Statistical Mechanics: lecture 5

Spontaneous emission

Spontaneous emission is the process through which an atom emits a photon to the electromagnetic field. It is the first pretty picture you saw when you first learned about the Bohr model in quantum mechanics. And is the simplest (and most elegant) example of thermalization.

Recommended reading: Scully and Zubairy, "Quantum optics", chapters 1 (for field quantization) and 8 (for spontaneous emission).

In these notes I set $\hbar = 1$. This only means that $\hbar\omega \to \omega$. That is, frequency and energy have the same units.

Field quantization in a nutshell

The EM field can be decomposed into different modes. A mode is specified by the wavevector $\vec{k}$ and the polarization λ (which can take on two values; e.g. $\lambda = 1$ for x polarized and $\lambda = 2$ for y polarized). We characterize radiation by specifying such modes.

For each $\vec{k}$, the associated frequency is determined by the dispersion relation

$$\omega_{k\lambda} = c|\vec{k}| = c\sqrt{k_x^2 + k_y^2 + k_z^2} \tag{1}$$

where c is the speed of light.

The EM field is quantized. This means that each mode $\vec{k}$ may have a discrete number of photons in them, 0, 1, 2, ... We describe this mathematically using the same creation and annihilation operators of the harmonic oscillator. But we now have one operator for each $\vec{k}$. That is, we introduce

$$a_{\vec{k}\lambda} = \text{operator that annihilates a photon with wave vector } \vec{k} \text{ and polarization } \lambda \tag{2}$$

We then have an infinite number of operators, one for each $\vec{k}$. These operators satisfy

$$[a_{\vec{k}\lambda}, a^\dagger_{\vec{k}'\lambda'}] = \delta_{\vec{k},\vec{k}'}\,\delta_{\lambda\lambda'} \tag{3}$$

meaning, for $\vec{k}' = \vec{k}$, it's like the QHO, $[a, a^\dagger] = 1$. But for $\vec{k}' \neq \vec{k}$, they commute. In the language of tensor products, $a_{\vec{k}}$ and $a_{\vec{k}'}$ live on different Hilbert spaces.

The operator $a^\dagger_{\vec{k}} a_{\vec{k}}$ counts the number of photons with wavevector $\vec{k}$. Its eigenstuff are, as with the QHO,

$$a^\dagger_{\vec{k}\lambda} a_{\vec{k}\lambda} |n_{\vec{k}\lambda}\rangle = n_{\vec{k}\lambda} |n_{\vec{k}\lambda}\rangle \quad , \; n_{\vec{k}} = 0, 1, 2, 3, \ldots \qquad (4)$$

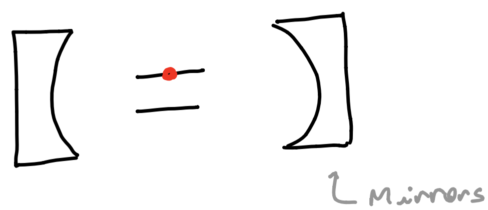
Since we have multiple values of $\vec{k}$, a complete basis for the Hilbert space is then given by

$$|n_{\vec{k}_1\lambda_1}, n_{\vec{k}_2\lambda_2}, n_{\vec{k}_3\lambda_3}, \ldots\rangle \qquad (5)$$

where $\vec{k}_i$ is just a pictorial way of referring to all the infinite number of allowed values of $\vec{k}$. The states (5) are called Fock states.

The Jaynes - Cummings model

The goal of this lecture is to discuss the spontaneous emission of radiation by an atom. Usually, the atom emits at multiple values of $\vec{k}$. But you can force it to emit at a single mode by confining the field in a tiny cavity:



If the cavity is sufficiently small, only a single mode will matter. In this case the interaction between the atom and this mode will be described by the Jaynes- Cummings model:

$$\boxed{H = \omega a^\dagger a + \Omega \sigma_+ \sigma_- + g(a^\dagger \sigma_- + a\sigma_+)} \tag{6}$$

Here a is the annihilation operator for the mode in question, which we assume has some frequency ω. The 2^{nd} term is the atomic part, which we assume has gap Ω. We introduce a set of basis states $|g\rangle, |e\rangle$ for the atom, for the ground and excited state respectively. Then we can write $\sigma_- = |g\rangle\langle e|$, which lowers the state from $|e\rangle \to |g\rangle$. We also have, thus, that $\sigma_+ = \sigma_-^\dagger = |e\rangle\langle g|$ is the raising operator.

