2S\ln 2S - (S+M)\ln(S+M) - (S-M)\ln(S-M) \right\} \tag{34}$$

We now change variables to

$$m = \frac{2M}{N} = \frac{M}{S} \tag{35}$$

the sum (34) will then be from -1 to 1 in steps of

$$\Delta m = \frac{2}{N} = \frac{1}{S} \tag{36}$$

we then get

$$Z = \sum_{m=-1}^{1} \exp\left\{ \beta J S m^2 + 2\beta h S m - S(1+m)\ln(1+m) - S(1-m)\ln(1-m) \right\} \tag{37}$$

To convert the sum to an integral, we multiply by the "convenient 1"

$$1 = S\Delta m$$

we then get

$$Z = S \sum_{m} \Delta m \, \exp\{\cdots\}$$

which is in the form of a Riemann sum. When $S \to \infty$ ($N \to \infty$) we may then write

$$Z = S \int_{-1}^{1} dm \, \exp\left\{ \beta J S m^2 + 2\beta h S m - S(1+m)\ln(1+m) - S(1-m)\ln(1-m) \right\} \tag{38}$$

This is an alternative way of expressing Z as an integral.

To show that this is equivalent to (26) we change variables to y, defined as

$$m = \tanh(\beta h + \beta J y)$$

We thus get

$$dm = \operatorname{sech}^2(\beta h + \beta J y)\, \beta J\, dy$$

Moreover,

$$(1+m)\ln(1+m) + (1-m)\ln(1-m) = \ln(1-m^2) + m\ln\left(\frac{1+m}{1-m}\right)$$

$$= \ln \operatorname{sech}^2(\beta h + \beta J y) + 2m(\beta h + \beta J y)$$

Eq (38) thus becomes

$$Z = S\beta J \int_{-\infty}^{\infty} dy \ \operatorname{sech}^2(\beta h + \beta J y)\ \exp\Big\{ \beta J S m^2 + 2\beta h S m$$
$$- S \ln \operatorname{sech}^2(\beta h + \beta J y) - 2(\beta h + \beta J y) S m \Big\}$$

$$= S\beta J \int_{-\infty}^{\infty} dy\ \exp\Big\{ \beta J S (m^2 - 2my) + (1-S)\ln \operatorname{sech}^2(\beta h + \beta J y)\Big\}$$

Completing squares in the first term, we get

$$Z = S\beta J \int_{-\infty}^{\infty} dy \, \exp\Big\{ \beta J S \, (m - y)^2 - \beta J S y^2 + \\ + (1-S) \ln \operatorname{sech}^2 (\beta h + \beta J y) \Big\}$$