Exercises for Introduction to Quantum Optics

Prof. Cecilia Cormick - February 2019

1. *Orders of magnitude*

   (a) Consider the 852 nm wavelength corresponding to a transition of the cesium atom. Calculate the frequency of this radiation and the equivalent in temperature ($\hbar\omega = kT$).

   (b) The most important transition in sodium (^{23}Na, m = 22,99 u, u = 1,66 10^{27} kg) has a wavelength of 589 nm. How much is this energy in Joules? Which velocity of the atom would give it this amount of kinetic energy? Assume that the atom is initially at rest and in an electronic excited state. If it decays emitting a photon of this wavelength, imposing conservation of energy and momentum, give the final velocity of the atom and its kinetic energy.

2. *Optical pumping*

   Consider a laser with propagation direction along z and circular polarization σ_+. Take the z axis as direction of quantization for angular momentum. Consider an atom whose ground state has zero orbital angular momentum, $l = 0$, and spin $s = 1/2$. We call the two ground states $|g_{\pm 1/2}\rangle$ according to the projection of the spin along z. We assume that the intensity of the laser is weak enough to neglect the population of the excited state (if the atom gets excited it rapidly decays through spontaneous emission).

   (a) We consider that the laser is close to resonance with a transition between the ground state $|g_{\pm 1/2}\rangle$ and an excited state with $l = 1$, $s = 1/2$, $j = 1/2$ (once more with two sublevels m_j given by the projection of total angular momentum along z). Using the Clebsch-Gordan coefficients, write the two sublevels of the excited state in terms of the eigenstates of L_z and S_z, characterized by the quantum numbers m_l, m_s.

   (b) Remember the selection rules for optical transitions due to fields with plane wavefronts: upon absorption of a photon, the spin state does not change, and the projection along z of orbital angular momentum changes according to:

   - σ_+ polarization: $\Delta m_l = +1$
   - σ_- polarization: $\Delta m_l = -1$
   - Linear polarization in $\hat{z}$: $\Delta m_l = 0$

   (for a field propagating along z, polarization must be in the x-y plane and can always be written as combination of σ_+ and σ_-).

   Draw in a scheme the two ground and the two excited states and, using these rules, indicate which transitions can be induced by the laser in the eigenbasis of the atomic Hamiltonian. Notice that these transitions are the same for absorption and stimulated emission.

   (c) Taking into account that spontaneous emission can be in principle in any propagation direction and with any polarization, draw the possible atomic transitions obtained as a result of spontaneous decay.

   (d) Using the previous results, explain with the the asymptotic state of the atom.

   (e) Repeat the same idea but for the case when the resonant excited state has quantum numbers $l = 1$, $s = 1/2$, $j = 3/2$.

3. *Coherent evolution of a two-level atom*

   Consider an atom with two internal levels, $|g\rangle$ (the ground state) and $|e\rangle$ (an excited state). The atom is under the action of a laser such that, in the frame rotating with the laser, the Hamiltonian for the evolution of the internal state is of the form:

$$H = \frac{\hbar}{2}[-\Delta\sigma_z + \mathrm{Re}(\Omega)\sigma_x - \mathrm{Im}(\Omega)\sigma_y]$$

   where $\Omega \in \mathbb{C}$ is proportional to the amplitude of the laser and Δ is the detuning with respect to the atomic transition, $\Delta = \omega_L - \omega_a$.

(a) Rewrite the Hamiltonian in the form $H = (\hbar/2)\,\tilde{\Omega}\,\hat{n}\cdot\vec{\sigma}$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\hat{n}$ a norm 1 vector, and $\tilde{\Omega} \in \mathbb{R}$, giving $\tilde{\Omega}$ and $\hat{n}$ as a function of Δ and Ω.

(b) Show that $(\hat{n}\cdot\vec{\sigma})^2 = \mathbb{I}$.

(c) Using that property and the power expansion of the exponential, show that the evolution operator in this frame is:

$$U(t) = e^{-iHt/\hbar} = \cos(\tilde{\Omega}t/2) - i\sin(\tilde{\Omega}t/2)\,\hat{n}\cdot\vec{\sigma}$$

(d) Calculate the probability to find the atom in the excited state as a function of time, assuming it is initially in the ground state. Plot for different values of Δ and Ω. Indicate which is the maximum population that can be transferred to the excited state given Ω and Δ.