The last term in (6) is the atom-field interaction. It describes the exchange of excitations: $a^\dagger \sigma_-$ lowers the state of the atom and creates a photon. This is emission. Similarly, $a\sigma_+$ represents absorption.

The $\sigma_\pm$ operators satisfy $\sigma_-^2 = \sigma_+^2 = 0$. Moreover

$$\begin{aligned} \sigma_+ \sigma_- &= |e\rangle\langle g | g\rangle\langle e| = |e\rangle\langle e| \\ \sigma_- \sigma_+ &= |g\rangle\langle e | e\rangle\langle g| = |g\rangle\langle g| \end{aligned} \tag{7}$$

Using the completeness relation $|e\rangle\langle e| + |g\rangle\langle g| = 1$, we see that

$$\{\sigma_+, \sigma_-\} = 1 \tag{8}$$

where $\{A, B\} = AB + BA$ is the anti-commutator. This is to be contrasted with the property of the a operators

$$[a, a^\dagger] = 1 \tag{9}$$

The algebra determines the property of the operators. Eq (8) gives the infinite dimensional Fock state, whereas (9) gives a 2-dimensional space.

Notwithstanding, I will leave for you to check that

$$\begin{aligned} [a^\dagger a, a] &= -a & [a^\dagger a, a^\dagger] &= a \\ [\sigma_+ \sigma_-, \sigma_-] &= -\sigma_- & [\sigma_+ \sigma_-, \sigma_+] &= \sigma_+ \end{aligned} \tag{10}$$

Thus, $\sigma_\pm$ act as lowering and raising operators for $\sigma_+ \sigma_-$ in the same way as $a, a^\dagger$ act for $a^\dagger a$.

The interaction term in (6) satisfies the very special symmetry

$$\boxed{[a^\dagger a + \sigma_+\sigma_- \,,\, H] = 0} \qquad (11)$$

which means that the total number of quanta is conserved. The logic is that $a^\dagger\sigma_-$ and $a\sigma_+$ take a quanta from one system and put it in the other, so the total number of quanta is conserved. We can verify that (11) indeed works

$$[a^\dagger a, a^\dagger\sigma_- + a\sigma_+] = [a^\dagger a, a^\dagger]\sigma_- + [a^\dagger a, a]\sigma_+$$
$$= a^\dagger\sigma_- - a\sigma_+$$

$$[\sigma_+\sigma_-, a^\dagger\sigma_- + a\sigma_+] = a^\dagger[\sigma_+\sigma_-, \sigma_-] + a[\sigma_+\sigma_-, \sigma_+]$$
$$= -a^\dagger\sigma_- + a\sigma_+$$

The two therefore cancel, leading to Eq (11)

—//—

A convenient set of basis vectors for the Hamiltonian (6) is

$$\{ |n, g\rangle, |n, e\rangle \} \qquad n = 0, 1, 2, \dots \qquad (12)$$

where n are the eigenstates of $a^\dagger a$ and $|e/g\rangle$ of $\sigma_+\sigma_-$:

$$a^\dagger a |n\rangle = n|n\rangle$$

$$\left.\begin{aligned} &\sigma_+\sigma_- |g\rangle = 0 \\ &\sigma_+\sigma_- |e\rangle = |e\rangle \end{aligned}\right\} \text{ see Eq (7)} \qquad (13)$$

The symmetry (11) affords us a massive simplification. Since the number of quanta is conserved, there can be no terms in H connecting sectors with different quanta. Order the states (12) lexicographically, as

$$\underbrace{|0,g\rangle}_{0 \text{ quanta}}, \underbrace{|0,e\rangle, |1,g\rangle}_{1 \text{ quanta}}, \underbrace{|1,e\rangle, |2,g\rangle}_{2 \text{ quanta}}, |2,e\rangle, \ldots \tag{14}$$

the states are naturally divided in sectors (subspaces). Using (6) we find

$$H|n-1,e\rangle = [\omega(n-1)+\Omega]|n-1,e\rangle + g\sqrt{n}\,|n,g\rangle \tag{15}$$

$$H|n,g\rangle = \omega n\,|n,g\rangle + g\sqrt{n}\,|n-1,e\rangle$$

Thus, we see that H only connects states in the same sector.

