Quantum Information - Problem set 2

Professor: Gabriel T. Landi

Deadline: 10/06 (monday)

1. **Entanglement in a GHZ states**

   Consider 4 qubits prepared in the GHZ state

$$|\psi\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}. \tag{1}$$

   The goal of this exercise is to characterize entanglement from the perspective of different *bipartitions*. For this you will use the idea of Schmidt decomposition introduced in Sec. 4.1 of the lecture notes. The starting point is always a decomposition of the form

$$|\psi\rangle = \sum_{ij} \psi_{ij}\, |i\rangle_A \otimes |j\rangle_B, \tag{2}$$

   and the realization that ψ_{ij} may be viewed as a matrix, to which one can apply a Singular Value Decomposition. The difference in the case of the state (1), compared to what we studied in class, is that we have four and not two qubits. We can still apply the same ideas, but now a 4-qubit systems can be partitioned in more than two ways.

   (a) Consider first the bipartition $A = \{1, 2\}$ and $B = \{3, 4\}$. That is, define a basis $|i\rangle_A = |00\rangle, |01\rangle, |10\rangle, |11\rangle$, and analogously for B. Find the matrix ψ_{ij} and perform the Schmidt decomposition to find the Schmidt form of the state.

   (b) Next repeat the procedure for a bipartition $A = \{1, 2, 3\}$ and $B = \{4\}$.

   (c) Compute the reduced density matrices of 1, 2 and 3 qubits. The GHZ state is quite dramatic because as soon as you trace over a single qubit, the resulting state is almost the maximally mixed state, with zero quantum correlations of any kind.

   (d) Compute the mutual information $\mathcal{I}(1:23)$ and $\mathcal{I}(1:2)$. Recall that

$$\mathcal{I}(A\!:\!B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

   Thus $\mathcal{I}(1:23) = S(\rho_1) + S(\rho_{23}) - S(\rho_{123})$ and $\mathcal{I}(1:2) = S(\rho_1) + S(\rho_2) - S(\rho_{12})$.

2. **Entanglement in *W* states**

   Consider now the W-state

$$|\psi\rangle_{\mathrm{W}} = \frac{|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle}{2}. \tag{3}$$

   This state is fundamentally different from the GHZ. In fact, it can be shown that it is impossible to go from one to the other using only local operations. Repeat the same calculations of the previous exercise for the W state.

3. **Qubit quantum channels**

   Qubits can be conveniently described in terms of the position in Bloch's sphere,

$$\boldsymbol{r} = (r_x, r_y, r_z), \qquad r_i = \mathrm{tr}(\sigma_i \rho). \tag{4}$$

The goal of this exercise is to explore the effects of different single-qubit channels on a typical Bloch sphere vector $\boldsymbol{r}$. To accomplish that, discuss how $\boldsymbol{r}$ changes upon application of the following channels:

(a) Bit-flip channel:

$$M_0 = \sqrt{p}\, I, \qquad M_1 = \sqrt{1-p}\sigma_x. \tag{5}$$

(b) Phase-flip channel:

$$M_0 = \sqrt{p}\, I, \qquad M_1 = \sqrt{1-p}\sigma_z. \tag{6}$$

(c) Depolarizing channel:

$$M_0 = \sqrt{1-3p/4}\, I, \qquad M_1 = \sqrt{\frac{p}{4}}\sigma_x, \qquad M_2 = \sqrt{\frac{p}{4}}\sigma_y, \qquad M_3 = \sqrt{\frac{p}{4}}\sigma_z. \tag{7}$$

In this case, verify also that the channel may be written as

$$\mathcal{E}(\rho) = \frac{p}{2} I + (1-p)\rho. \tag{8}$$

The depolarizing channel is therefore very special as it simply mixes ρ with the maximally mixed state $I/2$.

4. **Lindblad equation for the finite temperature amplitude damping**

Consider the Lindblad master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \gamma(1-f)\Big[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\Big] + \gamma f\Big[\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}\Big]. \tag{9}$$

(a) Check that this equation preserves the trace of any density matrix.

(b) Parametrize ρ in any convenient way you wish and find the solution of Eq. (9) for an arbitrary initial state.

(c) Consider now the so-called Finite Temperature Amplitude Damping (FTAD) channel, described by the Kraus operators,

$$M_0 = \sqrt{F}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \qquad M_1 = \sqrt{F}\begin{pmatrix} 0 & \sqrt{\lambda} \\ 0 & 0 \end{pmatrix}, \tag{10}$$

$$M_2 = \sqrt{1-F}\begin{pmatrix} \sqrt{1-\lambda} & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_3 = \sqrt{1-F}\begin{pmatrix} 0 & 0 \\ \sqrt{\lambda} & 0 \end{pmatrix}, \tag{11}$$

Show that the solution of Eq. (9) has the form of a FTAD channel and find the connection between the parameters (γ, f) and (λ, F).