Obs. 1: $|\Omega|$ is the so-called "Rabi frequency", since it is the frequency of the oscillations in the population for the resonant case $\Delta = 0$, but the evolution operator $U(t)$ is periodic with half the frequency, so some authors use Rabi frequency for $|\Omega|/2$ (notice that on resonance $U(t = 2\pi/\tilde{\Omega}) = -\mathbb{I}$).

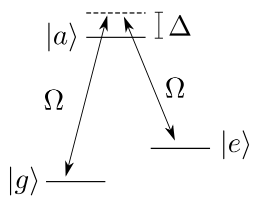
Obs. 2: I fix the basis $\{|e\rangle, |g\rangle\}$, so the Pauli matrices are

$$\sigma_x = |g\rangle\langle e| + |e\rangle\langle g|, \quad \sigma_y = i(|g\rangle\langle e| - |e\rangle\langle g|), \quad \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$$

This satisfies that $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ correspond to raising and lowering operators, i.e. σ_+ takes the atom to the excited level and σ_- does the opposite.

4. *Adiabatic elimination and two-photon transitions in a three-level atom*

Consider an atom with three internal levels, which we call $|g\rangle$, $|e\rangle$, and $|a\rangle$. The levels $|g\rangle$ and $|e\rangle$ are connected with $|a\rangle$ through dipolar transitions, so there is no dipolar transition between them. One can anyway generate and effective two-level system using $|a\rangle$ in an intermediate step, with two lasers far away from the resonance. This is called a Λ-system (see figure, but notice that for this treatment it doesn't matter if $|g\rangle$ and $|e\rangle$ have lower energies than $|a\rangle$ or not). In this excercise the effective Hamiltonian will be derived using a procedure called "adiabatic elimination". There are many methods to do this, and when standard perturbation theory is applicable, then it is also equivalent, but adiabatic elimination is used in more general situations (most importantly for open systems), so it's worth learning what it is.



Setting the zero of energy at level $|g\rangle$ and using a rotating wave approximation, the system Hamiltonian is given by:

$$H = E_e|e\rangle\langle e| + E_a|a\rangle\langle a| + \hbar\frac{\Omega}{2}(|a\rangle\langle g|e^{-i\omega_1 t} + |g\rangle\langle a|e^{i\omega_1 t}) + \hbar\frac{\Omega}{2}(|a\rangle\langle e|e^{-i\omega_2 t} + |e\rangle\langle a|e^{i\omega_2 t})$$

where $\omega_1 = E_a/\hbar + \Delta$, $\omega_2 = (E_a - E_e)/\hbar + \Delta$. We take Ω real (in this case with no loss of generality, why?) and the same for the two lasers to simplify the calculation. The detuning $|\Delta|$ is assumed to be much larger than Ω and that the spontaneous decay rate from level $|a\rangle$ (so we neglect spontaneous emission). Then, the transitions involving level $|a\rangle$ are off-resonant but two-photon processes are resonant.

(a) Move to a frame rotating with $H_0 = E_e|e\rangle\langle e| + E_a|a\rangle\langle a|$, so that the Hamiltonian $H'(t)$ in this frame is given by:

$$H'(t) = U_0(t)^\dagger H U_0(t) - H_0$$

with $U_0(t) = e^{-iH_0 t/\hbar}$. Calculate $H'(t)$ and show that all its terms oscillate at a frequency which is much higher than the time scale associated with the Hamiltonian amplitude.

(b) The procedure that follows is based on that separation of time scales. To make things clear at least the first time, it is convenient to make an expansion in the small dimensionless parameter $\epsilon = \Omega/\Delta$. For this, we define a dimensionless time variable $\tau = t\Delta$, so that the Hamiltonian in the previous item transforms as $H' \to \tilde{H} = H'/\Delta$. Write $\tilde{H}(\tau)$ and see that its terms are proportional to $\hbar\epsilon$ and oscillate in a time scale of order 1.

(c) The evolution operator $\tilde{U}(\tau)$ in this representation can be written as an expansion in powers of ϵ as:

$$\begin{aligned}\tilde{U}(\tau) &= \sum_{n=0} \tilde{U}_n(\tau) \\ &= \mathbb{I} - \frac{i}{\hbar}\int_0^\tau d\tau_1 \tilde{H}(\tau_1) - \frac{1}{\hbar^2}\int_0^\tau d\tau_2 \int_0^{\tau_2} d\tau_1 \tilde{H}(\tau_2)\tilde{H}(\tau_1) + \sum_{n>2}\left(\frac{-i}{\hbar}\right)^n \int_0^\tau d\tau_n \ldots \int_0^{\tau_2} d\tau_1 \tilde{H}(\tau_n)\ldots\tilde{H}(\tau_1)\end{aligned}$$

with $\tilde{U}_n(\tau)$ proportional to ϵ^n (and depending on the dimensionless time τ).