This means that in the basis $|n, g/e\rangle$ the Hamiltonian is already block diagonal:

$$H = \begin{pmatrix} 0 & & & & \\ & H_1 & & & \\ & & H_2 & & \\ & & & H_3 & \\ & & & & \ddots \end{pmatrix} \tag{16}$$

The 1^{st} "0" is the state $|0,g\rangle$ and the other H_n are 2×2 matrices.

From (15) we find

$$
H_m = \begin{array}{c} \\ \\ \end{array}
\begin{array}{cc}
|m-1,e\rangle & |m,g\rangle \\
\left(\begin{array}{cc} \omega(m-1)+\Omega & g\sqrt{m} \\ g\sqrt{m} & \omega m \end{array}\right)
\end{array}
\begin{array}{c} \langle m-1,e| \\ \langle m,g| \end{array}
\tag{17}
$$

This is the Hamiltonian in the subspace with m excitations.

In order to diagonalize the full H, we thus only need to worry about these 2x2 blocks. The H_m can be diagonalized as

$$
H_m |\phi_m^\pm\rangle = \lambda_m^\pm |\phi_m^\pm\rangle \tag{18}
$$

where

$$
\lambda_m^\pm = \frac{\Omega}{2} + \omega(m - 1/2) \pm \frac{\Delta_m}{2} \tag{19}
$$

$$
\Delta_m = \sqrt{(\omega-\Omega)^2 + 4g^2 m}
$$

and eigenvectors

$$
|\phi_m^+\rangle = \begin{pmatrix} \cos\theta_m/2 \\ \operatorname{sen}\theta_m/2 \end{pmatrix} \qquad |\phi_m^-\rangle = \begin{pmatrix} -\operatorname{sen}\theta_m/2 \\ \cos\theta_m/2 \end{pmatrix} \tag{20}
$$

$$
\tan\theta_m = \frac{2g\sqrt{m}}{\Omega - \omega}
$$

Sanity check: If $g=0$, $\Delta = |\omega - \Omega|$ and the eigenvalues (42) tend to

$$
\frac{\Omega}{2} + \omega(m - 1/2) \pm \frac{(\omega-\Omega)}{2} = \begin{cases} \omega m & (\,|m,g\rangle \\ \Omega + \omega(m-1) & (\,|m-1,e\rangle \end{cases}
$$

The eigenvectors (20) are written in the basis $\{|n-1,e\rangle, |n,g\rangle\}$, as in (17).
Thus, what (20) actually means is

$$
\begin{aligned}
|\psi_n^+\rangle &= \cos\frac{\theta_n}{2}|n-1,e\rangle + \sin\frac{\theta_n}{2}|n,g\rangle \\
|\psi_n^-\rangle &= -\sin\frac{\theta_n}{2}|n-1,e\rangle + \cos\frac{\theta_n}{2}|n,g\rangle
\end{aligned}
\tag{21}
$$

Note also that these states are entangled (they cannot be written as a product.

Spontaneous emission:

Let us suppose that the field started in the vacuum $|0\rangle$ (no photons) and the atom started in the excited state $|e\rangle$. Then the entire dynamics will be restricted to the subspace $\{ |0,e\rangle, |1,g\rangle \}$. We can decompose the initial state as

$$|\psi_0\rangle = |0,e\rangle = |\phi_1^+\rangle\langle\phi_1^+|0,e\rangle + |\phi_1^-\rangle\langle\phi_1^-|0,e\rangle$$

since $\langle\phi_n^\pm|0,e\rangle = 0$ for all other values of n. Using (21) we get

$$|\psi_0\rangle = \cos\frac{\theta_1}{2}\,|\phi_1^+\rangle - \sin\frac{\theta_1}{2}\,|\phi_1^-\rangle \tag{22}$$

