Quantum Information - Problem set 1

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Deadline: 29/04 (Monday)

1. **Similarity transformations**

   (a) Let A be a Hermitian matrix and $B = SAS^{-1}$, where S is arbitrary. Relate the eigenvalues and eigenvectors of B and A. This is called a similarity transformation. Unitaries are a particular case, in which $S^{-1} = S^{\dagger}$. We can also invert the argument on our previous result: if A and B are two operators which share the same eigenvalues, then there must exist a similarity transformation between them. If A and B are both Hermitian, then this transformation can be accomplished by a unitary.

   (b) Show that $Sf(A)S^{-1} = f(SAS^{-1})$: similarity transformations infiltrate, just like unitaries!

2. **Positive semidefinite matrices**

   (a) Let C be an arbitrary (not necessarily Hermitian) operator. Show that $C^{\dagger}C$ is positive semidefinite.

   (b) Now invert the argument: given a density matrix ρ (which is Hermitian and positive semidefinite), show that one can always find a matrix C such that $\rho = C^{\dagger}C$. This is the matrix equivalent of taking a square root. Tip: use the eigenstructure of ρ.

3. **Functions of operators.** The goal of this exercise is to show you that, when manipulating functions of operators, all that matters is the *algebra*. That is, we don't need to know what are the actual matrix elements or even if the matrix is finite or infinite. All properties follow only from the abstract algebra between operators.

   (a) Let A be an operator such that $A^3 = 1$. Find $e^{\alpha A}$, where α is a constant. The result is a bit ugly, but with Mathematica it's super easy.

   (b) Let A be an operator such that $A^2 = 0$. Find $e^{\alpha A}$.

   (c) Consider the angular momentum operators $S_{x,y,z}$ satisfying $[S_i, S_j] = i\epsilon_{i,j,k}S_k$. Compute $e^{\alpha S_x} S_z e^{-\alpha S_x}$.

4. **Dephasing channel.** We saw in the lectures that the most general operation taking density matrices into density matrices is a quantum channel (CPTP map) of the form

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^{\dagger}, \qquad \sum_k M_k^{\dagger} M_k = 1. \tag{1}$$

   An important example of a quantum channel is the dephasing channel, characterized by the set of Kraus operators

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \tag{2}$$

   where $\lambda = 1$.

   (a) Investigate the action of the dephasing channel on a general qubit density matrix.

(b) Provide a geometric interpretation in terms of Bloch's sphere.

(c) Suppose the system starts in a pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Compute the von Neumann entropy before and after applying a dephasing channel.

(d) Interpreting the set of Kraus operators $\{M_k\}$ in (2) as the measurement operators for a generalized measurement, discuss the measurement backaction on a general pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

5. **Freedom in the Kraus representation.**

(a) The choice of Kraus operators $\{M_k\}$ is not unique. Show that if two sets of Kraus operators are related by

$$M_k = \sum_q U_{k,q}, N_q, \tag{3}$$

where $U_{k,q}$ is a unitary ($UU^\dagger = U^\dagger U = 1$), then the quantum channel (1) remains exactly the same.

(b) Use the result from (a) to show that the dephasing map (2) can also be written in terms of the Kraus operators

$$N_0 = \sqrt{\alpha}\, \mathbb{I}_2, \qquad N_1 = \sqrt{1-\alpha}\, \sigma_z. \tag{4}$$

Find the relation between α and λ.

(c) Interpreted as a measurement, what is the backaction of this new set of Kraus operators. Is it the same as that of the representation (2)?

6. **Dephasing using a CNOT.** Consider two qubits, S and E (here E stands for "environment"), with S prepared in some arbitrary state ρ_S and E prepared in a statistical mixture $\rho_E = p|+\rangle\langle+| + (1-p)|-\rangle\langle-|$. The two systems then interact according to a CNOT having E as the control bit:

$$U_{\text{CNOT}} = |0\rangle\langle 0|_E \otimes \mathbb{I}_S + |1\rangle\langle 1|_E \otimes \sigma_x^S. \tag{5}$$

Consider now the map obtained by first evolving SE and then taking the partial trace over the environment:

$$\rho_S' = \text{tr}_E\Big\{U(\rho_S \otimes \rho_E)U^\dagger\Big\}. \tag{6}$$

Show that this map produces the dephasing channel (2) for certain values of p. Relate p to λ. Eq. (6) is called a **Stinespring dilation** of the quantum channel: it is a way of representing a quantum channel as some unitary evolution of a system plus an environment, and then tracing out the environment.

7. **SWAP gate.** The SWAP gate acts on two qubits in the following way:

$$U_{\text{SWAP}}|i,j\rangle = |j,i\rangle. \tag{7}$$

(a) Write down the matrix elements of U_{SWAP} in the computational basis.

(b) Verify that one may write

$$U_{\text{SWAP}} = \frac{1}{2}(1 + \boldsymbol{\sigma}^A \cdot \boldsymbol{\sigma}^B), \tag{8}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. This form is originally due to Pauli.

(c) Show that the SWAP can be constructed by the application of 3 CNOT channels (5), alternating who the target qubit is.