Evaluate $\tilde{U}_1(\tau)$ and show that it is a purely oscillatory function of τ, so that after a time τ of order 1 (i.e. a time t of order Δ^{-1}) this function no longer increases, and is of order ϵ at all times.

(d) Evaluate now the second-order term $\tilde{U}_2(\tau)$ and show that it has two components: one term proportional to τ, and one oscillatory term. Indicate which is the condition on τ which guarantees that the linear term in $\tilde{U}_2$ is much larger than the oscillatory terms in $\tilde{U}_1$. Show that this condition also implies that the oscillatory terms in $\tilde{U}_2$ can be neglected. Rewrite this condition in terms of the original time variable t.

(e) In similar form one can go on with the remaining terms in the expansion. All odd terms result in oscillatory contributions, while even powers of ϵ give some non-oscillating terms. These can be identified with the terms arising from a power series for the evolution given by an effective time-independent Hamiltonian $\tilde{H}_{\text{eff}}$. In particular, the linear term in $\tilde{U}_2$ corresponds to the first order of the effective evolution, $\tilde{U}_{1,\text{eff}}(\tau)$, and so on. Infer from $\tilde{U}_{1,\text{eff}}(\tau)$ the form of the effective Hamiltonian $\tilde{H}_{\text{eff}}$.

(f) Identify the effects present in the effective Hamiltonian. Notice that the coupling to level $|a\rangle$ disappears, but there is an effective coupling between the two other levels, as well as an AC-Stark shift.

(g) Come back to the initial representation with time t (i.e. multiply the effective Hamiltonian by Δ). Indicate the value of the AC-Stark shift for each of the levels, and the frequency of the transition between $|g\rangle$ and $|e\rangle$ (remember that for this treatment this frequency must be much smaller than Ω).

(h) Think about how you could have solved the same system using standard perturbation theory.

5. *Jaynes-Cummings model*

We consider a two-level atom in a cavity such that the atom is coupled with only one mode of the field. Using the rotating wave approximation, the Hamiltonian can be written as $H = H_0 + V$, with $H_0 = \hbar\omega_0|e\rangle\langle e| + \hbar\omega_c\, a^\dagger a$ and $V = \hbar g(a\sigma_+ + a^\dagger\sigma_-)$ (it's standard to use g also for the coupling... the meaning of the letter should be clear from the context).

(a) Show that the number of excitations is conserved, i.e. $[H, N] = 0$ with $N = a^\dagger a + |e\rangle\langle e|$.

(b) From the previous result, the Hilbert space can be split in two-level subspaces with bases: $\{|g, n+1\rangle, |e, n\rangle\}$, according to the eigenvalues of the operator N, and the time evolution does not couple the different subspaces (the state $|g, 0\rangle$ is decoupled from all others). Write the Hamiltonian as a 2×2 matrix for each of these subspaces.

(c) Draw schematically the spectrum for subspace $\{|g, n+1\rangle, |e, n\rangle\}$ as a function of the detuning $\Delta_c = \omega_c - \omega_0$ (leaving either ω_c or ω_0 fixed). Indicate the form of eigenstates and eigenvalues on resonance and very far away from it.

(d) Consider the resonant case, $\Delta_c = 0$, and calculate all eigenstates and eigenvalues. Make a scheme of the seven lowest values of the energy (remember $\omega_0 \gg g$) indicating the corresponding eigenvalues. Calculate the evolution of a system initially in state $|g, n+1\rangle$, and give the probability to find the atom in the excited state as a function of time. Indicate the oscillation frequency for this probability.

(e) Consider again the resonant case but assume that the initial state is of the form $|g, \alpha\rangle$ with the atom in the ground state and the field in a coherent state $|\alpha\rangle$. Calculate the time evolution of the state and the probability to have an excited atom as a function of time. Plot this probability for the cases $|\alpha|^2 = 10, 20, 30, 50$. Discuss the limit $|\alpha|^2 \gg 1$ and compare with the semiclassical treatment.

6. *Bloch equations*
Consider the evolution of a two-level atom as in problem 3, but now subjected also to spontaneous decay, introducing a non-unitary term in the evolution of the form:

$$\mathcal{L}_\Gamma \rho = \frac{\Gamma}{2}(2\sigma_- \rho \sigma_+ - \{\sigma_+\sigma_-, \rho\}).$$

(notice that the form of this non-unitary term is the same in the frame rotating with the laser and in the non-rotating frame).