It is now easy to compute the state at any time t:

$$|\psi_t\rangle = \cos\frac{\theta_1}{2}\,e^{-i\lambda_1^+ t}\,|\phi_1^+\rangle - \sin\frac{\theta_1}{2}\,e^{-i\lambda_1^- t}\,|\phi_1^-\rangle \tag{23}$$

Let us now compute the reduced density matrix of the atom:

$$\begin{aligned}
\rho_A &= \mathrm{tr}_F\, |\psi_t\rangle\langle\psi_t| \\
&= \mathrm{tr}_F \Big\{ \cos^2\frac{\theta_1}{2}\,|\phi_1^+\rangle\langle\phi_1^+| + \sin^2\frac{\theta_1}{2}\,|\phi_1^-\rangle\langle\phi_1^-| \\
&\quad - \sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\Big[e^{-i(\lambda_1^+-\lambda_1^-)t}\,|\phi_1^+\rangle\langle\phi_1^-| + e^{i(\lambda_1^+-\lambda_1^-)t}\,|\phi_1^-\rangle\langle\phi_1^+| \Big] \Big\}
\end{aligned} \tag{24}$$

Using (21) we get, for instance

$$\mathrm{tr}_F\, |\psi_m^+\rangle\langle\psi_m^+| = \mathrm{tr}_F \left[\cos\frac{\theta_m}{2}|m-1,e\rangle + \sin\frac{\theta_m}{2}|m,g\rangle\right]\left[\cos\frac{\theta_m}{2}\langle m-1,e| + \sin\frac{\theta_m}{2}\langle m,g|\right]$$

$$= \cos^2\frac{\theta_m}{2}\, \mathrm{tr}_F\, |m-1,e\rangle\langle m-1,e| + \sin^2\frac{\theta_m}{2}\, \mathrm{tr}_F\, |m,g\rangle\langle m,g|$$

$$+ \sin\frac{\theta_m}{2}\cos\frac{\theta_m}{2}\, \mathrm{tr}_F\, |m-1,e\rangle\langle m,g| + \sin\frac{\theta_m}{2}\cos\frac{\theta_m}{2}\, \mathrm{tr}_F\, |m,g\rangle\langle m-1,e|$$

$$= \cos^2\frac{\theta_m}{2}\, |e\rangle\langle e| + \sin^2\frac{\theta_m}{2}\, |g\rangle\langle g|$$

Proceeding similarly we obtain

$$\mathrm{tr}_F\, |\psi_m^+\rangle\langle\psi_m^+| = \cos^2\frac{\theta_m}{2}\, |e\rangle\langle e| + \sin^2\frac{\theta_m}{2}\, |g\rangle\langle g|$$

$$\mathrm{tr}_F\, |\psi_m^-\rangle\langle\psi_m^-| = \sin^2\frac{\theta_m}{2}\, |e\rangle\langle e| + \cos^2\frac{\theta_m}{2}\, |g\rangle\langle g| \qquad (25)$$

$$\mathrm{tr}_F\, |\psi_m^+\rangle\langle\psi_m^-| = \mathrm{tr}_F\, |\psi_m^-\rangle\langle\psi_m^+|$$

$$= \sin\frac{\theta_m}{2}\cos\frac{\theta_m}{2}\, \left(|g\rangle\langle g| - |e\rangle\langle e|\right)$$

Thus, Eq (24) reduces to

$$\rho_A = \cos^2\frac{\theta_1}{2}\left[\cos^2\frac{\theta_1}{2}\,|e\rangle\langle e| + \sin^2\frac{\theta_1}{2}\,|g\rangle\langle g|\right] +$$

$$+ \sin^2\frac{\theta_1}{2}\left[\sin^2\frac{\theta_1}{2}\,|e\rangle\langle e| + \cos^2\frac{\theta_1}{2}\,|g\rangle\langle g|\right]$$

$$+\left(\sin\frac{\theta_1}{2}\cos\frac{\theta_1}{2}\right)^2\left(e^{-i\Delta_1 t} + e^{i\Delta_1 t}\right)\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$= \left(\cos^4\frac{\theta_1}{2} + \sin^4\frac{\theta_1}{2}\right)|e\rangle\langle e| + 2\left(\frac{\sin\theta_1}{2}\right)^2|g\rangle\langle g|$$

$$2\left(\frac{\sin\theta_1}{2}\right)^2\cos(\Delta_1 t)\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

$$\therefore \quad \boxed{\rho_A = \frac{1+\cos^2\theta_1}{2}|e\rangle\langle e| + \frac{\sin^2\theta_1}{2}|g\rangle\langle g| + \frac{\sin^2\theta_1}{2}\cos(\Delta_1 t)\left(|e\rangle\langle e| - |g\rangle\langle g|\right)} \qquad (26)$$

