Solid State Physics 2 - Problem set 2

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1. Consider the Heisenberg ferromagnetic model in one dimension:

$$H = -J \sum_{n=1}^{N} \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} \tag{1}$$

   (a) Construct the Heisenberg equations of motion for each spin component,

$$\frac{\mathrm{d}\langle S_n^\alpha \rangle}{\mathrm{d}t} = i\langle [H, S_n^\alpha] \rangle \tag{2}$$

   where $\alpha = x, y, z$.

   (b) Linearize the previous equations by assuming that we can replace $S_n^z \simeq S$. That is, we are looking for solutions which represent small deviations from the fully magnetized state.

   (c) Show that you can solve your linearized equations using the ansatz

$$\langle S_n^x \rangle = A e^{ikx_n} \sin \omega t, \qquad \langle S_n^y \rangle = A e^{ikx_n} \cos \omega t \tag{3}$$

   Find the dispersion relation $\omega(k)$.

2. In this problem I want you to investigate the magnetic frustration in a square lattice with antiferromagnetic interaction between nearest and second nearest-neighbors. The Hamiltonian is

$$H = \sum_{i,j} J(\boldsymbol{R}_i - \boldsymbol{R}_j) \boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{4}$$

   where

$$J(\pm \hat{\boldsymbol{x}}) = J(\pm \hat{\boldsymbol{y}}) = J_1 > 0, \qquad J(\pm \hat{\boldsymbol{x}} \pm \hat{\boldsymbol{y}}) = J_2 > 0 \tag{5}$$

   The reason why this system is frustrated is as follows: if we have only the AFM nearest-neighbor interaction J_1, then second nearest-neighbors will tend to align ferromagnetically. But $J_2 > 0$ wants to make second nearest-neighbors anti-parallel, so there is a competition between the two terms.

   (a) Find the tight-binding dispersion relation $J(\boldsymbol{q})$.

   (b) Find the vector $\boldsymbol{Q}$ which minimizes $J(\boldsymbol{q})$ and discuss the corresponding magnetic configuration. The special point here is at $J_2 = J_1/2$. Separate the analysis into $J_2 < J_1/2$ and $J_2 > J_1/2$. Discuss the physics of these two regimes.

(c) Now consider specifically the case $J_2 = J_1/2$. Show that in this case there is an infinite number of $\boldsymbol{Q}$ vectors which minimize $J(\boldsymbol{q})$. This means that at this point the ground-state is massively degenerate and the system is magnetically frustrated.

3. Consider the Landau free energy for a superconductor,

$$F = \int \mathrm{d}^3\boldsymbol{r} \left\{ \frac{\boldsymbol{B}^2}{8\pi} + \frac{\hbar^2}{2m}\left|\nabla\psi - \frac{2ie}{\hbar c}\boldsymbol{A}\psi\right|^2 + \frac{a}{2}|\psi|^2 + \frac{b}{4}|\psi|^4 \right\} \tag{6}$$

where $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is the magnetic field and, unlike in the lecture notes, I'm using CGS units because I based this problem on Landau's book and I am too lazy to change to SI.

(a) Find the equations that minimize F. To do that, you need to use some calculus of variations to vary the functional $F[\psi, \psi^*, A_x, A_y, A_z]$ (you should treat ψ and ψ^* as independent variables). Your equation for ψ will look like a non-linear Schrödinger equation. But the cool part is that the equation for A_i will give you Maxwell's equations

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} \tag{7}$$

but with a current which is given by

$$\boldsymbol{J} = -\frac{ie\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{2e^2}{mc}|\psi|^2\boldsymbol{A} \tag{8}$$

which is the expression for the probability current we get in quantum mechanics.

(b) Suppose now that ψ is given by its equilibrium value, $|\psi| = \sqrt{-a/b}$. Use this and substitute Eq. (8) into Eq. (7) to arrive at London's equation:

$$\nabla^2\boldsymbol{B} = \frac{\boldsymbol{B}}{\delta^2} \tag{9}$$

where δ is the London penetration depth. Compute it in terms of a, b and the other fundamental constants and show that it diverges at $T = T_c$ (recall that $a \sim (T - T_c)$).

4. In this problem I want you to study a model recently investigated experimentally by Landig *et. al.* in Nature, **532** 476 (2016). Their system consists of a square lattice with K sites, each described by a bosonic operator b_i. The Hamiltonian is

$$H = \sum_i \left\{ \frac{U_s}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i \right\} - J\sum_{\langle i,j\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) - \frac{U_\ell}{K}\Theta^2 \tag{10}$$

where

$$\Theta = \sum_{i \in e} \hat{n}_i - \sum_{i \in o} \hat{n}_i \tag{11}$$

Except for the last term in Eq. (10), this Hamiltonian is exactly the Bose-Hubbard Hamiltonian we studied in class. The new term is the last one, which is a long range interaction between all sites in the lattice. The idea is that we divide the square lattice into even (e) and odd (o) sites, like a chess board. The operator Θ is the *imbalance* operator: it measures the imbalance between the number of particles in the odd sites and the number of particles in the even sites. The energy term $-\frac{U_\ell}{K}\Theta^2$ therefore favors an imbalanced configuration.

Study this problem in the mean-field approximation. For the long-range part, approximate

$$\Theta^2 \simeq 2\langle\Theta\rangle\Theta - \langle\Theta\rangle^2 \tag{12}$$

For the hopping term, use the same mean-field approximation as in the Bose-Hubbard model. But now introduce two order parameters $\psi_e = \langle b_i \rangle$ for $i \in e$ and $\psi_o = \langle b_i \rangle$ for $i \in o$. Your model will therefore have a total of 3 order parameters. Write down the effective Hamiltonian within the mean-field approximation and show that it can reduced to a system of two bosonic modes (recall that in the Bose-Hubbard model we reduced our problem to a single bosonic mode. Now we have to distinguish between even and odd sub-lattices, so we need two bosonic modes). Challenge: construct the phase diagram numerically. See, for instance, Dogra *et. al.*, PRA, **94**, 023632 (2016).