(a) Calculate the equations for the evolution of the elements of the density matrix of the atom (notice that only two of the four elements are independent).

(b) Rewrite the equations in terms of the real variables appearing in the Bloch equations, $(u, v, w) = (1/2)\langle\vec{\sigma}\rangle$ (always in the rotating frame).

(c) Solve the equations and describe the evolution for the cases when:

i. $\Omega = 0$.

ii. $\Gamma = 0$.

iii. $\Delta = 0$.

(d) Find the stationary state (in the rotating frame) for arbitrary values of Ω, Γ, Δ (notice that one must have $\Gamma \neq 0$ in order to have a stationary state).

(e) Move back to the non-rotating frame and write the state as a function of time in the asymptotic regime $t \to \infty$.

7. *Coherent coupling between atoms and electromagnetic field*

Find information about at least two different research groups that have setups coupling atomic transitions with the electromagnetic field at the quantum level (i.e. with a non-negligible single-photon coupling). Identify:

(a) Do they use one or many atoms, are the atoms neutral or charged?

(b) Which is the wavelength of the field and which part of the spectrum does it belong to? (i.e. microwave, optical, etc).

(c) Which is the atomic transition coupling with the field, and which are the lifetimes of the corresponding levels?

(d) How much is the coupling strength between the field and atom(s)? (if this is not given, find related information if any, for instance how many repetitions of the experiment are required for the absorption of a single photon).

(e) Does the experiment run on resonance or away from it? If it's the latter, how large is the detuning?

(f) Do they use the field in a cavity? If so, which is the linewidth of the cavity? (the definition of the quality factor can vary a bit but it is basically $Q \propto \omega/\kappa$). Do they mention the size of the cavity?

8. *Mediated interactions between ions*

To build a quantum computer with a collection of two-level systems (qubits), it is enough to be able to implement single-qubit gates and at least one kind of two-qubit entangling gate. This exercise shows in a simplified form how one can apply an entangling operation between the internal states of two ions using a normal oscillation mode as mediator. We will treat each ion as a three-level system: two of the levels, $|g\rangle$ and $|e\rangle$, correspond to the computational states, and the third, $|e'\rangle$, is auxiliary. The oscillation of the center-of-mass mode of the two-ion chain is treated as a quantized harmonic oscillator with creation and annihilation operators $a^\dagger$ and a.

With a laser acting on ion j ($j = 1$ or 2) with the right frequency we can generate a Hamiltonian which, in a rotating frame with respect to the free Hamiltonian H_0, takes the form:

$$H_j = \hbar\frac{\Omega}{2}\left[(|e\rangle\langle g|)_j\, a + (|g\rangle\langle e|)_j\, a^\dagger\right]$$

(so it is a realization of the Jaynes-Cummings Hamiltonian, without involving level $|e'\rangle$). Alternatively, changing the frequency and/or polarization of the laser we can generate a similar Hamiltonian H'_j but which involves level $|e'\rangle$ instead of $|e\rangle$.

Calculate for each step the effects of the following sequence, assuming the initial state:

$$|\psi_0\rangle = \frac{|g\rangle + |e\rangle}{\sqrt{2}} \frac{|g\rangle + |e\rangle}{\sqrt{2}} |0\rangle$$

which is a state without entanglement between the electronic states of the ions, and with the vibrational mode in the ground state.

(a) Hamiltonian H_1 is applied during a time π/Ω, which implements on ion 1 and the mode the transformation:

$$|g\rangle|1\rangle \to -i|e\rangle|0\rangle, \quad |e\rangle|0\rangle \to -i|g\rangle|1\rangle$$

while the state $|g\rangle|0\rangle$ is not changed.

(b) Hamiltonian H_2' is applied for a time $2\pi/\Omega$, implementing on ion 2 and the mode the transformation:

$$|g\rangle|1\rangle \to -|g\rangle|1\rangle, \quad |e'\rangle|0\rangle \to -|e'\rangle|0\rangle$$

while the state $|g\rangle|0\rangle$ is not modified (and neither any state in which ion 2 is in $|e\rangle$).

(c) The same pulse as in the first step is repeated.

Show that in the final state the mode is back at the vacuum state, while ions 1 and 2 are in a two-qubit maximally entangled state, so that the two-qubit state is pure but each of the qubits has a reduced density matrix equal to $(|g\rangle\langle g| + |e\rangle\langle e|)/2$.