The prob. of finding the atom in the ground state is

$$P_g(t) = \langle g|\rho_A|g\rangle = \frac{\sin^2\theta_1}{2} - \frac{\sin^2\theta_1}{2}\cos(\Delta_1 t)$$

$$= \sin^2\theta_1\sin^2\Delta_1 t/2$$

Using (20) we get

$$\tan\theta_1 = \frac{2g}{\Omega-\omega} \quad \Rightarrow \quad \sin\theta = \frac{2g}{\sqrt{4g^2+(\Omega-\omega)^2}}$$

whence

$$\boxed{P_g = \sin^2\theta_1 \sin^2 \Delta_1 t = \frac{4g^2}{4g^2 + (\Omega-\omega)^2} \sin^2 \Delta_1 t/2} \qquad (27)$$

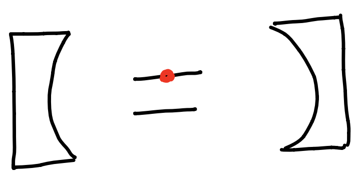
where $\Delta_1 = \sqrt{(\Omega-\omega) + 4g^2}$.

This starts at 0 when $t=0$ and then oscillates back and forth between 0 and $\sin^2\theta_1$. This maximum value is affected by whether or not the atom and field are resonant. If they are $(\omega=\Omega)$ then

$$\sin^2\theta_1 = 1$$

In this case there are, therefore, instants of time when $P_g = 1$, meaning the atom fully emitted.

The reason why this oscillates back and forth is because the emitted photon may be reabsorbed by the atom. This is a consequence of the fact that the field is confined to a cavity.



In order to observe "true" spontaneous emission, we have to consider the interaction with an infinite number of modes, as we will do below.

Thermal properties:

The Hamiltonian is written in Block form as in (40). Consequently, the same will also be true for $e^{-\beta H}$. The reason is that Block diagonal matrices never mix. The rules of matrix multiplication also hold for blocks so if

$$H = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{30}$$

then

$$H^2 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A^2 & 0 \\ 0 & B^2 \end{pmatrix} \tag{31}$$

which is true irrespective of the size of the blocks A and B. As a consequence, for any function $f(x)$

$$f(H) = \begin{pmatrix} f(A) & 0 \\ 0 & f(B) \end{pmatrix} \tag{32}$$

This means that, due to the structure of H in Eq (40),

$$e^{-\beta H} = \begin{pmatrix} 1 & & & \\ & e^{-\beta H_1} & & \\ & & e^{-\beta H_2} & \\ & & & \ddots \end{pmatrix} \tag{33}$$

The partition function is the sum of the diagonal entries, so

$$
\begin{aligned}
Z &= 1 + \sum_{n=1}^{\infty} \operatorname{tr} e^{-\beta H_n} \\
&= 1 + \sum_{n=1}^{\infty} \left(e^{-\beta \lambda_n^{+}} + e^{-\beta \lambda_n^{-}} \right)
\end{aligned}
\tag{34}
$$

The final formula, unfortunately, is ugly. But in principle now that we have Z all thermodynamic quantities are accessible.

Spontaneous emission onto free space

We now turn to the problem of an atom emitting onto free space. This means it will now interact with all modes of radiation and not just a single one, like in the Jaynes-Cummings model (6). The Hamiltonian is naturally generalized to

$$\boxed{H = \Omega \sigma_+ \sigma_- + \sum_k \omega_k a_k^\dagger a_k + \sum_k g_k (a_k^\dagger \sigma_- + a_k \sigma_+)} \tag{35}$$

For simplicity of notation I label the modes by k instead of $(\vec{k}, \lambda)$. Each mode k has frequency ω_k and interacts with the atom with strength g_k. Lucky for us, we won't need to specify exactly how g_k looks like (but if you are curious, $g_k \sim \omega_k^{1/2}$ so that high frequency modes tend to couple more strongly).

We assume, as before, that the system starts in the excited state $|e\rangle$ and the field starts in the vacuum $|0\rangle$ (no photons in any of the modes):

$$|\Psi_0\rangle = |0, e\rangle \tag{36}$$

The Hamiltonian (35) also conserves the number of excitations [c.f. Eq (11)]:

$$[H, \sigma_+ \sigma_- + \sum_k a_k^\dagger a_k] = 0 \tag{37}$$

This means that since the initial state (36) has 1 excitation, the evolution will be restricted to 1 excitation for all future times.

Let us define a set of states

$$|1_k\rangle = a_k |0\rangle \tag{38}$$

that represent a single photon in mode k. In terms of tensor structure, this looks like

$$|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle$$

i.e., tons of zeros and a single lonely "1" at mode k.

The state of atom + field at any time will be restricted to a subspace spanned by the states $|0,e\rangle$ and $|1_k, g\rangle$. Thus, we may decompose

$$|\Psi_t\rangle = c_0(t)|0,e\rangle + \sum_k c_k(t)|1_k, g\rangle \tag{39}$$

where $c_0(t)$ and $c_k(t)$ are coefficients. The initial condition (36) implies

$$c_0(0) = 1, \qquad c_k(0) = 0 \tag{40}$$

We now use (39) to solve Schrödinger's equation

$$\partial_t |\Psi_t\rangle = -iH|\Psi_t\rangle \tag{41}$$

Using (35) we get

$$\begin{aligned} H|0,e\rangle &= \left(\Omega \sigma_+ \sigma_- + \sum_k w_k a_k^\dagger a_k\right)|0,e\rangle + \sum_k g_k (a_k^\dagger \sigma_- + a_k \sigma_+)|0,e\rangle \\ &= \Omega|0,e\rangle + \sum_k g_k a_k^\dagger \sigma_- |0,e\rangle \\ &= \Omega|0,e\rangle + \sum_k g_k |1_k, g\rangle \end{aligned}$$

and

$$H|1_k, g\rangle = \left(\Omega\sigma_+\sigma_- + \sum_q \omega_q a_q^\dagger a_q\right)|1_k, g\rangle + \sum_q g_q\left(a_q^\dagger\sigma_- + a_q\sigma_+\right)|1_k, g\rangle$$

$$= \omega_k|1_k, g\rangle + \sum_q g_q \underbrace{a_q\sigma_-|1_k, g\rangle}$$

The only annihilation that will be successful is when $q = k$.

$$= \omega_k|1_k, g\rangle + g_k|0, e\rangle$$

thus

$$H|\Psi_t\rangle = c_0 H|0, e\rangle + \sum_k c_k H|1_k, g\rangle$$

$$= c_0\left[\Omega|0, e\rangle + \sum_k g_k|1_k, g\rangle\right] +$$

$$+ \sum_k c_k\left[\omega_k|1_k, g\rangle + g_k|0, e\rangle\right] \qquad (42)$$

$$= \left[\Omega c_0 + \sum_k g_k c_k\right]|0, e\rangle + \sum_k\left[\omega_k c_k + g_k c_0\right]|1_k, e\rangle$$

Plugging this into (41) and exploiting the fact that $\{|0,e\rangle, |1_k,g\rangle\}$ form an orthonormal basis, we can get an equation for each coefficient

$$\boxed{\begin{aligned} \dot{c}_0 &= -i\left(\Omega c_0 + \sum_k g_k c_k\right) \qquad (43) \\ \dot{c}_k &= -i\left(\omega_k c_k + g_k c_0\right) \qquad (44) \end{aligned}}$$

This is now a system with an infinite number of coupled differential equations.

To proceed we use a naughty trick. Consider a simple differential equation like

$$\frac{dx}{dt} = ax + f(t) \tag{45}$$

where $f(t)$ is some known function. The solution is

$$x(t) = x_0 e^{at} + \int_0^t dt' \, e^{a(t-t')} f(t') \tag{46}$$

If we focus now on (44), you will see it looks like (45). The difference is that "$f(t)$" is not a given function, but is part of the thing we want to find, $c_0(t)$. Notwithstanding, the solution (46) still holds on a formal level, so we may write

$$c_k(t) = c_k(0) e^{-i\omega_k t} - i g_k \int_0^t dt' \, e^{-i\omega_k(t-t')} c_0(t') \tag{47}$$

This is formally true, although not immediately useful since we don't know $c_0(t)$ (yet). Since $c_k(0) = 0$ the first term drops out and we are left with

$$c_k(t) = -i g_k \int_0^t dt' \, e^{-i\omega_k(t-t')} c_0(t') \tag{48}$$

We now plug this in Eq (43), leading to

$$\dot{c}_0 = -i\Omega c_0 - \sum_k g_k^2 \int_0^t dt' \, e^{-i\omega_k(t-t')} c_0(t') \tag{49}$$

This is now an equation for c_0 only. But the equation is integro-differential. It is convenient to define a new variable

$$\tilde{c}_0 = e^{i\Omega t} c_0 \tag{50}$$

Then

$$\dot{\tilde{c}}_0 = i\Omega \tilde{c}_0 + e^{i\Omega t} \dot{c}_0$$

$$= \underbrace{i\Omega \tilde{c}_0}_{\text{cancel out}} + \underbrace{e^{i\Omega t} \Big[-i\Omega c_0}_{\text{cancel out}} - \sum_k g_k^2 \int_0^t dt' e^{-i\omega_k (t-t')} c_0(t') \Big]$$

$$= - \sum_k g_k^2 \int_0^t dt' \, e^{i\Omega t} e^{-i\omega_k (t-t')} e^{-i\Omega t'} \tilde{c}_0(t')$$

Thus

$$\dot{\tilde{c}}_0 = - \sum_k g_k^2 \int_0^t dt' \, e^{i(\Omega - \omega_k)(t-t')} \tilde{c}_0(t') \tag{51}$$

We can make this even cleaner by introducing

$$\boxed{\chi(t-t') = \sum_k g_k^2 e^{i(\Omega - \omega_k)(t-t')}} \tag{52}$$

so that we arrive at

$$\boxed{\dot{\tilde{c}}_0 = - \int_0^t dt' \chi(t-t') \tilde{c}_0(t')} \tag{53}$$

This equation shows that $\chi(t-t')$ is a memory kernel. It weights the contribution of $\tilde{c}_0(t')$ from all past times t'.

So far we have said absolutely nothing about the structure of the field, such as the functions ω_k and g_k. Quite remarkably, we see that all properties of the field are contained in the memory kernel χ. The detailed values of g_k and ω_k are not important. The only thing that matters is the specific combination of Eq. (52).

The electromagnetic modes k actually vary continuously. It is therefore useful to introduce the spectral density

$$\boxed{J(\omega) = 2\pi \sum_k g_k^2 \, \delta(\omega - \omega_k)} \tag{54}$$

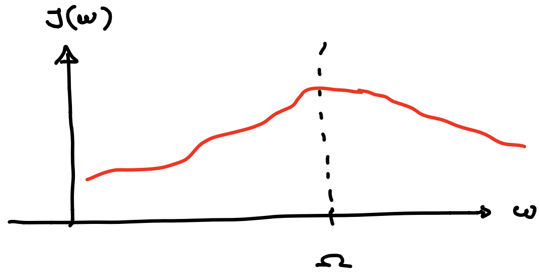
This is the most important quantity characterizing an environment. The idea is that $J(\omega)$ is a δ-comb, but since the ω_k are so close together, the comb becomes a kind of smooth function.

In terms of $J(\omega)$ we can rewrite (52) as

$$\chi(t) = \int_0^\infty \frac{d\omega}{2\pi} \, J(\omega) \, e^{i(\Omega - \omega)t} \tag{55}$$

To understand why, just substitute (54) here and you will see you get back (52).

Now let's look at (55). There is an integral over all frequencies, of $J(\omega)$. But the exponential has $e^{i(\Omega-\omega)t}$, where Ω is the frequency of the atom. When ω is really far from Ω, $e^{i(\Omega-\omega)t}$ will oscillate very very fast and things will thus have a tendency to cancel out. The main contribution should therefore come from $J(\omega)$ close to Ω.



Since all that will matter is $J(\Omega)$, we can substitute $J(\omega) \to J(\Omega)$ in (55), which can then be taken outside the integral.

$$\chi(t) \simeq J(\Omega) \int_0^\infty \frac{d\omega}{2\pi} e^{i(\Omega-\omega)t} \tag{56}$$

This is called the Wigner-Weisskopf approximation. A more rigorous analysis shows that this approximation is good when $J(\omega)$ is smooth and $J(\Omega) << \Omega$. For simplicity of notation, we write

$$J(\Omega) := \gamma \tag{57}$$

The resulting integral in (56) looks like a δ-function. Recall the identity

$$\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tag{58}$$

Now change variables in (56) to $\omega' = \omega - \Omega$:

$$x(t) = \gamma \int_{-\Omega}^{\infty} \frac{d\omega'}{2\pi} \, e^{-i\omega' t} \qquad (59)$$

This is not (58) only because of the lower limit of integration. But it can be shown that the correction that comes from this is small, so we may approximate

$$\boxed{x(t) \simeq \gamma \, \delta(t)} \qquad (60)$$

Going back to (53), we now get

$$\dot{\tilde{c}}_0 = -\int_0^t dt' \, x(t-t') \, \tilde{c}_0(t') = -\gamma \int_0^t dt' \, \delta(t-t') \, c_0(t')$$

$$= -\frac{\gamma}{2} \tilde{c}_0(t)$$

where the factor of $1/2$ is because we only get $1/2$ of the δ, since it is evaluated at the limit of integration. To summarize

$$\boxed{\dot{\tilde{c}}_0 = -\frac{\gamma}{2} \tilde{c}_0 \quad \rightsquigarrow \quad \tilde{c}_0(t) = e^{-\gamma t/2}} \qquad (61)$$

where I also used $c_0(0) = 1$. But c_0 is the coefficient multiplying $|0, e\rangle$. Thus $c_0(t)$ is precisely the probability of finding the atom in the excited state

$$\boxed{P_e = |c_0|^2 = |\tilde{c}_0|^2 = e^{-\gamma t}} \qquad (62)$$

Et voila, Émission spontanée: The prob. that the atom is found in the excited state decays exponentially in time with a coefficient γ which depends on the spectral density [c. f. Eqs (57) and (54)]. That is, γ depends on how the atom couples to the electromagnetic field.

It is remarkable that it couples even with the vacuum. Quantum systems are always open!

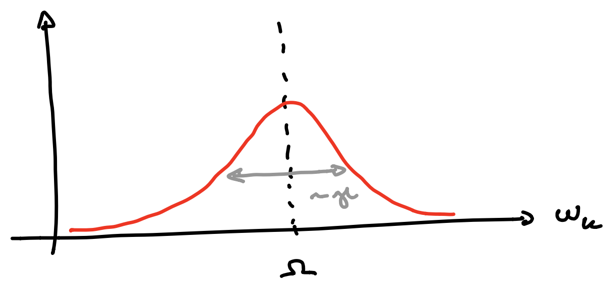
Now, where does the atom emitt to? To answer that we go back to Eq (48):

$$
\begin{aligned}
c_n(t) &= -i g_n \int_0^t dt\, e^{-i\omega_n (t-t')} c_0(t') \\
&= -i g_n \int_0^t dt\, e^{-i\omega_n (t-t')} e^{-i\Omega t'} e^{-\gamma t'/2} \\
&= -i g_n\, e^{-i\omega_n t} \left\{ \frac{1 - e^{-\left[\frac{\gamma}{2} + i(\omega_n - \Omega)\right]t}}{\frac{\gamma}{2} + i(\omega_n - \Omega)} \right\}
\end{aligned} \qquad (63)
$$

For large times the 2^{nd} term vanishes because of $e^{-\gamma t/2}$ and we get

$$\boxed{|c_n(\infty)|^2 = \frac{g_n^2}{\frac{\gamma^2}{4} + (\omega_n - \Omega)^2}} \qquad (64)$$

This is a Lorentzian distribution centered in Ω and with width γ



The atom therefore does not emit on the same frequency Ω, but it actually emits on a set of frequencies around Ω. However, in most systems $\gamma \ll \Omega$ so that this peak is very sharp.